Towards an Analytical Solution for Agent Based Models: an Application to a Credit Network Economy*

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Abstract

In this paper we present an analytically solvable model of the credit market with heterogeneous and interacting firms and banks. The economy is modelled as a network, a theoretical structure which is particularly suitable to represent the interactions among different agents. In this credit network firms interact directly with banks and, indirectly, among themselves. The main novelty is the use of the master equation to perform the aggregation over a population of heterogeneous firms and to describe the endogenous evolution of the network. The

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asymptotic solution of the master equation provides a system of coupled equations which governs the dynamics of growth and fluctuations of the aggregate output and the network degree.

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1 Introduction

The mainstream approach, as it is formalised in the DSGE (dynamic stochastic general equilibrium) models, is based on the process of intertemporal maximization of utility in the market clearing context of the standard competitive equilibrium theory. It is built upon the representative agent (RA) framework, which rules out direct interactions among agents by assumption. Its fundamental hypotheses trivially lead to conclusions that there can be no inefficiencies and any pathology in general[1].

In short, the RA framework of the DSGE models adopts the most extreme form of conceptual reductionism in which macroeconomics and the financial network is reduced to the behaviour of an individual agent[2]. The deep understanding of the interplay between the micro and macro levels is ruled out, as well as any "pathological" problems, such as coordination failure. The RA in economics is tantamount to stating that "macroeconomics equals microeconomics".

We believe that a change of focus is necessary: an appropriate microfoundation should consider the interaction at the agent based level. This is feasible if the economy is represented by a framework with heterogeneous agents. The development of sound micro-founded models should also involve

[1]There are attempts of reconciling the empirical evidence with the mainstream model by introducing various imperfections. Unfortunately, small departures from the perfect information hypothesis have been shown to undermine most of the key propositions of the standard competitive model (Greenwald and Stiglitz, 1986).

[2]In natural sciences, the notion of reductionism is much more limited since it amounts to represent the nature of macro-phenomena by analysing the constitutive elements, whose interaction allows for emergent phenomena, i.e. characteristics that are not present in the single element.

the links among agents in a networked economy. This approach can provide better insights on how a crisis emerges from the microeconomic interaction and how it propagates in the economic system.

The literature provided, up to now, two different tools for dealing with interaction and heterogeneity: game theory and computational economics (CE). In the game theory framework, each agent takes into account the behaviour of every other, which is supposed to be known. But a complete network populated by rational and fully informed agents would require infinite computational ability, which is beyond any plausible assumption. On the other hand, there has been a strong development of CE models, populated by interacting and boundedly rational agents. In these models, macro outcomes are very sensitive to different configurations of the parameters. Furthermore, the relationship between results and initial conditions is undefined, not to mention the issue of the trade-off between tractability and realism.

The idea of this paper is neither to ignore interactions between agents, nor to get hopelessly mired in complicated details by trying to model those interactions in their completeness, but rather to strike for some middle ground in which the consequences of interconnectedness can be at least crudely assessed. To achieve this, we start with a "minimal" model of the economy as a credit network, in which firms interact directly with banks. Rather than simulating these interactions, we represent them by means of probabilities, and derive equations describing the evolution of the network and how its structure changes with time. The equations we derive provide some qualitative insights into the systemic fragility of the credit network.

The results of the present paper are based on the solution algorithms proposed by Di Guilmi et al. (2012) and are here presented without the full detail of the derivation. The complete demonstrations are available upon request. The remainder of the paper is structured as follows: section 2 presents the general hypotheses of the model, introducing firms and banks; section 3 illustrates the solution of the model dynamics; section 4 interprets the results and proposes some policy implications; section 5 offers some final remarks.

2 The model

This section introduces the main hypotheses about the structure of the model, concerning the firms, the banks and their match on the credit market.

2.1 The firms sector

The productive sector is modelled along the lines of Delli Gatti et al. (2010). Each firm sets the optimal quantity of output on the basis of its financial condition, according to the following rule

$$Q_{f,t} = \alpha A_{f,t}^{\beta} \tag{1}$$

where $\alpha > 0$ and $\beta \in (0, 1)$ are constant parameters and A is the net worth of firm f. As shown in Greenwald and Stiglitz (1993), equation (1) comes from a profit maximisation problem, when bankruptcy is costly and the probability of default is inversely related to the net worth. For the sake of simplicity all firms are assumed to have the same linear production function, defined as

$$Q_{f,t} = (1/\varpi)N_{f,t}, \tag{2}$$

where Q is the physical output, N is the quantity of labour and $\varpi > 0$ is the inverse of labour productivity, assumed to be a constant parameter. From (1) and (2) it follows that $N_{f,t} = \varpi\alpha A_{f,t}^{\beta}$. The wage bill W is

$$W_{f,t} = wN_{f,t} \tag{3}$$

where the nominal wage w is constant and uniform across firms.

Firms do not know in advance the quantity of goods that will be demanded and this can produce uncertainty about the final price. For this reason, following Greenwald and Stiglitz (1993), we model the selling price for each firm u_f as a stochastic variable coming from a uniform distribution, defined in the interval $[u_{min}; u_{max}]$.

Because of imperfect information there is a hierarchy in the sources of firms financing, from internal to external finance. The firms that can finance their whole wage bill with internal resources are defined as *self financing* (SF), while the others are *non self financing* (NSF). The latter ones resort to the credit market as, by assumption, they are completely rationed on the equity market. The demand for credit for a NSF firm is given by

$$D_{f,t} = W_{f,t} - A_{f,t}, \tag{4}$$

while $D_{f,t} = 0$ for SF firms. Firms' profits $\pi_{f,t}$ are defined as

$$\pi_{f,t} = u_{f,t}Q_{f,t} - W_{f,t} - r_{f,t}D_{f,t}, \tag{5}$$

where r is the rate of interest. The net worth of firms is the sum of past profits:

$$A_{f,t+1} = A_{f,t} + \pi_{f,t} \tag{6}$$

If $A < 0$ the firm goes bankrupt and it is replaced by a new one, such that the total number of firms is constant in time.

2.2 The banking sector

A bank can lend to different firms and sets unilaterally the interest rate for each of them. In particular the bank b asks to the firm f an interest rate which is inversely related to the financial soundness of borrower and lender, according to the formula

$$r^f_{b,t} = a\left[A^{-a}_{b,t} + \left(\frac{D_{f,t}}{A_{f,t}}\right)^a\right] = a\left[A^{-a}_{b,t} + v^a_{f,t}\right] \tag{7}$$

where $a > 0$, $A_{b,t}$ is the bank's net worth and $v_{f,t}$ is the firm leverage ratio.

The dependence on $A_{b,t}$ represents the fact that, for a bank, the lower is its equity, the higher is its probability of default. Consequently, the owners of the bank's capital will demand a premium for risk which is inversely proportional to its internal financial resources. As a consequence, a higher net worth means a lower premium paid by the bank to investors and allows the bank to charge a lower interest rate to customers (Gambacorta, 2008). The second factor in the interest rate formula, $(v_{f,t})^a$, quantifies the higher risk premium requested by the bank to highly-leveraged firms.

This formula analytically devices the mechanism described by Stiglitz and Greenwald (2003, 145): "*The high rate of bankruptcy is a cause of the high interest rate as much as a consequence of it*". This is because the demise of one or more firms generates bad debt for the lender and, thus, brings about a deterioration in its financial conditions (a lower $A_{b,t}$). As a consequence, the bank will raise the interest rates, worsening the positions of its customers and, possibly, leading to the bankruptcy of some of them. The new bad debt has further negative impact on the financial soundness of the lender in a downward spiral.

The profits of the bank b are given by

$$\pi_{b,t} = \sum_f (1 + r^f_{b,t}) D^f_{b,t}, \tag{8}$$

where $D^f_{b,t}$ is the credit supplied by the bank b to the firm f. The bank net worth is computed as

$$A_{b,t+1} = A_{b,t} + \pi_{b,t} - BD_{b,t} \tag{9}$$

where BD is the bad debt, which is the debt that cannot be paid back due to the bankruptcy of borrowers. If $A_{b,t} < 0$, the bank goes bankrupt, and it is replaced by a new one. Thus, also the number of banks is constant.

2.3 Partner selection and network evolution

A NSF firm signs a one-period credit contract with one bank. In order to capture the fact that firms have limited information about the credit market, at each point in time a NSF firm can demand for credit only to a subset of banks. If we indicate with B the total number of banks in the economy, the number of banks that are "visible" to a firm is given by $\lfloor mB \rfloor$, where $m \in (0, 1]$ is a parameter constant across firms and in time. The subset of banks for each firm is randomly selected at each period. Hence every firm possibly surveys a different pool of banks every period.

The firm sorts the banks in its randomly selected pool according to the proposed interest rates and then sends a signal to the bank which offers the lowest interest rate, demanding for credit.

Credit rationing can arise as the banks must comply with a regulatory framework. In particular, the monetary authority determines an adequacy ratio along the lines of the Basel II accord. Consequently, the total lending of the bank b, denoted by $L_{b,t}$, must be lower than $\bar{\Theta} A_{b,t}$, where $\bar{\Theta} \geq 1$ is a constant parameter. Let us define

$$\bar{L}_{b,t} = A_{b,t}\bar{\Theta} \tag{10}$$

as the limit for lending that the bank b can supply.

In order to select how many and which requests for credit will be satisfied, the bank adopts a prudential criterion. It first sorts the potential customers in ascending order of leverage ratios v. Then it considers the demand for credit starting from the firm with the lowest v and accepts their demand as long as condition (10) is satisfied.

When a firm is refused credit from a bank, it sends a signal to the bank which follows in its list, that is a bank who offers a higher interest rate. If all the banks in the firm's pool decline to supply credit, the firm must reduce its production in order to meet its financial constraint. It can pay salaries only with its internal resources, and, accordingly, the quantity produced by a fully credit rationed firm is equal to $q_{f,t} = A_{f,t}/(w\varpi)$.

The structure of the network is represented by a series of islands, or cliques: each island is composed by the lending bank and its borrowing firms. The composition and the number of cliques vary from a period to another. The composition changes as a NSF firms can become SF (therefore not connected to any bank) or change its lender. The number of cliques is variable as a bank can have no customers at a given time (and so not be included in the network) and been chosen by one or more firms in the following period (forming a clique).

3 Solution of the model

This section introduces and applies the analytical techniques for the solution of the model. We have already partitioned the population of firms into two groups: SF and NSF. The evolution of the density for the number of agents belonging to a particular cluster can be modelled by means of the master equation (ME). The ME is a differential equation which describes the variation of the probability of observing a certain number of agents in a given group; it can be solved under asymptotic conditions. In this treatment, the final outcome of the solution is a system of equations which describes the evolution of the network and the degree distribution. To this aim we first need to quantify the number of NSF firms, as they compose the network, and then use this result to study the evolution of the network. We set up two master equations:

- the first one (NSF-ME) to describe the evolution of the number of NSF firms; the solution of this ME will be plugged in
- the second one (K-ME) which models the dynamics of the degree and, through further passages, makes possible the identification of the degree distribution.

One of the novelties is in that the first ME is *nested* into the second to model the dependence between the two.

In the first subsection the NSF-ME is introduced and solved. It is worth noticing that the transition rates for this ME are endogenous and dependent upon the financial conditions of firms. The second subsection presents the K-ME and its solution.

3.1 Stochastic evolution of firms

The evolution of two different groups of firms is modelled along the lines of Di Guilmi et al. (2010), who study an analogous problem using a ME to model the dynamics of the densities of different groups of firms. The ME is a function of the transition probabilities for the agents to move between the two groups. In this model, each firm has a different transition probability, which is dependent on its financial condition (the equity A) and on the price shock u. In order to make the problem analytically tractable we need to quantify an average probability for each of the two transitions. Therefore we identify two *representative* firms, one SF and one NSF by taking the average net worth, indicated respectively by A_0 for SF and A_1 for NSF firms. This reduction in the degrees of freedom of the problem is defined as *mean-field approximation* in statistical mechanics. From these two values we can compute the targeted productions, the costs and the financial needs for the two average firms by using equations (1)-(4).

3.1.1 The transition rates and the NSF-ME

The probability ι for a NSF firm f to become SF depends on the capacity of the firm of having at time $t-1$ a profit large enough to pay the salary bill at time t. This condition can be expressed as

$$A_{f,t-1} + \pi_{f,t-1} \geq W_{f,t},$$

which, using equation (5), becomes

$$A_{f,t-1} + u_{f,t-1}Q_{f,t-1} - W_{f,t-1} - r_{f,t-1}D_{f,t-1} \geq W_{f,t}. \tag{11}$$

The only exogenous variable in (11) is the price u_f and thus it is convenient to specify the probability of switching as a function of it. In particular, since the distribution of u is known, it is possible to quantify the minimum price threshold above which the NSF firm can obtain a profit sufficient to become SF. We denote this threshold with $\bar{u}$. Rearranging equation (11) and using the subscript 1 for the mean-field variables of the representative NSF firm, we can write

$$u_{1,t-1} \geq \frac{W_{1,t} + W_{1,t-1} + r_1 D_{1,t-1} - A_{1,t-1}}{Q_{1,t-1}} = \bar{u}_t. \tag{12}$$

Using the uniform probability function of u, we can write ι as

$$\iota_t = 1 - F(\bar{u}_t) = 1 - \frac{\bar{u}_t - u_{min}}{u_{max} - u_{min}}. \tag{13}$$

In the same fashion, we can specify the condition for the mean-field SF firm to become NSF as

$$A_{0,t-1} + u_{0,t-1} Q_{0,t-1} - W_{0,t-1} < W_{0,t}, \tag{14}$$

which can be written as follows

$$u_{0,t-1} < \frac{W_{0,t} + W_{0,t-1} - A_{0,t-1}}{Q_{0,t-1}} = \underline{u}_t, \tag{15}$$

Where $\underline{u}$ is the upper threshold of the price shock. Denoting with ζ the probability of becoming NSF and making use of the known uniform probability function of u, we can write

$$\zeta_t = F(\underline{u}_t) = \frac{\underline{u}_t - u_{min}}{u_{max} - u_{min}}. \tag{16}$$

In order to obtain the transition rates, the transition probabilities need to be conditioned on the probability of being NSF or SF. In particular, the probability to find a firm belonging to one particular group is higher the larger is the size of that group. This fact can be modelled by means of the two following environmental externality functions

$$\begin{aligned} \psi_{1,t} &= \psi_1 \left(\tfrac{N_{1,t}-\vartheta}{N}\right) = \frac{b_1 + b(N_{1,t} - \vartheta)}{N}, \\ \psi_{0,t} &= \psi_0 \left(\tfrac{N_{1,t}}{N}\right) = \frac{b_0 + b[N - (N_{1,t} + \vartheta)]}{N}, \end{aligned} \tag{17}$$

where $b > 0$, $b_1 > -bN_1$, $b_0 > -b(N - N_1)$, $\vartheta = \{0, 1\}$ is the observed variation in N_1 and ψ_1 and ψ_0 are constants. Accordingly the transition rates β and γ are given by the following homogeneous functions

$$\begin{aligned} N\beta_t \left(\frac{N_{1,t} - \vartheta}{N}\right) &= N\zeta_t \left[\frac{b_1 + b(N_{1,t} - \vartheta)}{N}\right] \left[\frac{N - (N_{1,t} - \vartheta)}{N}\right], \\ N\delta_t \left(\frac{N_{1,t} + \vartheta}{N}\right) &= N\iota_t \left\{\frac{b_0 + b[N - (N_{1,t} + \vartheta)]}{N}\right\} \left(\frac{N_{1,t} + \vartheta}{N}\right). \end{aligned} \tag{18}$$

The NSF-ME is a differential equation which quantifies the variation in the probability to observe a given number of NSF firms $N_{1,t}$. It is given by the probability of observing $N_{1,t}+1$ or $N_{1,t}-1$ and having a transition of one firm, respectively, out from or in the NSF condition, less the probability of already having a number $N_{1,t}$ of NSF firms and observing any transition. Consequently, we have

$$\frac{dP_t(N_{1,t})}{dt} = \underbrace{[\beta_t(N_{1,t}-1)P_t(N_{1,t}-1) + \delta_t(N_{1,t}+1)P_t(N_{1,t}+1)]}_{inflow\ probabilities} + \\ - \underbrace{[(\beta_t(N_{1,t}) + \delta_t(N_{1,t}))P_t(N_{1,t})]}_{outflow\ probabilities}. \quad (19)$$

3.1.2 ME solution and dynamics of the proportions of firms

The solution algorithm involves three main steps:

1. split the state variable N_1 in two components:
   - the *drift* (ϕ), which is the expected value of $n_1 = N_1/N$;
   - the *spread* (ϵ), which quantifies the aggregate fluctuations around the drift.
     Accordingly, the state variable is re-formulated in the following way (Aoki, 1996):
     $$N_{1,t} = N\phi_t + \sqrt{N}\epsilon_t; \quad (20)$$
2. expand in Taylor's series the modified master equation;
3. equate the terms with the same order of power for N.

This process yields an ordinary differential equation, known as macroscopic equation, which describes the dynamics of the trend, and a stochastic partial differential equation, known as Fokker-Planck equation, which describes the dynamics of the density R of the fluctuations (see Aoki, 1996; van Kampen, 1992). Hence, the final solution is given by the mean-field system of coupled equations

$$\begin{cases} \dot{\phi} = \Delta_t(\phi) = \beta_t(\phi) - \delta_t(\phi) = \rho_t\phi - \rho_t\phi^2 \\ \partial_t R_t(\epsilon) = -\partial_\phi \Delta_t(\phi)\partial_\epsilon(\epsilon R_t(\epsilon)) + \frac{1}{2}\Sigma_t(\phi)\partial_\epsilon^2 R_t(\epsilon) \end{cases} \quad (21)$$

where $\rho_t = b(\zeta_t - \iota_t)$ and $\Sigma_t(\phi) = \beta_t(\phi) + \delta_t(\phi)$.

The first of the (21) is a logistic equation with two equilibrium points: $\phi^* = \{0,1\}$. The system approaches the two boundaries without hitting them. Indeed, the quadratic term $-\rho_t\phi^2$ acts as a break with increasing intensity as ϕ_t approaches one of the equilibrium points. Both equations depend on the transition rates and, therefore, on the average financial conditions of the firms.

Following van Kampen (1992), we substitute the formulation of the transition rates (18) into (21) and integrate the resulting expression. The final solution, with the trend dynamics ϕ and the distribution probability for the fluctuation component ϵ, can be written as

$$\begin{cases} \phi(t) = \left[1 + \left(\frac{1}{\phi_0} - 1\right)\exp\left(-\rho_t t\right)\right]^{-1}, & \\ \epsilon(t) \overset{i.i.d}{\to} \mathcal{N}(\mu_{\epsilon,t}, \sigma^2_\epsilon, t) & s.t. \\ & \mu_{\epsilon,t} = \langle\epsilon_0\rangle\, e^{\Delta'_t(\phi)} \\ & \sigma_{\epsilon^2,t} = \langle\epsilon^2_e\rangle\,(1 - e^{2\Delta'_t(\phi)t}), \end{cases} \tag{22}$$

with $\phi_0 \in [0,1)$ and $\langle\epsilon^2_e\rangle = -\Sigma_t(\phi)/(2\Delta'_t(\phi))$.

By using the mean-field values of the average production of the firms in the two groups Q_1 and Q_0, it is possible to obtain the trend and the fluctuations of the aggregate output, as for equation (20). The total output can be expressed as

$$\begin{aligned} Q_t &= N_{1,t}Q_{1,t} + (N - N_{1,t})Q_{0,t} = \\ &= N\{[\phi_t + N^{-1/2}\epsilon_t]Q_{1,t} + [1 - \phi_t - N^{-1/2}\epsilon_t]Q_{0,t}\} = \\ &= N\{Q_{0,t} - [\phi_t + N^{-1/2}\epsilon_t][Q_{0,t} - Q_{1,t}]\}. \end{aligned} \tag{23}$$

It is possible to show that $Q_1 < Q_0$, therefore the dynamics of trend and fluctuations of aggregate production are dependent upon ϕ and ϵ in system (22).

3.2 Stochastic evolution of the network: K-ME

In this section we develop the ME for the network degree. The NSF-ME is nested into the K-ME; to the best of our knowledge, this method has never been developed before. The problem is analysed by studying the evolution of the probability for two firms to be financed by the same bank. This (indirect) connection between two firms defines an (indirect) link in the network. The solution algorithm makes use of the concept of *giant component*, which is the node with the highest number of connection. In this model, it represents the bank which has the largest number of customers[3].

As done for the NSF-ME, in the same fashion we split the state variable in the trend and fluctuations components. The volume of NSF firms N_l that are borrowers from a bank with degree K_l is assumed to be given by

$$N_{l,t} = N_{1,t}[K_l \phi_{l,t} + \sqrt{K_l} \epsilon_{l,t}], \tag{24}$$

where ϕ_l is the expected value and ϵ_l is the fluctuations component. Equation (24) shows a direct correlation between N_1 and the expected number of borrowers for a bank. It can be rearranged and written in intensive form as

$$n_{l,t} = \frac{N_{l,t}}{N_{1,t}} = K_l \phi_{l,t} + \sqrt{K_l} \epsilon_{l,t}. \tag{25}$$

3.2.1 The transition rates and the K-ME

The transition probabilities in this setting concern the creation or destruction of a link between two firms. We introduce the variable ω and set $\omega_{i,j} = 1$, if there is a link between the two firms i and j (they share the same bank), and $\omega_{i,j} = 0$ otherwise. Accordingly, the creation and destruction rates are equal to, respectively,

$$\begin{aligned} \mathbb{P}(\omega_{t+1,i,j} = 1 | \omega_{t,i,j} = 0) = \zeta, \\ \mathbb{P}(\omega_{t+1,i,j} = 0 | \omega_{t,i,j} = 1) = \iota. \end{aligned} \tag{26}$$

Analogously to the NSF-ME case, two externality functions need to be defined in order to quantify the transition rates. These functions are assumed to be dependent on the size of the giant component. In particular, the market share of the largest bank (the giant component) is given by $\gamma_t = S_t/N_{1,t}$, where S_t is the number of its customers. Due to the interest rate formula in equation (7), the size of the giant component impacts on the morphology of the network by exerting a *gravitational* effect. The larger is a bank, the higher is its capacity to attract new borrowers by offering a lower interest rate. Thus, the greater is the giant component, the more it attracts firms and, consequently, the smaller are the chances of an inflow of firms into another component of the network. Accordingly, the externality function $\psi_{1,l,t}$

[3]In the solution algorithm, without loss of generality, we consider it as an exogenous stochastic random variable.

for the inflows into a generic component (bank) is assumed to be an inverse function of γ_t and $k_l = K_l^{-1}$. Symmetrically, the outflow externality function $\psi_{0,l,t}$ is a direct function of the two quantities. The externality functions can be specified as follows

$$\psi_{1,l,t} = \exp(p_1^2(1-\gamma_t)(\phi_l^0/k_l)), \tag{27}$$

$$\psi_{0,l,t} = \exp(p_1^2\gamma_t(\phi_l^0/k_l)), \tag{28}$$

where p_1 is the firm-bank matching probability[4]. Consequently the formulations of the transition rates are the following

$$\begin{aligned} \beta_t(n_{l,t} - \vartheta\rho_{l,t}) &= \lambda_{l,t}[1 - (n_{l,t} - \vartheta\rho_{l,t})] \ , \\ \delta_t(n_{l,t} + \vartheta\rho_{l,t}) &= \mu_{l,t}[n_{l,t} + \vartheta\rho_{l,t}], \end{aligned} \tag{30}$$

where $\lambda_{l,t} = \zeta\psi_{1,l,t}$ and $\mu_{l,t} = \iota\psi_{0,l,t}$. The term $\pm\vartheta\rho_{l,t}$ introduces a correction to take into account that the number of firms in this case is variable, being represented by the NSF firms. In section 3.1, the total population of firm is constant and equal to N; for the network dynamics we need to consider only the firms who enter the credit market, whose number comes from the solution of the NSF-ME. In particular, we indicate with ϑ the observed variation in $n_{l,t}$ and $\rho_{l,t} = \rho(K_{l,t}, N_{1,t})$.

The K-ME describes the evolution of the probability distribution for the degree in each level. It can be expressed as

$$\begin{aligned} \frac{dP_t(n_{l,t})}{dt} = \ & \underbrace{\beta_t(n_{l,t}-1)P_t(n_{l,t}-1) + \delta_t(n_{l,t}+1)P_t(n_{l,t}+1)}_{\text{inflow probability}} - \\ & \underbrace{(\beta_t(n_{l,t}) + \delta_t(n_{l,t}))P_t(n_{l,t})}_{\text{outflow probability}}. \end{aligned} \tag{31}$$

[4]This probability can be quantified as the product of the probability for a bank to be in a firm's pool, m, and the probability of being chosen by the firm. The latter should be different for each bank, as it is dependent on $\bar{\Theta}$ and A_b. Since the probability of matching needs to be defined at the mean-field level, we express it as function only of $\bar{\Theta}$ and write it as: $1 - exp(-c\bar{\Theta})$, with $c > 0$. Accordingly, p_1 is given by

$$p_1(\bar{\Theta}) = m\left[1 - exp(-c\bar{\Theta})\right]. \tag{29}$$

As a consequence, the probability for two firms to be connected to the same bank is p_1^2.

3.2.2 ME solution: dynamics of the degree and degree distribution

In order to solve (31), we adopt the same methodology used for the NSF-ME. Also in this case, the final solution is a system analogous to (21). Splitting the state variable as in (24) and following the steps of the solution algorithm defined in subsection 3.1.2, we obtain the following equation for the trend

$$\phi_{l,t} = (\phi_l^0 - \phi_l^*) \exp\left\{-\rho_{l,t}\left[\lambda_{l,t} + \mu_{l,t}\right] t\right\} + \phi_l^* \tag{32}$$

where

$$\phi_l^* = \left[1 + \mu_{l,t}/\lambda_{l,t}\right]^{-1} \tag{33}$$

is the steady state value of the degree.

Finally, the Fokker-Planck equation provides a Gaussian law for the fluctuations about the expected l-th degree level with mean $\mu_{\epsilon_{l,t}} = \langle \epsilon_l^0 \rangle \exp(-\rho_{l,t}\Delta_t'(\phi_l)t)$ and variance $\sigma^2_{\epsilon,l,t} = \langle \epsilon_l^{*2} \rangle \left[1 - \exp(2\rho_{l,t}\Delta_t'(\phi_l))t\right]$, where ϵ_l^{*2} is the stationary value of fluctuations in the l-th degree level and ϵ_l^0 is the initial condition.

The trend equation and the distribution of fluctuations can be used to compute the degree distribution and its evolution in time. If we define a vector of possible initial starting points ϕ_0 for the average degree, there will be a different trajectory for each starting point according to the dynamics described by equation (32). Consequently, at each point in time, the empirical distribution of the degree is obtained by the different values of ϕ generated by the different trajectories.

4 Results

The systems of equations that compose the solution illustrate the role that the levels of indebtedness and concentration play in shaping the dynamics of the network evolution. The description of the network by means of the ME makes possible an analytical representation of the concept of *too big to fail* (proxied by the giant component) and *too interconnected to fail* (the degree). The model is also able to endogenously generate a dynamics for the proportion of NSF and SF firms, and consequently, of aggregate production. These dynamics affect the size and the evolution of the network.

4.1 Interpretation

The analytical solution of the model describes the effects of the interaction among agents in the system. Indeed, the dynamics of the degree is modelled as dependent on the interaction among firms through the banking system. In particular, the solution shows how the interaction can cause coordination failures; as a consequence, the system oscillates between different steady states. The transition from an equilibrium point to another can be originated by avalanches of bankruptcies of firms and banks when the level of concentration becomes critical.

In order to identify the effects of the level of concentration on the market structure, we need to study the effect of the size of the giant component on the equilibrium solution. We can substitute equations (30) into (32) and take the derivative with respect to γ. It can be demonstrated that this derivative is always positive. As a consequence, both the value of the degree during the adjustment and its steady value positively depend on the size of the giant component. The variance of the fluctuations of the degree is directly proportional to its steady state value and, therefore, to the giant component. Consequently, a market with a relatively high level of concentration (large giant component and large degree) will display a higher volatility, due to the expected larger fluctuations in the average degree. Therefore, the degree distribution will appear as platycurtik. Since in the model the degree distribution is a proxy of the size distribution of banks, this implies larger instability, due to the possibility of rapid and deep modifications in the market structure. This analytical result is confirmed by the numerical analysis of the model[5] as shown by figure 1.

In respect of the aggregate output of the economy, the solution shows that instability in the credit market structure brings about higher volatility in output. An increase in the number of NSF firms has a negative effect on aggregate output, as shown by equation (23), and brings about an increase in the average degree, as for equation (24). Moreover, due to the solution equations, a larger N_1 causes a larger volatility in output due to equation (23); in the same way, the bigger is the average degree, the larger is the variance of its fluctuations. Accordingly, there is a positive correlation between variance

[5]Computer simulations of the model with full degree of heterogeneity for banks and firms have been performed in order to provide some insights on the dynamics generated by the behavioural rules of agents. It is worth stressing that, in our study, numerical simulations are used just as a test of the analytical outcomes.

of the degree and variance of output.

With regard to the dynamics of the network, the increasing concentration follows the growth of the economy: as firms and banks profit, interest rates decrease due to the accumulation of internal finance. Sounder banks are able to attract customers and grow faster. This virtuous cycle can lead to the emergence of a *big bank* which controls the biggest share of the credit market (represented by the giant component). This builds the set-up for the subsequent crisis, according to a pattern analogous to the one envisaged by Minsky (1982): during a boom credit becomes cheaper and firms are led to increase their production, as they accumulate profits. This growth increases the concentration in the credit market and makes firms and banks more vulnerable to negative shocks. Indeed, the presence of a big bank can have a destabilising effect on the system through the variance of the distribution of the fluctuations of the degree. Every bank is potentially subject to large shocks, which consequently impact on the conditions of credit for firms through equation (7); as a consequence, borrowers can experience substantial variation in the cost of debt and, in the case of a significant negative shock, become insolvent, worsening the conditions of other banks and eventually spiralling down the whole system. The network economy displays an endogenous cyclical behaviour, in which the tendency to concentration heightens the probability for the system to be hit by systemic financial distress (figures 2 and 3).

4.2 Policy implications

Despite the fact that the representation of the credit market is simplified, the model captures the basic features of a credit network economy, such as the emergence of nodes with systemic relevance and the possibility of crises through propagation effects, when these nodes are in financial distress (De Masi et al., 2010).

The model provides all the measures of systemic risk measures identified by the European Central Bank: (i) the degree of connectivity; (ii) the degree of concentration; (iii) the size of exposures. Furthermore it analytically describes their dynamics and highlights the causal relationship among them. For these reasons, the framework can be helpful for a preliminary assessment of a stabilisation policy.

The main policy tool embedded in the model is the adequacy ratio requirement $\bar{\Theta}$. By handling this parameter, the policy maker can influence the dynamics of the model and reduce the probability of a systemic collapse through different channels. The capital requirement directly influences the structure of the market in two ways. On the one hand, a reduction in $\bar{\Theta}$ defines a ceiling for the size of the giant component, diminishing the chances of the emergence of a big bank. As detailed in subsection 4.1, a smaller giant component reduces the variance of the distribution of fluctuations of the degree. The final effect is a smaller chance of large and sudden modifications in the market structure and a higher stability. On the other hand, the average degree (and thus the average size of banks) is directly proportional[6] to Θ. This effect is amplified by the fact that a smaller $\bar{\Theta}$ is also likely to increase the competition in the credit market, lowering the interest rate spread among banks. The ultimate outcome is therefore a higher dispersion in the market and no big banks. Hence, the model illustrates that the size limit is also a limit on the interconnectivity, linking the concepts of *too big to fail* and *too interconnected to fail*. The introduction of a capital requirement allows the policy maker to shape the network topology and the market structure.

The parameter $\bar{\Theta}$ indirectly affects the probability of bankruptcy for firms. In fact, with stricter lending limits for banks, a firm with high leverage ratio is likely to be credit rationed, due to the banks' selection process, and thus to reduce its production and financial needs, consequently lowering its chances of bankruptcy.

At the micro level, $\bar{\Theta}$ indirectly influences the transition rates. A lower lending limit will bring about, on average, smaller interest rates, through equation (7), as only the NSF firms with the lowest debt ratio will be financed. The threshold $\bar{u}$ in (12) will be lower, increasing the probability for a NSF firms to become SF. This chain of effects is particularly relevant as it allows the policy maker to influence the path of evolution of the system between different equilibriums. Indeed the economic policy, by influencing the transition rates, can impact on the dynamics described by the first of the (22). In this way, it can drive the proportion of NSF between the two limits 0 and 1 in order to set the economy on the preferred equilibrium path. The impact of the variations in $\bar{\Theta}$ in the numerical simulations are illustrated by figure 4.

Summarising, with reference to the three measures of systemic risk, the model shows that the capital requirement threshold influences: (i) the degree of connectivity, as it impacts on the average degree of network; (ii) the degree of concentration, as it directly determines the maximum size of banks and,

[6]See equation (29).

indirectly, their average size; (iii) the level of exposures, by imposing a limit to lending. The solution of the model makes possible a quantitative and qualitative analysis of the impact of such a policy on the average degree, on the shape of the degree distribution and on the resilience of the credit market structure.

Through the stabilization of the credit market, policy makers can also influence the dynamics of production. An increase in the probability for NSF firms to become SF brings about a lower number of NSF firms in steady state and, consequently, a lower variance of the fluctuation component ϵ in equation (22). Equation (23) demonstrates that it causes a higher level and a smoother dynamics for aggregate output. Indeed, as the dynamics of output is dependent on the density of NSF firms, a lower variance of this density is accompanied by a smaller variance of output.

5 Concluding remarks

In this paper we propose a technique for the analytical solution for models with heterogeneous and interacting agents and apply it to a credit network model. In particular, we describe the dynamics of the behaviour of the agents by means of two MEs, one nested into the other. Their asymptotic solutions yield the trend and fluctuations of the two state variables: the proportion of NSF firms and the network degree.

The solution identifies some emerging properties of a credit network. We find that rising economic output, and the consequent increase in the overall wealth of firms, turns out to be proportional to how much the loans in the system come to be concentrated among a few banks. In network terms, this concentration can be measured by the average degree for banks. There is a natural tendency for this quantity to rise as the economy expands and banks and firms profit. This rise in concentration is potentially destabilising for the system: the failure of a single bank can bring trouble to a large number of firms, which pass it on to other banks, leading to further failures, in a downward spiral. Cascades of failures put financial pressure on all firms, raising the costs of borrowing and slowing down the economy.

The model is able to endogenously generate feedback between economic growth and rising interconnectedness which leads to cycles of booms and busts. The solution of the model highlights the causal links among micro, meso and macro-variables. In this perspective, the present work provides a starting point for the development of more refined models of the credit network in order to test possible stabilisation policies.

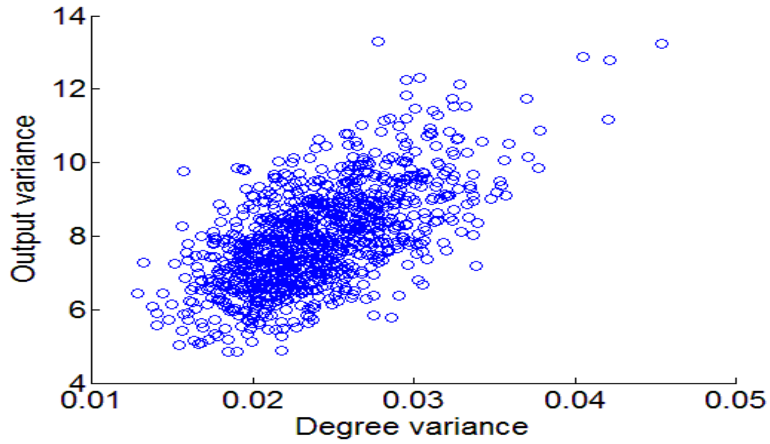

Figure 1: Variance of the degree and of the output. Monte Carlo simulation (correlation: 0.62).

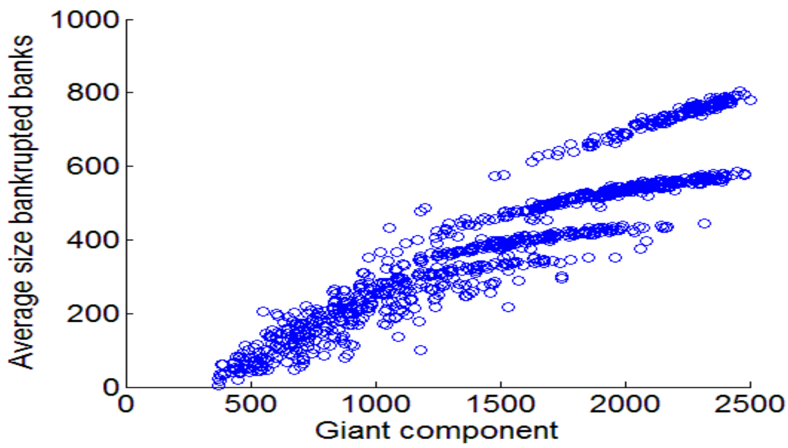

Figure 2: Giant component and average lending of bankrupted banks. Single simulation (correlation in Monte Carlo simulations: 0.71).

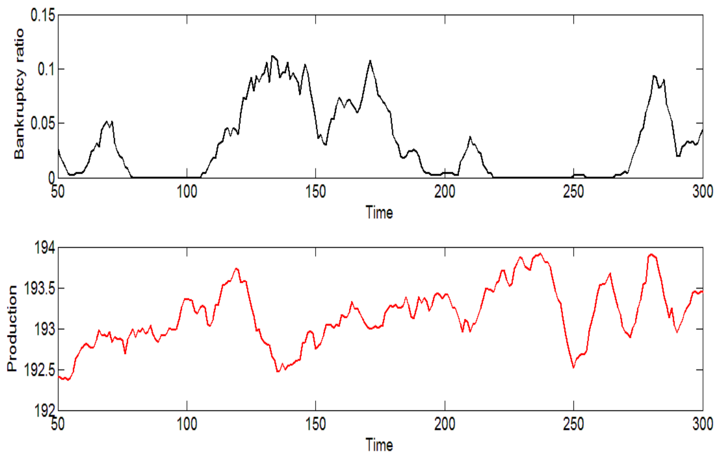

Figure 3: Banks bankruptcy ratio and aggregate output.

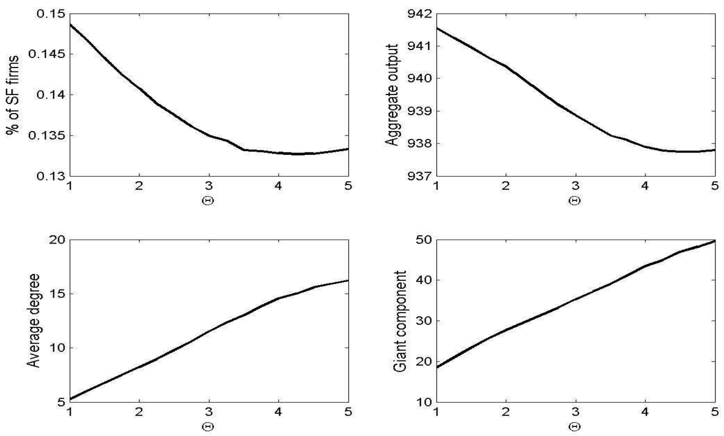

Figure 4: Average proportion of SF firms, output, degree and giant component for different values of θ. Monte Carlo simulation.

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Analytical proofs for the paper *Towards an Analytical Solution for Agent Based Models: an Application to a Credit Network Economy*

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1 Master equation modelling: general set-up and solution

This section introduces and develops the general solution for the master equation. The method will be then applied to study the dynamics of the number of NSF firms (section 2) and the network structure (section 3). We refer to the economic model with complete heterogeneity as the agent based model (ABM). The approach presented here can be considered as an analytical stochastic approximation to the numerical results of the ABM, obtained by means of computer simulation. The ABM is set up with its own microscopic behavioural rules, and it is used here as the scientist uses the laboratory. That is, from the one hand, the simulations are performed to visualise the macroscopic results that will be investigated in a phenomenological way by means of statistical mechanics tools. On the other hand, the outcomes of the analytical solution can be compared with the simulation results to test its accuracy.

1.1 Elementary notions

Let us consider a system $\mathcal{S}_N$ made of $N \gg 1$ heterogeneous interacting agents (HIAs) and set X to be a macroscopic random quantity changing its values through time $t \in \mathbb{T}$ over a state space $\Lambda \subset \mathbb{Q}$. In other words we define the following stochastic process

$$\chi = \{X(t) : \Omega \to \Lambda, t \in \mathbb{T}\} \tag{1}$$

Now consider that $\Omega = \{\omega_j \ : \ j \leq M\}$. It follows that $X(\omega, t) \in \Lambda \ \forall \omega \in \Omega$. Without loss of generality we can assume that $\Omega = \{\omega_0, \omega_1\}$ where ω_1 is our state of interest and ω_0 can represent every other state the HIAs can belong to.
According to this representation $\{X_1(t) = X(\omega_1, t)\}_{t \in \mathbb{T}}$ is a random trajectory on Λ: as such it can yield different values (i.e. realizations). Let us assume that

$$\underline{X}_1 = \min_{t \in \mathbb{T}} \{X(\omega_1, t)\} \in \Lambda \quad : \quad \underline{X}_1 = \min \Lambda \tag{2}$$

and suppose that $X_1(t)$ moves from one level to another with jumps of constant size which can be evaluated by the following variation

$$|X_1(t + \Delta t) - X_1(t)| = r(\Delta t) = r \quad : \quad \Delta t = \text{fixed} \tag{3}$$

This allows us to provide a representation for Λ

$$\Lambda = \{\underline{X}_1 + hr \quad : h = 0, 1, \ldots, H\} \tag{4}$$

so the boundaries of Λ are

$$\begin{cases} (\underline{X}_1 = \min_t \Lambda) \leq X_1(t) \leq (\overline{X}_1 = \max_t \Lambda) \\ \underline{X}_1 \leq X_1(t) \leq \overline{X}_1 = Hr + \underline{X}_1 \end{cases} \tag{5}$$

By assumption we consider Λ to be mechanically invariant, i.e. constant through time, therefore $\underline{X}_1$ and $\overline{X}_1$ exist and are constant as well.
Let us now define a counting measure $v : \Omega \to \{N_j : j \leq N\}$ such that

$$N_j = v(\omega_j) \in [0, N] \quad : \quad \sum_j^N = N \tag{6}$$

therefore (Ω, v) is termed countable space. It is therefore easy to see that

$$n_j = N_j/N \in [0, 1] \tag{7}$$

defines an empirical probability measure so let us endow Ω with a probability measure $\mathbb{P}$. On the other hand observe that $X_j(t)$ and $N_j(t)$ are macroscopic extensive observables for the system $\mathcal{S}_N$ so (7) and

$$\begin{cases} x_1 = X_1(t)/N \in [\underline{x}_1, \overline{x}_1] \;\; s.t. \\ \underline{x}_1 = X_1(t)/N \;\;,\;\; \overline{x}_1 = \overline{X}_1/N \;\;:\;\; \overline{X}_1 = Hr + \underline{X}_1 \end{cases} \tag{8}$$

are both intensive representations of $N_j(t)$ and $X_j(t)$ respectively. Micro interactions are not observable but it is possible to write that, for each agent i of the system at every time t,

$$(i \in \omega_0 | t \to i \in \omega_1 | t + \Delta t) \;\;;\;\; (i \in \omega_0 | t + \Delta t \leftarrow i \in \omega_1 | t) \tag{9}$$

Since we now concentrate our attention on $\omega_1 \in \Omega$ we can say that

$$\begin{cases} \omega_0 \to \omega_1 \;\; \text{is an inflow w.r.t. } \omega_1 \\ \omega_0 \leftarrow \omega_1 \;\; \text{is an outflow w.r.t. } \omega_1 \end{cases} \tag{10}$$

Since Ω is endowed with a probability measure let us define[1]

$$\psi_{1,t} = \mathbb{P}_t(i \in \omega_1) \;\;\wedge\;\; \psi_{0,t} = \mathbb{P}_t(i \in \omega_0) \tag{11}$$

as the state external influence probabilities. They can depend on time or being assumed constant: they can be stated *a priori* or they can depend on an external model as well as on the external environment. This is why they are termed externality functions by Aoki. If we then consider

$$\mathbb{P}_t(i \in \omega_0 \to i \in \omega_0) = \zeta_t \;\;\wedge\;\; \mathbb{P}_t(i \in \omega_0 \leftarrow i \in \omega_0) = \iota_t \tag{12}$$

to be, respectively, the probabilities for an inflow and an outflow w.r.t. ω_1, then we can define the following

$$\begin{cases} \lambda_t = \psi_{1,t}\zeta_t \;\; \text{birth rate} \\ \mu_t = \psi_{0,t}\iota_t \;\; \text{death rate} \end{cases} \tag{13}$$

These results are valid for every macroscopic quantity X we can measure on the system $\mathcal{S}_N$, therefore it applies to $N_j(t)$ too. The stochastic process χ defined in (1) is a jump Markov process with jumps' size defined by (3).

[1]Except for β_t, δ_t, Δ_t, Σ_t, P_t and Q_t, which are endogenously specified functions of the drifting trajectory m and the fluctuations about it s, as concerning a given quantity x we write $x(t)$ when it autonomously depends on time while x_t means that time is an indexing parameter determined by an exogenous event.

1.2 Transitory mechanics: state space, transition rates and ME

Let us now introduce two instrumental constants θ and ϑ, defined as follows

$$\vartheta = |\theta| \quad : \quad \theta = -1, 0, +1 \tag{14}$$

Now consider that

$$X_1(t) = X_h \quad : \quad X_h = \underline{X}_1 + hr \in \Lambda \tag{15}$$

is a generic realization of $X_1(t) = X(\omega_1, t)$. Therefore, if we fix a value X_h we can observe the following events

$$X(\omega_1, t \pm \Delta t) = X_h \pm \vartheta r = X_h + \theta r \in \Lambda \tag{16}$$

W.r.t. the state $X_h \in \Lambda$, we can observe gains (inflows) and losses (outflows) of magnitude r, which is precisely the allowed size of the jumps for the stochastic process. Figure 1 provides a graphical representation of the model. The quantity X_h is the realization value of the stochastic process $X(\omega_1, t)$ on

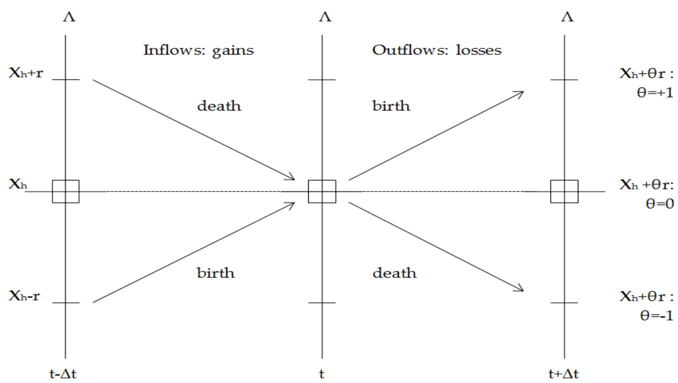

Figure 1: Representation of the stochastic transitory mechanics.

the state space Λ, that is: under the same environmental and internal conditions the process will provide $X(\omega_1, t) = X_h$. The quantity X_h is considered as a target state but we also consider what happens in its neighbourhood: this is the meaning of $X_h \pm \vartheta r \in \Lambda$. W.r.t. X_h we can have inflows (gains) and outflows (losses) as if we were gaining or loosing a fraction (or portion of volume) of the involved macroscopic extensive but transferable quantity

X. But an inflow or a gain can happen by means of a birth (death) from a lower (upper) level; similarly an outflow or a loss can happen with a death (birth) to a lower (upper) level. If the single agent i moves from ω_0 to ω_1 or *vice-versa*, brings its own portion of the total quantity X: to allow this we have to consider transferable quantities.
All these variations are due to unobserved microscopic interactions. We model this multiplicity of latent interactions of agents living on Ω in terms of transition rates on Λ. This representation is defined as mean-field interaction, a kind of indirect interaction at a meso-level, which is defined as an intermediate level of aggregation between the micro-level of ABM and the macro-level. Di Guilmi et al. (2012) explain this aspect in terms of $N-\ell$ complexity for ABM, $M-\ell$ complexity for mesoscopic modelling and $1-\ell$ for RA modelling.
The transition rates are defined as transition probabilities per (vanishing) reference unit of time. By using (14) we represent them as follows

$$w_t(X_h \pm (1-\vartheta) r | X_h \pm \vartheta r) = \begin{cases} w_t(X_h \pm r | X_h) & : \ \vartheta = 0 \ \Rightarrow \text{outflows} \\ w_t(X_h | X_h \pm r) & : \ \vartheta = 1 \ \Rightarrow \text{inflows} \end{cases} \tag{17}$$

Therefore, ϑ identifies outflows or losses and inflows or gains. Both in-outflows can happen by means of births and deaths. The *birth transition rates* can be expressed as

$$\beta_t(X_h - \vartheta r) = \begin{cases} \beta_t(X_h) = w_t(X_h + r | X_h) \ : \ \vartheta = 0 \ \Rightarrow \text{out-birth} \\ \beta_t(X_h - r) = w_t(X_h | X_h - r) \ : \ \vartheta = 1 \ \Rightarrow \text{in-birth} \end{cases} \tag{18}$$

In the same fashion, the *death transition rates* can be written as

$$\delta_t(X_h + \vartheta r) = \begin{cases} \delta_t(X_h) = w_t(X_h - r | X_h) \ : \ \vartheta = 0 \ \Rightarrow \text{out-death} \\ \delta_t(X_h + r) = w_t(X_h | X_h + r) \ : \ \vartheta = 1 \ \Rightarrow \text{in-death} \end{cases} \tag{19}$$

Then we can represent state probabilities as follows

$$P_t(X_h \pm \vartheta r) = P_t\left(X(\omega_1, t) = X_h \pm \vartheta r \in \Lambda\right) \tag{20}$$

where the state space Λ has been defined in (4) and $X(\omega_1, t) = X_1(t)$ for $\omega_1 \in \Omega$. By observing that

$$X_h = \underline{X}_1, \underline{X}_1 + r, \ldots, \underline{X}_1 + hr, \ldots (\underline{X}_1 + Hr = \overline{X}_1) \in \Lambda \tag{21}$$

we can set up the following master equation (ME from here on) w.r.t. our state of interest X_h

$$\begin{aligned} \frac{dP_t(X_h)}{dt} = & \underbrace{[\beta_t(X_h - r)P_t(X_h - r) + \delta_t(X_h + r)P_t(X_h + r)]}_{inflow\ probabilities} + \\ & - \underbrace{[(\beta_t(X_h) + \delta_t(X_h))P_t(X_h)]}_{outflow\ probabilities} \end{aligned} \tag{22}$$

with the following boundary conditions

$$\begin{cases} \frac{dP_t(\underline{X}_1)}{dt} = \delta_t(\underline{X}_1 + r)P_t(\underline{X}_1 + r) - \beta_t(\underline{X}_1)P_t(\underline{X}_1) \\ \frac{dP_t(\overline{X}_1)}{dt} = \beta_t(\overline{X}_1 - r)P_t(\overline{X}_1 - r) - \delta_t(\overline{X}_1)P_t(\overline{X}_1) \end{cases} \tag{23}$$

Finally we observe that in the transitory mechanics, the stayers, i.e. those which do not change state from X_h, do not play any role. The master equation (22) is a balance equation between in-out probability flows and it dynamically describes the evolution of the probability density function for the random quantity X over the state space. Indeed, (22) is a differential equation driving the dynamics of $P_t([X(\omega_1, t) = X_1(t)] = X_h)$ on Λ. Once it has been solved, this equation evaluates which is the probability of finding a portion X_h of the quantity X due to agents in state ω_1: in this respect the ME is a way to operate stochastic aggregation with HIAs.

1.3 Phenomenological model and the solvable ME

According to Aoki (1996, 2002); Aoki and Yoshikawa (2006) and van Kampen (1992), if we are confident that the quantity X has a unimodal distribution peaked about its expected value, a common and very suitable phenomenological model reads as

$$X(\omega_1, t) + \theta r = N \langle X(\omega_1, t)/N \rangle + \sqrt{N} s(t) + \theta r \tag{24}$$

We consider $X(\omega_1, t) = X_h$ to be fixed. Therefore, we specify the phenomenological model as follows[2]

$$\begin{cases} X_h + \theta r = Nm + \sqrt{N}s + \theta r \;\; s.t. \\ (i)\; N = |\mathcal{S}_N| \;\; (ii)\; m = \langle X_h/N \rangle \\ (iii)\; s = (X_h - Nm)N^{-1/2} \overset{i.i.d.}{\to} F_s(\mu_s, \sigma_s^2) \end{cases} \tag{25}$$

From (25) it follows that

$$X_h + \theta r = Nm + \sqrt{N}s + \theta r \Rightarrow \frac{X_h + \theta r}{N} = m + \frac{1}{\sqrt{N}}\left(s + \frac{\theta r}{\sqrt{N}}\right) \tag{26}$$

so we can define

$$m_{(\theta)} = m + \frac{1}{\sqrt{N}} s_{(\theta)} \;\; : \;\; s_{(\theta)} = s + \frac{\theta r}{\sqrt{N}} \tag{27}$$

[2]The constant θ has been defined in eq. (14).

According to (20) and (24) we can also observe that

$$P_t(X(\omega_1, t)) = Q_t(s(t)) \tag{28}$$

is always possible to be set since, according to (25-iii), we can write

$$s(t) = (X(\omega_1, t) - Nm(t))N^{-1/2} \tag{29}$$

Therefore,

$$\frac{dP_t(X(\omega_1, t))}{dt} = \frac{\partial Q_t(s(t))}{\partial t} + \frac{ds(t)}{dt}\frac{\partial Q_t(s(t))}{\partial s} \tag{30}$$

and, for the fixed state realization $X(\omega_1, t) = X_h$, it follows that[3]

$$\frac{ds}{dt} = -\sqrt{N}\frac{dm}{dt} \tag{31}$$

As a conseqence, the l.h.s. of the ME (22) reads as

$$\frac{dP_t(X_h)}{dt} = \frac{\partial Q_t(s)}{\partial t} - \sqrt{N}\frac{dm}{dt}\frac{\partial Q_t(s)}{\partial s} \tag{32}$$

As usual in this stream of modelling, time is rescaled as follows

$$t = f(N)\tau \quad : \quad f(N) = N \Rightarrow [t] \neq [\tau] \tag{33}$$

for an arbitrary function $f()$ such that

$$dt = f(N)d\tau \Rightarrow dt = Nd\tau \;\text{ iff }\; f(N) = N \tag{34}$$

Since $f(N) = N$, it must be that $[t] \neq [\tau]$. By using (33) and (34), the (32) reads as

$$\frac{1}{N}\frac{dP_t(X_h)}{dt} = \frac{1}{N}\frac{\partial Q_t(s)}{\partial t} - \frac{1}{\sqrt{N}}\frac{dm}{dt}\frac{\partial Q_t(s)}{\partial s} \tag{35}$$

Following Aoki, the transition rates in (18) and (19) need to be specified as homogeneous functions w.r.t. a system size parameter, which in our case is $N = |\mathcal{S}_N| \gg 1$. Thus, by using (27) and (14) we have

$$\beta_t(X_h - \vartheta r) = N\beta_t\left(\frac{X_h - \vartheta r}{N}\right) = N\beta_t\left(m + \frac{1}{\sqrt{N}}\left(s - \frac{\vartheta r}{\sqrt{N}}\right)\right) = N\beta_t(m_{(-\vartheta)}) \tag{36}$$

[3]Even though X_h is fixed, fluctuations (i.e. volatility) about the value of this state are not fixed, rather they vary through time according to the probability distribution $Q_t(s)$. These fluctuations are due to the unobservable multiplicity of interactions of HIAs in the state ω_1 and their transitions on Ω. Fluctuations are the macroscopic effect of microscopic interactions. Therefore, they are endogenous to the macroscopic model. Moreover, they are an emergent property of the system whose probability distribution will be estimated.

$$\delta_t(X_h+\vartheta r) = N\delta_t\left(\frac{X_h+\vartheta r}{N}\right) = N\delta_t\left(m+\frac{1}{\sqrt{N}}\left(s+\frac{\vartheta r}{\sqrt{N}}\right)\right) = N\delta_t(m_{(+\vartheta)}) \tag{37}$$

In the same way we rewrite the state probabilities (20)

$$P_t(X_h\pm\vartheta r) = P_t(Nm+\sqrt{N}s\pm\vartheta r) = P_t\left(Nm+\sqrt{N}\left(s\pm\frac{\vartheta r}{\sqrt{N}}\right)\right) = Q_t(s_{(\pm\vartheta)}) \tag{38}$$

which is in accordance with (28). By using (36) and (37) we can now rewrite the ME in (22) where the l.h.s is now coherent with (35), that is

$$\begin{aligned}\frac{1}{N}\frac{dP_t(X_h)}{dt} = &\left[\beta_t(m_{(-)})Q_t(s_{(-)}) + \delta_t(m_{(+)})Q_t(s_{(+)})\right] + \\ &- \left[(\beta_t(m_{(0)}) + \delta_t(m_{(0)}))Q_t(s_{(0)})\right]\end{aligned} \tag{39}$$

where $m_{(\pm\vartheta)}$ is given by (27) according to (14). We set

$$\Sigma_t(m_{(0)}) = \beta_t(m_{(0)}) + \delta_t(m_{(0)}) \tag{40}$$

and

$$\Delta_t(m_{(0)}) = \beta_t(m_{(0)}) - \delta_t(m_{(0)}) \tag{41}$$

Equation (39) can be reformulated in the following way.

$$\left\{\begin{array}{c}\frac{1}{N}\frac{\partial Q_t(s)}{\partial t} - \frac{1}{\sqrt{N}}\frac{dm}{dt}\frac{\partial Q_t(s)}{\partial s} = \\ \left[\beta_t(m_{(-)})Q_t(s_{(-)}) + \delta_t(m_{(+)})Q_t(s_{(+)})\right] - \left[\Sigma_t(m_{(0)})Q_t(s_{(0)})\right]\end{array}\right. \tag{42}$$

which is the solvable formulation of the ME.

1.4 Asymptotic expansion solution method

This section presents Aoki's method for the solution of ME. Its principle is rather simple even if algebraically heavy. It consists in expanding with Taylor polynomials both the transition rates about m and the density about s in the ME (42). To let the reader comprehend the technique we here develop all the needed calculations by introducing a suitable formalism which reduces the length of the expression. First of all, let us define a new instrumental constant to represent the system size parameter

$$\eta^{p/2} = N^{-p/2} \to 0^+ \;\; \forall p \geq 1 \;\text{ iff }\; N := |\mathcal{S}_N| \to +\infty \tag{43}$$

Taylor expansion terms for a generic function $(x) \mapsto f(x) \in \mathcal{C}^\infty$ are given by powers of the displacement of the state variable x about the target value x_0, that is $(x-x_0)^p/p!$ for $p \geq 0$. Therefore, as regarding transition rates in (42) we can summarize these polynomial's terms as follows

$$\left\{ \begin{array}{c} \frac{(m_{(\pm\vartheta)}-m)^p}{p!} = \left[\left(m + \frac{1}{\sqrt{N}} s_{(\pm\vartheta)} \right) - m \right]^p \frac{1}{p!} = \frac{(s_{(\pm\vartheta)}/N^{1/2})^p}{p!} = \\ \frac{N^{-p/2}}{p!} \left(s \pm \frac{\vartheta r}{\sqrt{N}} \right)^p = \frac{\eta^p/2}{p!} \left(s \pm \eta^{1/2} \vartheta r \right)^p \end{array} \right. \tag{44}$$

Then, indicating with $f(x)^{(p)}$ the p-th order derivative of f, we have the following Taylor's polynomials for transition rates about m

$$\beta_t(m_{(-)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} \left(s \boxed{-} \eta^{1/2} r \right)^p \beta_t^{(p)}(m) \ \ \vartheta = 1 \ \ \theta = -1 \tag{45}$$

$$\delta_t(m_{(+)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} \left(s \boxed{+} \eta^{1/2} r \right)^p \delta_t^{(p)}(m) \ \ \vartheta = 1 \ \ \theta = +1 \tag{46}$$

$$\Sigma_t(m_{(0)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} (s)^p \, \Sigma_t^{(p)}(m) \ \ \vartheta = 0 \ \ \theta = 0 \tag{47}$$

where signs are highlighted in boxes. In the same way, the density of fluctuations about s are expanded as follows

$$\frac{(s_{(\pm\vartheta)} - s)^p}{p!} = \left[\left(s \pm \frac{\vartheta r}{\sqrt{N}} \right) - s \right]^p \frac{1}{p!} = \frac{(\pm\vartheta r/N^{1/2})^p}{p!} = \frac{(\pm\eta^{1/2}\vartheta r)^p}{p!} \tag{48}$$

and the density is

$$Q_t(s_{(\pm)}) = \sum_{p \geq 0} \frac{(\pm\eta^{1/2} r)^p}{p!} Q_t^{(p)}(s) \ \text{ if } \ \vartheta = 1 \ \text{ or } \ Q_t(s) \ \text{ if } \ \vartheta = 0 \tag{49}$$

By plugging eq.s (43)-(49) into (42) with $p \leq 2$ and setting $\partial_x^{(p)} = (\partial/\partial x)^p$ and $\dot{x} = dx/dt$ we have[4]

$$\underbrace{\eta \partial_t Q_t - \sqrt{\eta}\dot{m}\partial_s Q_t}_{MElhs} =$$

$$\underbrace{\left\{ \begin{array}{c} \left[\beta_t + \sqrt{\eta}(s\boxed{-}\sqrt{\eta}r)\beta'_t + \frac{\eta}{2}(s\boxed{-}\sqrt{\eta}r)^2\beta''_t\right] \times \\ \left[Q_t\boxed{-}\sqrt{\eta}rQ'_t\boxed{+}\frac{\eta r^2}{2}Q''_t\right] \end{array} \right\}}_{CB1:-\vartheta=-1\Leftrightarrow\theta=-1} +$$

$$\underbrace{\left\{ \begin{array}{c} \left[\delta_t + \sqrt{\eta}(s\boxed{+}\sqrt{\eta}r)\delta'_t + \frac{\eta}{2}(s\boxed{+}\sqrt{\eta}r)^2\delta''_t\right] \times \\ \left[Q_t\boxed{+}\sqrt{\eta}rQ'_t\boxed{+}\frac{\eta r^2}{2}Q''_t\right] \end{array} \right\}}_{CB2:+\vartheta=1\Leftrightarrow\theta=1} +$$

$$-\underbrace{\left\{ \left[\Sigma_t + \sqrt{\eta}s\Sigma'_t + \frac{\eta}{2}s^2\Sigma''_t\right] \times Q_t \right\}}_{CB3:\vartheta=0\Leftrightarrow\theta=0} \tag{50}$$

If we now develop the products in curly brackets, we get the following 21 terms

$$= \left\{ \begin{array}{c} \beta_t Q_t - \sqrt{\eta}r\beta_t Q'_t + \frac{\eta r^2}{2}\beta_t Q''_t + \\ \sqrt{\eta}(s-\sqrt{\eta}r)\beta'_t Q_t - \eta r(s-\sqrt{\eta}r)\beta'_t Q'_t + \frac{\eta^{3/2}r^2}{2}(s-\sqrt{\eta}r)\beta'_t Q''_t \\ \frac{\eta}{2}(s-\sqrt{\eta}r)^2\beta''_t Q_t - \frac{\eta^{3/2}}{2}r(s-\sqrt{\eta}r)^2\beta''_t Q'_t + \frac{\eta^2 r^2}{4}(s-\sqrt{\eta}r)^2\beta''_t Q''_t \end{array} \right\}_{CB1} +$$

$$\left\{ \begin{array}{c} \delta_t Q_t + \sqrt{\eta}r\delta_t Q'_t + \frac{\eta r^2}{2}\delta_t Q''_t + \\ \sqrt{\eta}(s+\sqrt{\eta}r)\delta'_t Q_t + \eta r(s+\sqrt{\eta}r)\delta'_t Q'_t + \frac{\eta^{3/2}r^2}{2}(s+\sqrt{\eta}r)\delta'_t Q''_t \\ \frac{\eta}{2}(s+\sqrt{\eta}r)^2\delta''_t Q_t + \frac{\eta^{3/2}}{2}r(s+\sqrt{\eta}r)^2\delta''_t Q'_t + \frac{\eta^2 r^2}{4}(s+\sqrt{\eta}r)^2\delta''_t Q''_t \end{array} \right\}_{CB2} +$$

$$-\left\{ \Sigma_t Q_t + \sqrt{\eta}s\Sigma'_t Q_t + \frac{\eta}{2}s^2\Sigma''_t Q_t \right\}_{CB3}$$

[4]In eq. (50) the $CB\#$s indicate *curly brackets* and specifications about the values of the instrumental constants defined in eq. (14) are provided for clarity of exposition. Moreover, in boxes we highlighted changes in signs and suppress variables in transition rates, evaluated at m, and the density, evaluated at s. The reader not interested in the following algebraic calculations can skip them and go to eq. (51).

Computing all the involved products and then collecting terms according with the orders of power of the term ηr, we obtain

$$= \left\{ \ (\beta_t + \delta_t - \Sigma_t) Q_t \ \right\}_{CB(i)} +$$

$$\sqrt{\eta} \left\{ \ r(\delta_t - \beta_t) Q'_t + \left[(s - \sqrt{\eta} r)\beta'_t + (s + \sqrt{\eta} r)\delta'_t \right] Q_t - s\Sigma'_t Q_t \ \right\}_{CB(ii)} +$$

$$\eta \left\{ \begin{array}{c} \frac{r^2}{2}(\beta_t + \delta_t) Q''_t - r\left[(s - \sqrt{\eta} r)\beta'_t - (s + \sqrt{\eta} r)\delta'_t \right] Q'_t + \\ \frac{1}{2}\left[(s - \sqrt{\eta} r)^2 \beta''_t + (s + \sqrt{\eta} r)^2 \delta''_t - s^2 \Sigma''_t \right] Q_t \end{array} \right\}_{CB(iii)} +$$

$$\frac{\eta^{3/2}}{2} \left\{ \begin{array}{c} r^2 \left[(s - \sqrt{\eta} r)^2 \beta'_t + (s + \sqrt{\eta} r)^2 \delta'_t \right] Q''_t + \\ r\left[(s + \sqrt{\eta} r)^2 \delta''_t - (s - \sqrt{\eta} r)^2 \beta''_t \right] Q'_t \end{array} \right\}_{CB(iv)} +$$

$$\frac{\eta^2}{4} \left\{ \ r^2 \left[(s - \sqrt{\eta} r)^2 \beta''_t + (s + \sqrt{\eta} r)^2 \delta''_t \right] Q''_t \ \right\}_{CB(v)}$$

Observing that

1. $\Sigma_t^{(p)} = \beta_t^{(p)} + \delta_t^{(p)}$ $\forall p \geq 0$ due to eq. (40),
2. $\Delta_t^{(p)} = \beta_t^{(p)} - \delta_t^{(p)}$ $\forall p \geq 0$ due to eq. (41) and
3. $\frac{\eta^{3/2}}{2} = \frac{1}{2\sqrt{N^3}}$, $\frac{\eta^2}{4} = \frac{1}{4N^2} \to 0^+$ faster than $N \to +\infty$

we can see that $CB(i)$, $CB(iv)$ and $CB(v)$ can be set equal to zero. Accordingly, only $CB(ii)$ and $CB(iii)$ are considered. The development of the calculation yields

$$= \sqrt{\eta} \left\{ \ -r\Delta_t Q'_t + s(\beta'_t + \delta'_t - \Sigma'_t) Q_t + \sqrt{\eta} r(\delta'_t - \beta'_t) Q_t \ \right\} +$$

$$\eta \left\{ \begin{array}{c} \frac{r^2}{2}\Sigma_t Q''_t + r\left[s(\delta'_t - \beta'_t) \right] Q_t + \sqrt{\eta} r^2 (\delta'_t + \beta'_t) Q'_t + \\ \frac{s^2}{2}(\beta''_t + \delta''_t - \Sigma''_t) Q_t + \sqrt{\eta} rs(\delta''_t - \beta''_t) Q_t + \frac{\eta r^2}{2}(\beta''_t + \delta''_t) Q_t \end{array} \right\} =$$

$$\sqrt{\eta} \left\{ \ -r\Delta_t Q'_t - \sqrt{\eta} r \Delta'_t Q_t \ \right\} +$$

$$\eta \left\{ \ \frac{r^2}{2}\Sigma_t Q''_t - rs\Delta'_t Q'_t + \sqrt{\eta} r^2 \Sigma'_t Q'_t - \sqrt{\eta} rs \Delta''_t Q_t + \frac{\eta r^2}{2}\Sigma''_t Q_t \ \right\} =$$

$$\left\{ \begin{array}{c} -\sqrt{\eta} r \Delta_t Q'_t - \eta r \Delta'_t Q_t + \eta \frac{r^2}{2}\Sigma_t Q''_t - \eta rs \Delta'_t Q'_t + \\ + \left[\eta^{3/2} r^2 \Sigma'_t Q'_t - \eta^{3/2} rs \Delta''_t Q_t + \frac{\eta^2 r^2}{2}\Sigma''_t Q_t \right] \end{array} \right\}$$

The term in square brackets can be ignored since $\eta^{3/2} \to 0^+$ very fast as the system size N grows. As a consequence, we have that the l.h.s. of the above expression can be formulated as

$$= -\sqrt{\eta} r \Delta_t Q'_t - \eta r \Delta'_t (Q_t + s Q'_t) + \eta \frac{r^2}{2} \Sigma_t Q''_t$$

By observing that $\partial_s(sQ_t) = Q_t + sQ'_t$ we conclude that (51) reads as

$$\eta \partial_t Q_t - \sqrt{\eta} \dot{m} \partial_s Q_t = -\sqrt{\eta} r \Delta_t Q'_t - \eta \left[r \Delta'_t \partial_s (sQ_t) - \frac{r^2}{2} \Sigma_t Q''_t \right] \tag{51}$$

where $Q_t^{(p)} = \partial_s^p Q_t$ and $\Delta'_t = \partial_m \Delta_t$. Therefore we can now apply the polynomial identity principle and split the ME (51) in two equations comparing terms with powers of η of the same order

$$\begin{cases} \dot{m} = r \Delta_t(m) \\ \partial_t Q_t(s) = -r \partial_m \Delta_t(m) \partial_s (s Q_t(s)) + \frac{r^2}{2} \Sigma_t(m) \partial_s^2 Q_t(s) \end{cases} \tag{52}$$

The jumps size r is constant and can be set equal to 1 without loss of generality.

1.4.1 Macroscopic equation equilibrium solution

The first equation involved in system (52) is an ODE and it is called the macroscopic equation, or macroeconomic equation in Aoki's works. It does not depend on the second equation of the system to which it is coupled with, so we can define the following Cauchy problem

$$\dot{m} = r(\beta_t(m) - \delta_t(m)) \; : \; m(0) = m_0 \tag{53}$$

It admits an equilibrium solution

$$\dot{m} = 0 \Rightarrow \beta(m^*) = \delta(m^*) \tag{54}$$

which means that, at the equilibrium, births and deaths perfectly balance each other. Therefore, inflows and outflows, or gains and losses, balance as well.

1.4.2 Fokker-Planck equation stationary solution

The second equation is a Fokker-Planck (FP) equation, a partial differential equation of the second order of the parabolic type. It is a diffusion equation with a linear first order term. This equation drives the dynamics of the density of the fluctuations, around the drifting trajectory, due to those unobservable microscopic interactions of the HIAs on the state space. Let us rewrite it as

$$\partial_t Q_t(s) = -r\partial_m \Delta_t(m)\partial_s(sQ_t(s)) + \frac{r^2}{2}\Sigma_t(m)\partial_s^2 Q_t(s) \tag{55}$$

and then collect ∂_s as follows

$$\partial_t Q_t(s) = -\partial_s \left\{ r\partial_m \Delta_t(m)(sQ_t(s)) - \frac{r^2}{2}\Sigma_t(m)\partial_s Q_t(s) \right\} = -\partial_s S_t(s) \tag{56}$$

in order to get the continuity equation form

$$\partial_t Q_t(s) + \partial_s S_t(s) = 0 \quad : S_t(s) = r\partial_m \Delta_t(m)(sQ_t(s)) - \frac{r^2}{2}\Sigma_t(m)\partial_s Q_t(s) \tag{57}$$

being $S_t(s)$ the so called *current of probability*.
The stationarity condition is

$$S_t(s) = \text{const.} \Leftrightarrow \partial_t Q_t(s) = 0 \Leftrightarrow \partial_t Q_t(s) = 0 \Leftrightarrow \lim_{t \to +\infty} Q_t(s) = \text{const.} \tag{58}$$

By using the previous two equations we get

$$\Delta' sQ = \frac{r}{2}\Sigma \partial_s Q$$

where $\Delta' = \partial_m \Delta(m)$, $\Sigma = \Sigma(m)$ and $Q = Q(s)$,

$$2\frac{s}{r}\frac{\Delta'(m)}{\Sigma(m)} = \frac{\partial \log Q(s)}{\partial s}$$

By direct integration w.r.t. s, we have

$$\ln Q = K_1 + \frac{s^2}{r}\frac{\Delta'}{\Sigma}$$

being K_1 the usual integration constant. So we can conclude that

$$Q(s) = K \exp\left\{ \frac{s^2}{r}\frac{\Delta'(m)}{\Sigma(m)} \right\} \tag{59}$$

which looks like a Gaussian distribution. As a consequence, the general solution is Gaussian too. The normalizing constant K must be estimated in order for $Q(s)$ to properly represent a probability distribution.

Due to Gaussianity $Q(s)$ is symmetric w.r.t. to its expected value μ_s introduced in (25-iii) when saying that $s \overset{i.i.d.}{\to} F_s(\mu_s, \sigma_s^2)$, where we now know that F_s has a Gaussian shape. Accordingly, the normalizing condition can be written as

$$\left\{ \begin{array}{c} \int_{-M}^{+M} Q(s)ds = K \int_{-M}^{+M} e^{-s^2 R(m)} ds = 1 \;\; s.t. \\ -R(m) = \frac{\Delta'(m)}{r\Sigma(m)} \;\; , \;\; M \to \infty \end{array} \right. \tag{60}$$

It then follows that, by assuming $\langle s \rangle = 0$[5]

$$\int_{-M}^{+M} e^{-s^2 R} ds = 2 \int_0^{+M} e^{-s^2 R} ds = \frac{2}{\sqrt{R}} \int_0^{+M} e^{-s^2 R} d(\sqrt{R}s)$$

By using $u = \sqrt{R}s$ we than have

$$\frac{2}{\sqrt{R}} \int_0^{+\sqrt{R}M} e^{-u^2} du = \sqrt{\frac{\pi}{R}} \mathrm{Erf}(\sqrt{R}M)$$

where

$$\mathrm{Erf}(x) = \int_0^x e^{-t^2} dt \Rightarrow \lim_{x \to +\infty} \mathrm{Erf}(x) = 1$$

is the well known error function. Therefore, according to the normalizing condition (60) we conclude that

$$K = \sqrt{\frac{R}{\pi}} \tag{61}$$

and so the stationary solution of the FP equation reads as

$$\left\{ \begin{array}{c} Q(s) = \sqrt{\frac{R(m)}{\pi}} \exp\{-s^2 R(m)\} \;\; : \;\; R(m) = -\frac{\Delta'(m)}{r\Sigma(m)} \\ \Delta'(m) = \beta'(m) - \delta'(m) \;\; , \;\; \Sigma(m) = \beta(m) + \delta(m) \end{array} \right. \tag{62}$$

1.4.3 General solution of the FP equation

The FP equation (55) can be partially solved without specifying the transition rates. To this aim it can be rewritten by using the following substitutions

$$\left\{ \begin{array}{c} D_t^0(s) = r\Delta_t'(m)s = d_0(r)s \;\; : \;\; d_0(r) = r\Delta_t'(m) \\ D_t^1(s) = r^2\Sigma_t(m) = d_1(r) \;\; : \;\; d_1(r) = r^2\Sigma_t(m) \end{array} \right. \tag{63}$$

[5] It is possible since s are the fluctuations about the expected trajectory, see the following eq. (76). Moreover, assuming the symmetry w.r.t. to $\langle s \rangle = \mu_s$ the integral can be evaluated on $[0, +M]$ and then doubling the result: note that M is arbitrarily large.

where $d_0(r)$ is the drift coefficient (not to be confused with the drifting trajectory $\phi(t)$ coming from the macroscopic equation and the spread about this path) and $d_1(r)$ the diffusion coefficient: they are constant w.r.t. s on the plane (s,t) and they, in general, are dependent on jumps' size r. Therefore we have

$$\left\{\begin{array}{l} \partial_t Q_t(s) = -\partial_s[D_t^0(s)Q_t(s)] + \frac{1}{2}\partial_s^2[D_t^1(s)Q_t(s)] \\ \quad = -\partial_s\left(d_0(r)sQ_t(s) - \frac{d_1(r)}{2}\partial_s Q_t(s)\right) = -\partial_s S(s) \end{array}\right. \tag{64}$$

which is exactly eq. (56). The stationary condition (58) gives[6]

$$d_0(r)sQ_t(s) = \frac{d_1(r)}{2}Q_t'(s) \tag{65}$$

As the stationary solution of the FP equation is Gaussian, the same holds true for its general solution, so we can here provide this inferential result

$$s = (X_1(t) - Nm(t))N^{-1/2} \overset{i.i.d.}{\to} \mathcal{N}(\mu_s(t), \sigma_s^2(t)) \tag{66}$$

which qualifies the distribution $F_s(\mu_s, \sigma_s^2)$ provided in the phenomenological model (25). We only need the first two moments (the mean and the variance), along time. They can be obtained by computing the derivative in the FP equation (55) as follows

$$\partial_t Q_t = -d_0(r)Q_t - d_0(r)sQ_t' + \frac{d_1(r)}{2}Q_t'' \tag{67}$$

and the plug it into the following expression

$$\langle s^k \rangle = \int s^k Q_t(s)ds \Rightarrow \partial_t \langle s^k \rangle = \int s^k \partial_t Q_t(s)ds \quad : \quad k = 1, 2 \tag{68}$$

as suggested by van Kampen (1992).
First of all let us set $k = 1$ to find

$$\left\{\begin{array}{l} \partial_t \langle s \rangle = \int s\partial_t Q_t ds \\ \quad = -d_0(r)\int sQ_t ds - d_0(r)\int s^2 Q_t' ds + \frac{d_1(r)}{2}\int sQ_t'' ds \\ \quad = -d_0(r)\langle s \rangle - d_0(r)(s^2Q_t - 2\int sQ_t ds) + \frac{d_1(r)}{2}(sQ_t' - \int Q_t' ds) \\ \quad = d_0(r)\langle s \rangle - s(d_0(r)sQ_t - \frac{d_1(r)}{2}Q_t') + k_0 \\ \quad = d_0(r)\langle s \rangle + k_0 \end{array}\right. \tag{69}$$

[6] In general this gives $D_t^0(s)Q_t(s) = 1/2\partial_s(D_t^1(s))Q_t(s) + 1/2D_t^1(s)Q_t'(s)$, but since $D_t^1(s) = d_1(r)$ it follows that $\partial_s(D_t^1)(s) = 0$.

by using the stationary condition (65). Moreover, since[7]

$$k_0 = -\int Q'_t ds = -Q_t(s) \Rightarrow k_0 = 0 \tag{70}$$

therefore we have that

$$\partial_t \langle s \rangle = d_0(r) \langle s \rangle = -r\Delta'_t(m) \langle s \rangle \tag{71}$$

which is a differential equation whose solution will provide $\mu_s(t)$, transition rates involved in $\Delta'_t(m)$ are specified according to (41), as we show below. In the same way we set $k = 2$ in eq. (68) and compute

$$\left\{\begin{array}{l} \partial_t \langle s^2 \rangle = -\int s^2 Q_t ds - d_0(r) \int s^3 Q'_t ds + \frac{d_1(r)}{2} \int s^2 Q''_t ds \\ \quad = -d_0(r) \langle s^2 \rangle - d_0(r)(s^3 Q_t + 3\int s^2 Q_t ds) + \frac{d_1(r)}{2}(s^2 Q'_t - 2\int s Q'_t ds) \\ \quad = 2d_0(r) \langle s^2 \rangle - s^2(d_0(r) s Q_t - \frac{d_1(r)}{2} Q'_t) - d_1(r) \int s Q'_t ds \\ \quad = 2d_0(r) \langle s^2 \rangle - d_1(r)(s Q_t - \int Q_t ds) \\ \quad = 2d_0(r) \langle s^2 \rangle - d_1(r) k_1 \end{array}\right. \tag{72}$$

by using again the stationary condition (65). Therefore, by observing that

$$k_1 = (s Q_t - \int Q_t ds) = -1 \tag{73}$$

we conclude that

$$\partial_t \left\langle s^2 \right\rangle = 2d_0(r) \left\langle s^2 \right\rangle + d_1(r) = -2rs\Delta'_t(m) \left\langle s^2 \right\rangle + r^2 \Sigma_t(m) \tag{74}$$

which is a differential equation for the second moment.
In order to proceed further, the following system of coupled[8] differential equations needs to be solved

$$\left\{\begin{array}{l} \partial_t \langle s \rangle = d_0(r) \langle s \rangle \ : \ d_0(r) = -r\Delta'_t(m) \\ \partial_t \langle s^2 \rangle = 2d_0(r) \langle s^2 \rangle + d_1(r) \ : \ d_1(r) = r^2 \Sigma_t(m) \end{array}\right. \tag{75}$$

This system yields a fundamental result. The transition rates enter the distribution of spreading fluctuations about the drifting trajectory. Therefore, being transition rates the mean-field interaction functions of the HIAs, spreading fluctuations are really described as due to those unobservable microscopic interactions and transitions on Ω which macroscopic effects project onto Λ.

[7]Being s a continuous random variable, the value of the density at every point is $Q_t(s) = 0$.

[8]The coupling of eq. (71) and eq. (74) is due to m involved in the coefficients $d_0(r)$ and $d_1(r)$, defined in (63).

Let us now solve the system (75). First of all we integrate the equation for the first moment, i.e. the mean:

$$\partial_t \langle s \rangle = d_0(r) \langle s \rangle \ : \ \langle s(0) \rangle = \langle s_0 \rangle \Rightarrow \langle s \rangle = \langle s_0 \rangle e^{d_0(r)t} = \mu_s(t) \tag{76}$$

As regards the second moment we have

$$\begin{cases} \partial_t \langle s^2 \rangle = 2d_0(r) \langle s^2 \rangle + d_1(r) \ : \ \langle s(0)^2 \rangle = \langle s_0^2 \rangle \\ \Rightarrow \langle s^2 \rangle = \langle s_e^2 \rangle + (\langle s_0^2 \rangle - \langle s_e^2 \rangle) e^{2d_0(r)t} \ : \ \langle s_e^2 \rangle = -\frac{d_1(r)}{2d_0(r)} \end{cases} \tag{77}$$

Now let us compute the variance by using the above results

$$\begin{cases} \ll s \gg = \langle s^2 \rangle - \langle s \rangle^2 = \langle s_e^2 \rangle + (\langle s_0^2 \rangle - \langle s_e^2 \rangle) e^{2d_0(r)t} - \langle s_0 \rangle^2 e^{2d_0(r)t} \\ \quad = \langle s_e^2 \rangle (1 - e^{2d_0(r)t}) + (\langle s_0^2 \rangle - \langle s_0 \rangle^2) e^{2d_0(r)t} \ \ s.t. \ \ \langle s_0^2 \rangle = \langle s_0 \rangle^2 \\ \quad = \langle s_e^2 \rangle (1 - e^{2d_0(r)t}) = \sigma_s^2(t) \end{cases} \tag{78}$$

Note that if we set $\langle s_0 \rangle = 0$ then $\mu_s(t) = 0$ and $\sigma_s^2(t) = \langle s^2 \rangle$. Therefore, by using (63) we have

$$\begin{cases} \mu_s(t) = \langle s_0 \rangle \exp(-r\Delta_t'(m)t) \\ \sigma_s^2(t) = \langle s_e^2 \rangle [1 - \exp(2r\Delta_t'(m)t)] \ : \ \langle s_e^2 \rangle = -\frac{r}{2} \frac{\Sigma_t(m)}{\Delta_t'(m)} \end{cases} \tag{79}$$

and due the Gaussianity, we have that

$$Q_t(s) = \frac{1}{\sqrt{2\pi\sigma_t^2(t)}} \exp\left\{ -\frac{(s - \mu_s(t))^2}{2\sigma_s^2(t)} \right\} \tag{80}$$

Equation (80) is therefore a functional rather than a function, indeed we only know that $\mu_s(t)$ and $\sigma_s^2(t)$ are functions of transition rates in $\Delta_t'(m)$ and $\Sigma_t(m)$, both depending on the solution of the macroscopic equation.

1.4.4 Final inferential result: aggregate dynamics

Transition rates specifications are needed in order to derive a general solution to a specific problem. We involve the phenomenological model (24) and (25) to give the aggregate stochastic trajectory dynamics of the underlying stochastic process $X_1(t) = X(\omega_1, t)$ on the state space Λ we started with.

$$\begin{cases} X_1(t) = Nm(t) + \sqrt{N}s(t) \ s.t. \\ \dot{m} = \Delta_t(m) = \beta_t(m) - \delta_t(m) \ : \ m(0) = m_0 \ \Rightarrow m(t) = m(m_0, t) \\ s(t) \overset{i.i.d.}{\to} \mathcal{N}(\mu_s(t), \sigma_s^2(t)) \equiv Q_t(s) \text{ where} \\ \mu_s(t) = \langle s_0 \rangle \exp(r\Delta_t'(m)) \text{ and} \\ \sigma_s^2(t) = \langle s_e^2 \rangle [1 - \exp(2r\Delta_s'(m))] \ : \ \langle s_e^2 \rangle = -\frac{r}{2} \frac{\Sigma_t(m)}{\Delta_t'(m)} \end{cases} \tag{81}$$

It is worth noticing that

1. $\Delta'_t(m) < 0$ is a sufficient condition for the model to be consistent;
2. the size of jumps r is involved in the aggregate model.

Hence, once the ME has been set up and transition rates have been specified, the solution is completely given by eq. (81) as regards the aggregate state variable. From eqs. (30) and (35) we have that

$$\begin{cases} \partial_t P_t(X_1(t)) \equiv \partial_t Q_t(s) + \dot{s}\partial_s Q_t(s) \\ \eta\partial_t P_t(X_h) = \eta\partial_t Q_t(s) - \sqrt{\eta}\dot{m}\partial_s Q_t(s) \end{cases} \tag{82}$$

It is now possible to provide an expression for the r.h.s. of the ME which, once integrated w.r.t. time, provides the final solution to the original ME (22), that is

$$P_t(X_h) = Q_t(s) - \sqrt{\eta}\int \dot{m}\partial_s Q_t(s)dt \tag{83}$$

Equation (83), according to the phenomenological model (25) and (81), evaluates the probability of finding a fraction X_h of $X_1(t)$ in a state of Λ at time $t \in \mathbb{T}$.

2 The NSF-ME

2.1 Transitory mechanics, phenomenological model and ME

The NSF-ME concerns the dynamics of the NSF population which will determine the credit market network. The state space for these firms is $\Omega = \{\omega_0, \omega_1\}$ where $f \in \omega_1$ means that the firm is NSF, while $f \in \omega_0$ stands for SF. Accordingly, we have

$$N_1(t) = \#\left\{f \in \mathcal{F}_t^B\right\}, N_0(t) = N - N_1(t) = \#\left\{f \in \mathcal{F} - \mathcal{F}_t^B\right\} \tag{84}$$

being N the constant number of firms in the system. This means that the ABM is the DGP (data generating process) of the two stochastic processes in eq. (84). We define the state space Λ for $N_1(t)$ as done in eq.s (2)-(4) by observing that

$$\underline{N}_1 = \min_t\{N_1(t)\} \wedge \overline{N}_1 = \max_t\{N_1(t)\} \tag{85}$$

Then consider that a realization is

$$N_1(t) = N_h \in \Lambda = \{\underline{N}_1 + hr \ : \ h \leq H \in \mathbb{N}, \underline{N}_1 = 0\} \tag{86}$$

therefore

$$(\underline{N}_1 = 0) \leq N_h = [\underline{N}_1 + hr] \leq \overline{N}_1 = (\underline{N}_1 + Hr = N) \ : \ r = 1 \tag{87}$$

It is worth stressing that $r = 1$ follows from that $\underline{N}_1$, N_h, $\overline{N}_1$, h and N are all natural numbers: this is one of the reasons why in literature unitary jumps are involved.[9] We consider $\{N_1(t) : t \in \mathbb{T}\}$ a jump Markov process and that the variation

$$|N_1(t + \Delta t) - N_1(t)| = r = 1 \quad : \quad \Delta t \ \text{ fixed} \tag{88}$$

is the size of the jump as in eq. (3). In this section we explicitly define transition rates for the NSF-ME by using the theory developed in the model. Let us begin with birth transition rates. According to eq. (86) and by using eq. (18) with $X_1(t) = N_1(t)$ we have

$$\beta_t(N_h - \vartheta) = \begin{cases} \beta_t(N_h) = w_t(N_h + 1|N_h) \ : \ \vartheta = 0 \\ \beta_t(N_h - 1) = w_t(N_h|N_h - 1) \ : \ \vartheta = 1 \end{cases} \tag{89}$$

Since N is constant $\beta_t(N_1(t))$ depends on how many agents belong to $\omega_0 \in \Omega$ (how many SF firms are in the system), that is $N - N_1(t)$ being N_h a realization of $N_1(t)$.

Only firms in $\omega_1 \in \Omega$ will be involved in the network, indeed only NSF firms ask for credit. Therefore, ω_1 is our state of interest and N_h performs as the state variable on Λ to evaluate this volume of firms. Two environmental externality function are introduced in order to estimate the probability for a firm to be NSF or SF. For the first this function is defined as

$$\psi_{1,t} = \psi_1 \left(\frac{N_1(t)}{N} \right) \tag{90}$$

After that we introduce the probability for the transition event[10]

$$\mathbb{P}(\omega_0 \rightharpoonup \omega_1) = \zeta_t \tag{91}$$

Let us write

$$\beta_t(N_h - \vartheta) = \lambda_t(N - (N_h - \vartheta)) \ \ s.t. \ \ \lambda_t = \zeta_t \psi_{1,t} \tag{92}$$

[9]Other reasons concern the specific nature of *particles* and the time for transitions to happen. A treatment of these aspects goes far beyond the aims of this paper, hence we lefts them for further developments.

[10]See eq. (12). The variables ζ_t and ι_t are defined by eq.s (13) and (16) in the paper.

The function $\lambda(t)$ is said the birth rate and it can be constant w.r.t. time in simpler situations. The externality function is then defined to be

$$\psi_{1,t} = \psi_1 \left(\frac{N_1}{N} \right) |_{N_1(t)=N_h} = \frac{b_1 + bN_h}{N} \tag{93}$$

and, as obvious, $b_1 > -bN_h$ in order for ψ_1 to make sense. Therefore we can write

$$\beta_t(N_h - \vartheta) = \zeta_t \left(\frac{b_1 + b(N_h - \vartheta)}{N} \right) (N - (N_h - \vartheta)) \tag{94}$$

which, by the usual homogeneity assumption, provides

$$N\beta_t \left(\frac{N_h - \vartheta}{N} \right) = N\zeta_t \left(\frac{b_1 + b(N_h - \vartheta)}{N} \right) \left(\frac{N - (N_h - \vartheta)}{N} \right) \tag{95}$$

This formalization of $\beta_t(N_h - \vartheta)$ has been inspired by Aoki and Yoshikawa (2006) and Alfarano et al. (2005) and it is interesting to observe that

1. $[N - (N_h - \vartheta)]$ is linear with N_h and it can be seen as the resistance to transition from ω_0 to ω_1;

2. $(N_h - \vartheta)[N - (N_h - \vartheta)]$ is quadratic, therefore non-linear, with N_h and it can be seen as a frictional growth of NSF firms.

As previously remarked, $\vartheta = 0$ indicates outflows while $\vartheta = 1$ refers to inflow births according to eq. (17). Consider now that $N_1(t)$ follows a phenomenological model as the one in eq. (24) and (25) by setting $X_h = N_h$. Therefore, by using eq. (36) we have

$$\beta_t(N_h - \vartheta) = N\beta_t(\phi_{(-\vartheta)}) \tag{96}$$

being $\phi_{(-\vartheta)}$ defined according to eq. (27) with $r = 1$ and setting $m = \phi$.
By using eq. (19) the death transition rates can be defined as

$$\delta_t(N_h + \vartheta) = \begin{cases} \delta_t(N_h) = w_t(N_h - 1|N_h) & : \ \vartheta = 0 \\ \delta_t(N_h - 1) = w_t(N_h|N_h + 1) & : \ \vartheta = 1 \end{cases} \tag{97}$$

Being N constant $\delta_t(N_1(t))$ depends on how many agents we have in $\omega_1 \in \Omega$, that is how many non-self financing firms we have or, differently put, it depends on how many agents do not belong to $\omega_0 \in \Omega$.
Actually $f \in \omega_0$ is a *positive* state for a firm, and we hope the volume of SF

firms to be as higher as possible according to environmental externalities, so we introduce[11]

$$\psi_{0,t} = \psi_0 \left(\frac{N_1(t)}{N} \right) \tag{98}$$

The probability for a SF firm to become NSF is defined as[12]

$$\mathbb{P}(\omega_0 \leftharpoondown \omega_1) = \iota_t \tag{99}$$

Now we write

$$\delta_t(N_h + \vartheta) = \mu_t(N_h + \vartheta) \;\; s.t. \;\; \mu(t) = \iota_t \psi_0(t) \tag{100}$$

where μ_t is said the death rate function opposite to λ_t.
The externality function is defined as

$$\psi_{0,t} = \psi_0 \left(\frac{N_h}{N} \right) |_{N_1(t)=N_h} = \frac{b_0 + b(N - (N_h + \vartheta))}{N} \tag{101}$$

where $b_0 > -b(N - N_h)$ in order for $\psi_{0,t}$ to make sense, therefore we can write

$$\delta_t(N_h + \vartheta) = \iota_t \left(\frac{b_0 + b(N - (N_h + \vartheta))}{N} \right) (N_h + \vartheta) \tag{102}$$

so, by using homogeneity as previously done, we have

$$N\delta_t \left(\frac{N_h + \vartheta}{N} \right) = N\iota_t \left(\frac{b_0 + b(N - (N_h + \vartheta))}{N} \right) \left(\frac{N_h + \vartheta}{N} \right) \tag{103}$$

therefore we observe that

1. $(N_h + \vartheta)$ is linear with N_h and it represents resistance to transition from ω_1 to ω_0;
2. $(N_h + \vartheta)[N - (N_h + \vartheta)]$ is quadratic, therefore non-linear, and it mimics a controlled growth with some friction for the volume of SF firms.

Since $\vartheta = 0$ and $\vartheta = 1$ refer to outflow and inflow respectively, according to eq. (19), and since $N_1(t)$ follows the phenomenological model of eq. (24) and (25), we can now involve eq. (37) to write

$$\delta_t(N_h + \vartheta) = N\delta_t(\phi_{(+\vartheta)}) \tag{104}$$

[11] See eq. (11) and footnote 1.

[12] See eq. (12) and eq. (91). Note that ι_t, as well as ζ_t, is a random variable due to stochastic shock on prices, therefore these are exogenous time indexed quantities.

being $\phi_{(+\vartheta)}$ defined in eq. (27) with $r = 1$ and $m = \phi$.
The phenomenological model (25) can be explicitly formulated with $r = 1$, $m = \phi$ and $s = \epsilon$ in the following way

$$\left\{ \begin{array}{c} N_h + \theta = N\phi + \sqrt{N}\epsilon + \theta \ \ : \ \theta = -1, 0, +1 \ \ s.t. \\ (i) \ N = |\mathcal{S}_N| \ \ (ii) \ \phi = \langle N_h/N \rangle \\ (iii) \ \epsilon = (N_h - N\phi)N^{-1/2} \stackrel{i.i.d.}{\rightarrow} F_\epsilon(\mu_\epsilon, \sigma^2_\epsilon) \end{array} \right. \tag{105}$$

We now are able to write the ME, which will be solved as in section 1.4.1

$$\left\{ \begin{array}{c} \eta\partial_t Q_t(\epsilon) - \eta\dot{\phi}\partial_\epsilon Q_t(\epsilon) = \\ \left[\beta_t(\phi_{(-)})Q_t(\epsilon_{(-)}) + \delta_t(\phi_{(+)})Q_t(\epsilon_{(+)})\right] - \left[\Sigma_t(\phi_{(0)})Q_t(\epsilon_{(0)})\right] \end{array} \right. \tag{106}$$

where $\epsilon_{(\pm\vartheta)}$ follows eq. (27) and $\eta^{p/2} : p = 1, 2$ as defined in eq. (43).

2.2 Macroscopic equation: equilibrium and general solution

As shown in section 1.4.1, by means of eq. (53) with $r = 1$ we obtain

$$\dot{\phi} = \Delta_t(\phi) = \beta_t(\phi) - \delta_t(\phi) \ \ : \ \ \phi(0) = \phi_0 \tag{107}$$

Thus, we need an expression for $\beta_t(\phi)$ and $\delta_t(\phi)$, which are the intensive form representation of transition rates. According to eq. (96) we have

$$\beta_t(N_h - \vartheta) = N\beta_t(\phi_{(-\vartheta)}) \ : \ \phi_{(-\vartheta)} = \phi + \sqrt{\eta}(\epsilon - \sqrt{\eta}\vartheta) \ , \eta = N^{-1} \tag{108}$$

By using (95) with $\vartheta = 0$, $\eta = N^{-1}$ and $N_h = N\phi + \sqrt{N}\epsilon$ we can write

$$\left\{ \begin{array}{c} \beta_t(\phi + \sqrt{\eta}\epsilon) = \zeta_t \left\{ \left[\eta b_1 + b(\phi + \sqrt{\eta}\epsilon)\right] \left[1 - (\phi + \sqrt{\eta}\epsilon)\right] \right\} \Rightarrow \\ \Rightarrow \beta_t(\phi + \sqrt{\eta}\epsilon) \stackrel{\sqrt{\eta}\to 0^+}{\rightarrow} \beta_t(\phi) = \zeta_t b\phi(1 - \phi) \Rightarrow b > 0 \end{array} \right. \tag{109}$$

In the same way we derive a formulation for $\delta_t(\phi)$ by using eq. (104)

$$\delta_t(N_h + \vartheta) = N\delta_t(\phi_{(+\vartheta)}) \ : \ \phi_{(+\vartheta)} = \phi + \sqrt{\eta}(\epsilon + \sqrt{\eta}\vartheta) \ , \eta = N^{-1} \tag{110}$$

so, by using (103) with $\vartheta = 0$, $\eta = N^{-1}$ and $N_h = N\phi + \sqrt{N}\epsilon$ we have

$$\left\{ \begin{array}{c} \delta(\phi + \sqrt{\eta}\epsilon) = \iota_t \left\{ \left[\eta b_0 + b(1 - (\phi + \sqrt{\eta}\epsilon))\right] (\phi + \sqrt{\eta}\epsilon) \right\} \Rightarrow \\ \Rightarrow \delta_t(\phi + \sqrt{\eta}\epsilon) \stackrel{\sqrt{\eta}\to 0^+}{\rightarrow} \delta_t(\phi) = \iota_t b\phi(1 - \phi) \Rightarrow b > 0 \end{array} \right. \tag{111}$$

and we finally make explicit the macroscopic equation as follows

$$\dot{\phi} = \rho_t\phi(1 - \phi) \ : \ \rho_t = b(\zeta_t - \iota_t) \ , \ \phi(0) = \phi_0 \ , \ b > 0 \tag{112}$$

With regards to the time varying coefficient ρ_t defining the growth factor $\rho_t = b(\zeta_t - \iota_t)$, it is worth stressing that:

1. in order for the FP equilibrium solution in eq. (62) to make sense, it must be that $\Delta'(\phi^*) < 0$, therefore $b(\zeta_t - \iota_t)(1 - 2\phi^*) < 0$ with $b > 0$ by the nature of transition rates, hence

$$\begin{cases} \{\rho_t = b(\zeta_t - \iota_t) < 0 \ : \ b > 0\} \Rightarrow \\ \{\zeta_t < \iota_t \Leftrightarrow P(\omega_0 \leftharpoondown \omega_1) > P(\omega_0 \rightharpoonup \omega_1)\} \\ \Rightarrow \phi(t) \stackrel{t \to +\infty}{\to} \phi* = 0^+ \\ \\ \{\rho_t = b(\zeta_t - \iota_t) > 0 \ : \ b > 0\} \Rightarrow \\ \{\zeta_t > \iota_t \Leftrightarrow P(\omega_0 \leftharpoondown \omega_1) < P(\omega_0 \rightharpoonup \omega_1)\} \\ \Rightarrow \phi(t) \stackrel{t \to +\infty}{\to} \phi^* = 1^- \end{cases} \tag{113}$$

2. the growth factor ρ_t is negative (positive) when the price shock makes more probable the NSF (SF) state when $\phi^* < 1/2$ ($\phi^* > 1/2$) which implies $\phi^* = 0^+$ ($\phi^* = 1^-$);

3. note that the same shock on prices at time $t+h$ and $t+k$ induces $\rho_{t+h} = \rho_{t+k}$ but its effect on the trajectory $\phi(t)$ is different. It depends on how far the system is from the equilibrium and on which the equilibrium it is approaching.

The general solution of the macroscopic equation $\dot{\phi} = \rho_t \phi(1 - \phi)$ with its initial condition $\phi(0) = \phi_0$, that is a Cauchy problem, is

$$\phi(t) = \frac{1}{1 + \left(\frac{1}{\phi_0} - 1\right) \exp\left[-\rho_t t\right]} \in (0, 1) \ : \ \phi(0) = \phi_0 \in (0, 1) \tag{114}$$

The sign of $\rho_t = b(\zeta_t - \iota_t)$ drives the dynamics. If ζ_t and ι_t would be constant eq. (114) would provide the usual deterministic logistic trajectory:

- $\phi(t) = \vartheta \ \forall t \in \mathbb{T}$ iff $\phi(0) = \vartheta$, linear-constant on the equilibria $\phi^* \in \{0; 1\}$;

- $\phi(t) \in (0, 1) \ \forall t \in \mathbb{T}$ iff $\phi(0) \in (0, 1)$, with the S-shaped curve depending on the initial condition: increasing iff $\phi_0 \in (0.5, 1)$ or decreasing iff $\phi_0 \in (0, 0.5)$.

The probabilities ζ_t and ι_t are time indexed shock price probabilities, therefore they influence the trajectory of $\phi(t)$ of the NSF share of firms. As a consequence, $\phi(t)$ can be exogenously perturbed about a logistic paths under exogenous environmental shocks as shown in figure 2.

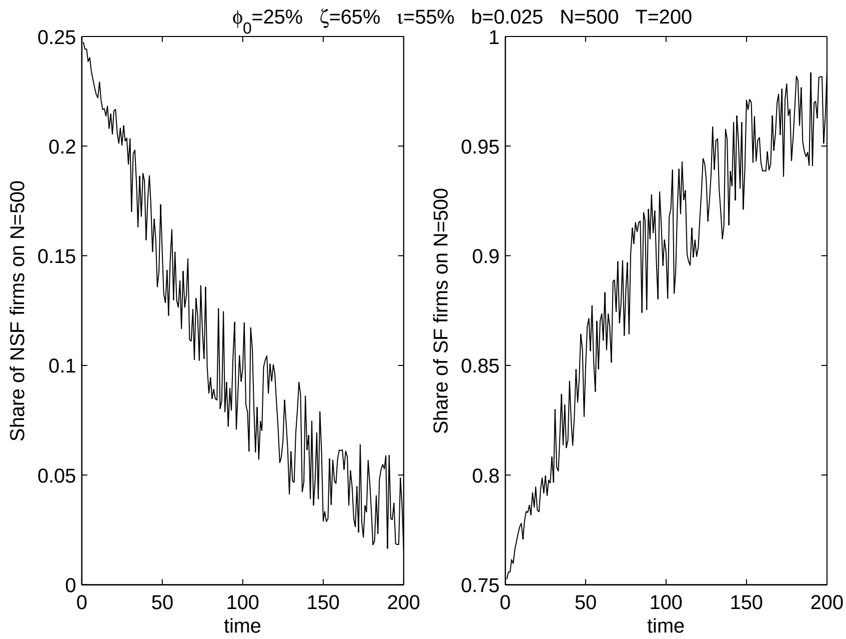

Figure 2: Macroeconomic equation solution: the perturbed drifting trajectory. NSF are $N_1(t) = N\phi(t)$. Left panel: $\phi_0 = 0.25$, $\zeta = 0.65$, $\iota = 0.55$, $b = 0.025$ and $N = 500$. Right panel: $\phi_0 = 0.75$, $\zeta = 0.65$, $\iota = 0.55$, $b = 0.025$ and $N = 500$.

The drifting trajectory which solves the macroeconomic equation enters transition rates as follows

$$\left\{\begin{array}{c}\Sigma_t(\phi)=\beta_t(\phi)+\delta_t(\phi)=b(\zeta_t+\iota_t)\phi(1-\phi)>0\\ \Delta(\phi)=\beta_t(\phi)-\delta_t(\phi)=b(\zeta_t-\iota_t)\phi(1-\phi)\left\{\begin{array}{l}>0\Rightarrow\zeta_t>\iota_t\\ <0\Rightarrow\zeta_t<\iota_t\end{array}\right.\end{array}\right\}\Rightarrow b>0 \tag{115}$$

Moreover, given that $\Delta'_t(\phi)<0$ must hold true in order for the FP equation to make sense we also have

$$\Delta'_t(\phi)=b(\zeta_t-\iota_t)(1-2\phi)<0\Rightarrow\left\{\begin{array}{l}\rho_t>0\wedge\phi_0\in(0.5,1)\\ \rho_t<0\wedge\phi_0\in(0,0.5)\end{array}\right. \tag{116}$$

Finally, observing that the only constant to be set is the initial condition ϕ_0, depending on its value probabilities ζ_t and ι_t can assume every value but they must satisfy $\Delta'_t(\phi)<0$ with $b>0$, therefore the extreme scenarios are

$$\left\{\begin{array}{l}0<\phi_0<0.5\ ,\ \zeta_t<\iota_t\Rightarrow\phi(t)\overset{t\to+\infty}{\to}0^+\text{ No NSF, only SF}\\ 0.5<\phi_0<1\ ,\ \zeta_t>\iota_t\Rightarrow\phi(t)\overset{t\to+\infty}{\to}1^-\text{ No SF, only NSF}\end{array}\right. \tag{117}$$

2.3 Fokker-Planck equation: stationary and general solution

According to eq.s (106), (109) and (111), the FP (55) reads as

$$\partial_t Q_t(\epsilon)=-\partial_\phi\Delta_t(\phi)\partial_\epsilon(\epsilon Q_t(\epsilon))+\frac{1}{2}\Sigma_t(\phi)\partial^2_\epsilon Q_t(\epsilon) \tag{118}$$

As long as we have two equilibria at $\phi^*_0=0$ and $\phi^*_1=1$ it follows that

$$\left\{\begin{array}{l}\Delta_t(\phi^*_i)=\beta_t(\phi^*_i)-\delta_t(\phi^*_i)=\rho_t\phi^*_i(1-\phi^*_i)\ :\ \rho_t=b(\zeta_t-\iota_t)\\ \Delta'_t(\phi^*_i)=\rho_t(1-2\phi^*_i)\\ \Sigma_t(\phi^*_i)=\beta_t(\phi^*_i)+\delta_t(\phi^*_i)=\xi_t\phi^*_i(1-\phi^*_i)\ :\ \xi_t=b(\zeta_t+\iota_t)\\ \phi^*_i=\theta_i=\left\{\begin{array}{l}0\text{ iff }i=0\\ 1\text{ iff }i=1\end{array}\right.\end{array}\right. \tag{119}$$

therefore the stationary[13] solution (59) now reads as

$$Q(\epsilon) = K_i \exp\left\{\epsilon^2 \frac{\rho_t}{\xi_t} \frac{1-2\phi_i^*}{\phi_i^*(1-\phi_i^*)}\right\} = K_i \exp\left\{\epsilon^2 \left(\frac{\zeta_t - \iota_t}{\zeta_t + \iota_t}\right)\left(\frac{1-2\phi_i^*}{\phi_i^*(1-\phi_i^*)}\right)\right\} \tag{120}$$

Now we observe that

$$\frac{1-2\phi_i^*}{\phi_i^*(1-\phi_i^*)} = \begin{cases} +\infty \text{ iff } i = 0 \Rightarrow \phi_0^* = 0 \\ -\infty \text{ iff } i = 1 \Rightarrow \phi_1^* = 1 \end{cases} \tag{121}$$

This shows that the stationarity condition to the Fokker-Planck equation at the eqilibria of the macroscopic equation does not make sense: indeed, as concerning the state variable, it means that all the firms are (N)SF steadily, hence no transition is allowed. However, these two limiting cases, although not impossible, are almost unlikely to be found. The system can approach an economy condition with all SF or NSF firms but this event is not so likely to happen when financial fragility really matters.
According to this, the stationarity of the Fokker-Planck equation is to be analyzed along the drifting path trajectory according to eq.s (62) which reads as

$$Q(\epsilon) = \sqrt{-\frac{1}{\pi}\frac{\Delta'(\phi)}{\Sigma(\phi)}} \exp\left\{-\epsilon^2 \frac{\Delta'(\phi)}{\Sigma(\phi)}\right\} \tag{122}$$

In order for (122) to make sense it must hold that

$$\frac{\Delta'(\phi)}{\Sigma(\phi)} < 0 \Leftrightarrow \left(\frac{\zeta_t - \iota_t}{\zeta_t + \iota_t}\right)\left(\frac{1-2\phi}{\phi(1-\phi)}\right) < 0 \tag{123}$$

as discussed in (116) and (117) with ϕ given in (114). The general solution of (118) can be derived by following van Kampen (1992) as shown in section 1.4.3. The functional form must be Gaussian since the stationary solution is Gaussian. As a consequence, only the first two moments are needed. By computing derivatives, according to (67), the FP equation (118) becomes

$$\partial_t Q_t(\epsilon) = -\Delta_t' Q_t(\epsilon) - \Delta_t' \epsilon Q_t' + \frac{1}{2}\Sigma_t Q_t'' \tag{124}$$

[13]Being a stationary solution time references are not necessary. Nevertheless they appear in ρ_t and ξ_t, as well as in their components, because these are random variables for which time is nothing but an index, they do not depend properly on time. Even though we consider to be very far from t_0, and state variables become stationary, random shocks still continue to be generated by the DGP, which is the ABM: this is why time enters ρ_t and ξ_t.

By using eqs. (63) with ϕ instead of m and $r = 1$, we get the following system

$$\begin{cases} \partial_t \langle \epsilon \rangle = d_0 \langle \epsilon \rangle \ : \ d_0 = \Delta'_t(\phi) \\ \partial_t \langle \epsilon^2 \rangle = 2d_0 \langle \epsilon^2 \rangle + d_1 \ : \ d_1 = \Sigma_t(\phi) \end{cases} \tag{125}$$

which is equivalent to the generic case (75). The solution of the ODE for the first moment is given by (76) which now reads as

$$\langle \epsilon \rangle = \langle \epsilon_0 \rangle e^{d_0 t} = \mu_\epsilon(t) \tag{126}$$

and by using (77) we get the solution of the ODE for the second moment, that is

$$\langle \epsilon^2 \rangle = \langle \epsilon_e^2 \rangle + (\langle \epsilon_0^2 \rangle - \langle \epsilon_e^2 \rangle) e^{2d_0 t} \ : \ \langle \epsilon_e^2 \rangle = -\frac{d_1}{2d_0} \tag{127}$$

The variance defined in eq. (78) now becomes

$$\ll \epsilon \gg = \langle \epsilon_e^2 \rangle (1 - e^{2d_0 t}) = \sigma_\epsilon^2(t) \tag{128}$$

Accordingly, the density for spreading fluctuations follows the Gaussian law

$$Q_t(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2(t)}} \exp\left[-\frac{(\epsilon - \mu_\epsilon(t))^2}{2\sigma_\epsilon^2(t)} \right] \tag{129}$$

2.4 NSF stochastic dynamics

Summarising all the results obtained so far, we can conclude that

$$\begin{cases} N_1(t) = N\phi(t) + \sqrt{N}\epsilon(t) \;\; s.t. \\ \phi(t) = \left[1 + \left(\frac{1}{\phi_0} - 1\right)\exp\left(-\rho_t t\right)\right]^{-1} \;\; : \phi_0 \in [0,1) \\ \epsilon(t) \stackrel{i.i.d}{\to} \mathcal{N}(\mu_\epsilon(t), \sigma_\epsilon^2(t)) \text{ where} \\ \\ \begin{cases} \mu_\epsilon(t) = \langle \epsilon_0 \rangle \exp\left(-\Delta'_t(\phi)t\right) \\ \sigma_\epsilon^2(t) = \langle \epsilon_e^2 \rangle \left[1 - \exp\left(2\Delta'_t(\phi)t\right)\right] \;:\; \langle \epsilon_e^2 \rangle = -\frac{\Sigma_t(\phi)}{2\Delta'_t(\phi)} \\ Q_t(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2(t)}} \exp\left[-\frac{(\epsilon - \mu_\epsilon(t))^2}{2\sigma_\epsilon^2(t)}\right] \text{ and} \end{cases} \\ \\ \begin{cases} \Delta'_t(\phi) = \beta'_t(t) - \delta'_t(\phi) = \rho_t(1 - 2\phi(t)) \\ \Sigma_t(\phi) = \beta_t(t) + \delta_t(\phi) = \xi_t\phi(t)(1 - \phi(t)) \text{ with} \end{cases} \\ \\ \begin{cases} \rho_t = b(\zeta_t - \iota_t) \\ \xi_t = b(\zeta_t + \iota_t) \text{ being} \end{cases} \\ \\ \begin{cases} \beta_t(\phi) = \zeta_t b\phi(1 - \phi) \\ \delta_t(\phi) = \iota_t b\phi(1 - \phi) \text{ where} \end{cases} \\ \\ \zeta_t, \iota_t > 0 \text{ are exogenously driven by shocks on prices } u_t \text{ and} \\ b > 0 \text{ is the only free parameter} \end{cases} \tag{130}$$

It is important to observe that the model makes sense if and only if

$$\begin{cases} \Sigma_t(\phi) = \xi b\phi(1 - \phi) > 0 \;:\; \xi_t = b(\zeta_t + \iota_t) \;,\; b > 0 \\ \Delta'_t(\phi) = \rho_t(1 - 2\phi) < 0 \;:\; \rho : t = b(\zeta_t - \iota_t) \;,\; b > 0 \end{cases} \tag{131}$$

Indeed, the first of (131) must hold because $\Sigma_t(\phi) = \beta_t(\phi) + \delta_t(\phi)$, where $\beta_t(\phi)$ and $\delta_t(\phi)$ are the transition rates (i.e. non negative transition probabilities per vanishing reference unit of time). As a consequence, $\Sigma_t(\phi) > 0$: if $\Sigma_t(\phi) \leq 0$ then (120) does not make sense because $R(\phi)$ is indefinite. The second of (131) must hold because $\Delta'_t(\phi)/\Sigma_t(\phi) < 0$ is essential. The condition $\Delta'_t(\phi) < 0$ is fundamental as already highlighted with reference to and eq. (116). As a consequence, if an inflow into NSF is more probable than an outflow, when the NSF are dominant in the long run, they will cover the largest share of firms. On the other hand, if a outflows from NSF is more probable than an inflow, when the SF are dominant, the NSF will tend to extinguish. The possible patterns of evolution are displayed in figure 3

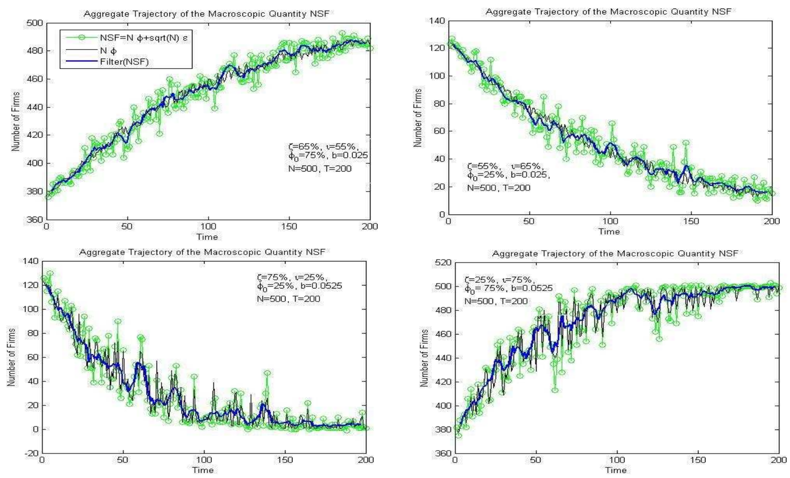

Figure 3: Four different solutions depending on the initial conditions on ϕ_0 and on probabilities ζ_t and ι_t.

3 The K-ME

3.1 Definitions

Let $\mathcal{F}_t^B = \{f_i \ : \ i \leq F \in \mathbb{N}\}$ be the set of $|\mathcal{F}_t^B|$ NSF firms and $\mathcal{B}_t = \{b_j \ : \ j \leq B \in \mathbb{N}\}$ a set of $B = |\mathcal{B}_t|$ banks. A NSF firm and a bank can give rise to three kinds of relationship[14]

1. *credit demand link* from $\mathcal{F}_t^B$ to $\mathcal{B}_t$

$$\delta : \mathcal{F}_t^B \otimes \mathcal{B}_t \to \{0,1\} \ s.t. \ (f,b) \mapsto \delta(f,b) = \begin{cases} 1 \Leftrightarrow f \rightharpoonup b \\ 0 \Leftrightarrow \text{otherwise} \end{cases} \tag{132}$$

2. *credit supply link* from $\mathcal{B}_t$ to $\mathcal{F}_t^B$

$$\sigma : \mathcal{F}_t^B \otimes \mathcal{B}_t \to \{0,1\} \ s.t. \ (f,b) \mapsto \sigma(f,b) = \begin{cases} 1 \Leftrightarrow f \leftharpoondown b \\ 0 \Leftrightarrow \text{otherwise} \end{cases} \tag{133}$$

3. *credit contract link* between $\mathcal{B}_t$ and $\mathcal{F}_t^B$

$$\lambda : \mathcal{F}_t^B \otimes \mathcal{B}_t \to \{0,1\} \ s.t. \ (f,b) \mapsto \lambda(f,b) = \begin{cases} 1 \Leftrightarrow f \rightharpoonup b \wedge f \leftharpoondown b \\ 0 \Leftrightarrow \text{otherwise} \end{cases} \tag{134}$$

[14]With $\otimes$ we indicate the interaction between the two sets.

The number of banks actively involved in the credit market, i.e. lending credit to NSF firms is $K_t = |\mathcal{B}_t| = \bar{B}_t$ being $\mathcal{B}_t^B = \bigcup_{f_i \in \mathcal{F}_t^B} \mathcal{B}_{f_i,t} \subseteq \mathcal{B}_t$. Each component is a clique and its cardinality $N_1^b(t)$ evaluates either the order of the clique and the degree of each client of a lender, that is $K_i = K(f_i)$ being f_i a node in a sub-network $\mathcal{G}_{b,t}$ of $\mathcal{G}_t$. Therefore, $K_i = N_1^b(t)$ for every firm borrowing from the b-th lender. The degree distribution is also the distribution of the cliqueness of the network $\mathcal{G}_t$. Moreover, the degree itself is a random variable taking values on a set of K_t natural numbers, being K_t the number of components in the network: degree's values range from 0 to S_t, being $S_t \in [1, N_1(t)]$ the number of nodes of the giant component.

$$K_i = \begin{cases} K(f_i) = K \in [0, S_t] \ : \ S_t \in [1, N_1(t)] \\ P(K,t) = P_t(K) \ : \ \sum_{K \in [0,S_t]} P_t(K) = 1 \end{cases} \tag{135}$$

3.2 Transitory mechanics, phenomenological model and ME

Consider a NSF firm $f \in \mathcal{F}_t^B$ with degree $K_t(f)$ at time t: $K_t(f)$ represents a random trajectory indexed by time over a certain state space

$$\Lambda_{K,t} = \{K(l,t) = l \leq S_t \in [0, N_1(t)]\} \tag{136}$$

made of S_t+1 discrete levels of degree between 0 and S_t. But, $S_t \in [0, N_1(t)]$ is the size of the giant component. It can change through time depending on the number of NSF firms in the graph $\mathcal{G}_t = \mathcal{G}(\mathcal{F}_t^B)$. It depends on the NSF-ME solution in eq. (130). Moreover, from the NSF-ME solution, $N_1(t) \in [0, N)$. This means that, from the one hand $\Lambda_{K,t}$ is not mechanically invariant[15] and, from the other hand, it is nevertheless ascribable to a mechanically invariant state space by considering its maximal extension: since $N_1(t) \in [0, N) \Rightarrow N_1(t) \in [0, N-1]$ it follows that $S_t \in [0, N_1(t)] \Rightarrow S_t \in [0, N-1]$, therefore

$$\Lambda_K = \{K_l = l \in [0, N-1]\} \tag{137}$$

is the mechanically invariant state space and it is made of N discrete levels of degree. For what concerns the general ME modelling of section 1 this means that eq. (5) reads as

$$[0 \leq \underline{K}(l,t) = \min_t \Lambda_{K,t}] \leq K(l,t) \leq [\overline{K}(l,t) = \max_t \Lambda_{K,t} = S_t] \tag{138}$$

The $N_1(t)$ NSF firms distribute on Λ_K according to $\mathcal{G}_t = \mathcal{G}(\mathcal{F}_t^B)$ which is made of $K_t \leq N$ non empty isolate clique-components (i.e. *islands*),

[15] An object is here said to be mechanically invariant if it is constant w.r.t. time.

therefore only $K_t \leq N$ levels can be occupied. An occupation number is $N^l(t) = K_l C_l(t)$ which means that

$$N^l(t) = \#\left\{K_t(f) = K_l \in \Lambda_K \ : f \in \mathcal{F}_t^B\right\} \subseteq [0, N-1] \tag{139}$$

that is the number of NSF firms with degree $K_l = l$: the occupation number for the l-th degree level among the N possible ones. According to this we can define a configuration as

$$\mathbf{N}(t) = \left(\left\{N^l(t) = K_l C_l(t) \ : \ K_l = l \in \Lambda_K\right\}\right) \ : \ \Lambda_K = [0, N-1] \tag{140}$$

which is a vector of occupation numbers: only $K_t \leq N$ of them are non zero, those concerning the levels of degree occupied by NSF firms in $\mathcal{G}_t = \mathcal{G}(\mathcal{F}_t^B)$. If we then divide each occupation number by the number of NSF firms we get

$$\mathbf{n}(t) = \left(\left\{n_l(t) = \frac{N^l(t)}{N_1(t)} \in [0, 1] \ : l \in \Lambda_K = [0, N-1]\right\}\right) \tag{141}$$

which evaluates the density of NSF firms in the l-th level of the degree or, if you prefer, the degree distribution: indeed $n_l(t)$ is the share of the $N_1(t)$ NSF firms with degree $K_l = l$.
Our problem is now to provide a model for $N^l(t)$ or $n_l(t) = N^l(t)/N_1(t)$ to develop a K-ME for the density of the degree. The definition of a component is

$$\mathcal{G}_{b,t} = \left\{f \in \mathcal{F}_t^b \ : \ \lambda(f, b) = 1, b \in \mathcal{B}_t^B \subseteq \mathcal{B}_t\right\} \subseteq \mathcal{G}_t \ : \ |\mathcal{G}_{b,t}| = N_1^b(t) \tag{142}$$

Among the $K_t \leq N$ non empty components consider $\mathcal{G}_{b,t}$ to be the one of interest. The dimension of the component also evaluates the degree of each node in it, that is, according to eq. (142), $N_1^b(t) = |\mathcal{G}_{b,t}| = K_l \in \Lambda_K$. The dimension of a component can change through time because of firms transitions within the network, but it can happen due to updates from/to the external environment as well as (see figure 4).

To represent this mechanism we define two types of transitions within the network:

- death transitions from $\mathcal{G}_{b,t}$: $X_b^{\rightarrow} = \#\left\{f \in \mathcal{G}_{b,t} \rightarrow f \in \mathcal{G}_{b',t+\Delta t}\right\}$ for which we have $N_1^b(t) = K_l \rightarrow N_1^b(t + \Delta t) = K_{l'}$ being $l' = l - X_b^{\rightarrow}$;
- birth transitions to $\mathcal{G}_{b,t+\Delta t}$: $X_b^{\leftarrow} = \#\left\{f \in \mathcal{G}_{b',t} \rightarrow f \in \mathcal{G}_{b,t+\Delta t}\right\}$ for which we have $N_1^b(t) = K_l \rightarrow N_1^b(t + \Delta t) = K_{l'}$ being $l' = l + X_b^{\leftarrow}$;

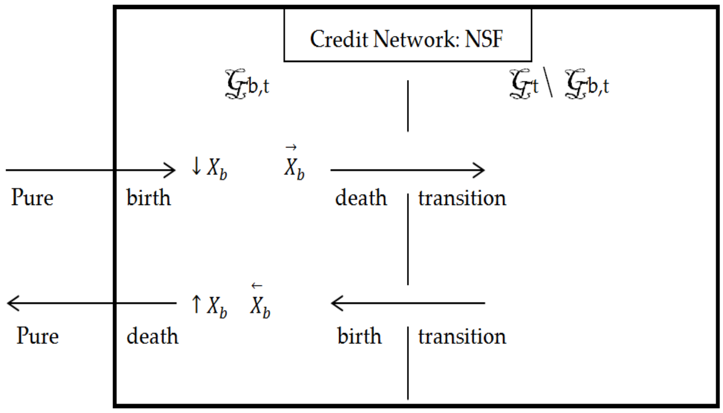

Figure 4: Transition within the network and updates from/to the external environment.

Similarly we define two kinds of updates:

- pure deaths from $\mathcal{G}_{b,t}$: $X_b^{\uparrow} = \#\{f \in \mathcal{G}_{b,t} \to f \notin \mathcal{G}_{t+\Delta t}\}$ for which we have $N_1^b(t) = K_l \to N_1^b(t+\Delta t) = K_{l'}$ being $l' = l - X_b^{\uparrow}$;
- pure births into $\mathcal{G}_{b,t+\Delta t}$: $X_b^{\downarrow} = \#\{f \notin \mathcal{G}_t \to f \in \mathcal{G}_{b,t+\Delta t}\}$ for which we have $N_1^b(t) = K_l \to N_1^b(t+\Delta t) = K_{l'}$ being $l' = l + X_b^{\downarrow}$;

According to these four sources of variation we define the following demographic law

$$N_1^b(t) = K_l \to N_1^b(t+\Delta t) = K_{l'} = K_l + \varphi_b^{nat}(t) + \varphi_b^{mig}(t)$$

where $\varphi_b^{nat}(t) = X_b^{\downarrow} - X_b^{\uparrow}$ is the natural balance and $\varphi_b^{mig}(t) = X_b^{\leftarrow} - X_b^{\rightarrow}$ the migratory balance, therefore the variation of the b-th sub-system is

$$|N_1^b(t+\Delta t) - N_1^b(t)| = |\varphi_b^{nat}(t) + \varphi_b^{mig}(t)| = |\varphi_b^{tot}(t)| \neq \text{const} \quad : \quad \Delta t \ \text{ fixed}$$

This quantity cannot be constant[16], neither through time nor across components since the volume of firms $N_1(t)$ in the credit network always updates, even when $\varphi_b^{mig}(t) = 0$. This means that the size of the jumps of the component's dimension changes and this configures the degree of a component as a stochastic process. In any case, if the problem is to specify a model for $N^l(t)$ it does not matter what happens within each single component

[16]In the case of NSF-ME we were allowed to put the variation as constant, see eq. (88).

since elementary constituents of sub-systems (i.e. firms in components) are perfectly indistinguishable. What really matters to our aim is the number components $C_l(t)$ in the K_l degree level on the state space Λ_K.
Since $C_l(t)$ is the number of components in the K_l degree level, and since its value changes according to the dynamics of the volume of $N_1(t)$ elementary constituents of the credit system, a phenomenological representation for it can be written as

$$C_l(t) = N_1(t)[\phi_l(t) + \sqrt{\eta_1(t)}\hat{\epsilon}_l(t)] \quad : \quad \eta_1(t)N_1(t) = 1 \tag{143}$$

Each component in K_l is also a l-clique, being $K_l = l \in \Lambda_K$. Therefore, to evaluate the number of firms in the l-th degree level we can consider the following phenomenological model

$$N^l(t) = K_l C_l(t) = K_l N_1(t)[\phi_l(t) + \sqrt{\eta_1(t)}\hat{\epsilon}_l(t)] \quad : \quad \eta_1(t)N_1(t) = 1 \tag{144}$$

In both phenomenological representations $\hat{\epsilon}_l(t)$ is unknown, but we can assume that

$$\hat{\epsilon}_l(t) = \sqrt{\frac{N_1(t)}{K_l}}\epsilon_l(t) \tag{145}$$

to get

$$N^l(t) = N_1(t)[K_l\phi_l(t) + \sqrt{K_l}\epsilon_l(t)] \tag{146}$$

from which we have

$$n_l(t) = \frac{N^l(t)}{N_1(t)} = K_l\phi_l(t) + \sqrt{K_l}\epsilon_l(t) \in [0,1] \tag{147}$$

Eq. (146) is the phenomenological model for the share of nodes in the K_l degree level: the sequence of these shares defines the degree distribution as in eq. (141). With this phenomenological model we develop the K-ME: the (sub-)system size parameter here is fixed to K_l. Moreover, note that in eq. (144) the solution of the NSF-ME enters the degree distribution model according to eq. (143). Therefore, setting up a K-ME for the state variable $n_l(t)$ in eq. (147) will involve the NSF-ME solution. This means that the NSF-ME is nested into the K-ME.

A component changes its degree level if at least one firm enters or exits within a small but fixed interval of time $\Delta t \to 0^+$. So we allow for unitary jumps on Λ_K. From eq. (147), the phenomenological model for $n_l(t)$ concerns an intensive quantity, indeed $n_l(t) \in [0,1]$ is the fraction of NSF firms in the network belonging to the K_l-th degree level. The (sub-)system size parameter is now constant being $K_l \in \Lambda_K$. We also know that jumps are allowed and, by assumption, one component at time changes its degree level. As a consequence, the size of the jump for the state variable $n_l(t)$ is represented in intensive form as $\rho_{l,t} = \rho(K_l, N_1(t))$. There could be at most $N = |\Lambda_K|$ components as well as N degree levels. Therefore we can define

$$\begin{cases} n_l(t) \pm \vartheta\rho_{l,t} = K_l\phi_l(t) + \sqrt{K_l}\epsilon_l(t) \pm \vartheta\rho_l(t) \ s.t. \\ \rho_l(t) = \rho(K_l, N_1(t)) \\ \phi_l(t) = \langle n_l(t)k_l\rangle \in [0, k_l] \quad : \quad k_l K_l = 1 \\ \epsilon_l(t) = (n_l(t) - K_l\phi_l(t))K_l^{-1/2} \overset{i.i.d}{\to} F_{\epsilon_l}(\mu_{\epsilon_l}(t), \sigma^2_{\epsilon_l}(t)) \end{cases} \tag{148}$$

This phenomenological model is estimated by following eq.s (24) and (25), where ϑ has been defined in eq. (14).
Since, by definition, $n_l(t) \in [0, 1]$ then $\phi_l(t) \in [0, k_l]$ being $k_l = 1/K_l$. Hence we observe that

$$\begin{cases} n_l(t) = 0 \Rightarrow K_l\phi_l(t) = -\sqrt{K_l}\epsilon_l(t) \Rightarrow \phi_l(t) = -\sqrt{k_l}\epsilon_l(t) \Rightarrow \\ \qquad \langle\phi_l(t)\rangle = 0 \text{ i.f.f. } \langle\epsilon_l\rangle = 0 \\ n_l(t) = 1 \Rightarrow K_l\phi_l(t) = 1 - \sqrt{K_l}\epsilon_l(t) \Rightarrow \phi_l(t) = k_l - \sqrt{k_l}\epsilon_l(t) \Rightarrow \\ \qquad \langle\phi_l(t)\rangle = k_l \text{ i.f.f. } \langle\epsilon_l\rangle = 0 \end{cases} \tag{149}$$

Therefore, since now, we can conclude that the fluctuations should have zero mean, that is

$$\mu_{\epsilon_l}(t) = \langle\epsilon_l(t)\rangle = 0 \tag{150}$$

This model coupled with eq.s (147), (146), (144) and (143) defines the complete model for the degree.

Now we specify transition rates according to the general set-up developed in section 1 by considering that transition rates are transition probabilities per (vanishing) reference unit of time and that they can be conceived as functions of three elements:

- probabilities for transition events to happen, see eq. (12);
- environment effects' or externality functions, see eq. (11);
- volumes of units in states.

A generic representation of the *a-priori* probabilities for transition events has been introduced in section 1.1 in eq. (12). In the present context they concern the creation and destruction of a link since a transition event, from a component of level K_l to another one, happens with links' creation or destruction. Hence, as concerning $C_l(t)$, a transition event is a jump on a higher/lower degree level and, due to this, transitions can be expressed in terms of probabilities. Define $t' = t + \Delta t$ and consider that between t and

t' a link between two firms can be created either if firms were linked or not, modifying the number of firms in a given degree level. On the other hand, a link destruction can happen if and only if two firms were linked at time t, and this modifies the cardinality of the degree level too. To make things as simple as possible we consider these probabilities to be constant through time independently of the way the link is created or destroyed.

$$\mathbb{P}(a_{i,j}^{t'} = 1 | a_{i,j}^{t} = 0) = \zeta \qquad \wedge \qquad \mathbb{P}(a_{i,j}^{t'} = 0 | a_{i,j}^{t} = 1) = \iota \tag{151}$$

We will call these probabilities as link creation and destruction rates respectively.

In every network $\mathcal{G}_t$ there is one giant component[17] $\mathcal{G}_{s,t}^{S}$ with dimension $S_t \overset{i.i.d}{\to} U([0, N-1])$. The giant component covers a fraction $\gamma_t = S_t/N_1(t)$ of nodes in the network and it represents the group of firms which are linked to the bank $s \in \mathcal{B}_t^B$ which has the largest number of clients. The dimension of the giant component is an information concerning reliability and convenience of the bank: this can induce an emulative behaviour among firms when choosing the bank to be linked to. This also means that the size of the giant component could modify the morphology of the network by exerting an attractive force[18]. Consider than that the more $\gamma_t \to 1^-$ the more $(1-\gamma_t) \to 0^+$: a firm tied to the giant component should have a low probability of abandoning it, while a firm tied to another component should have a high probability to leave its component to be tied with the giant one. Therefore, the presence of the giant component indirectly influences the process of creation and destruction of links. That is, the giant component $\mathcal{G}_{s,t}^{S}$ has a *gravitational* effect on the others $\mathcal{G}_{b,t}$. Therefore, the higher γ_t the more a firm should be tempted to leave its component $\mathcal{G}_{b,t}$ to enter $\mathcal{G}_{s,t}^{S}$, and the less it should be tempted to leave $\mathcal{G}_{s,t}^{S}$ to go to another bank if the firm belongs to the giant component. The influence of the giant component in the creation/destruction of links will be introduced in transition rates according to the externality functions $\psi_{0,t}$ and $\psi_{1,t}$ introduced in section 1.1, see eq. (11). Let us define them more precisely.

If we consider a component $\mathcal{G}_{b,t}$ to be the component of interest, an inflow from another component or an outflow to another component are influenced by the presence of the giant one. We introduce $\psi_{1,l,t}$ for the inflows and $\psi_{0,l,t}$

[17] There can be more one giant component but, in this case, they must have the same dimension. In what follows we consider only the case of one giant component.

[18] Actually every bank exerts to some extent this force but the giant component's one is the strongest. We consider only this one for simplicity of exposition of this innovative stream of modelling, leaving more complex structures to further developments.

for the outflows into/from a degree level $K_l = l \in \Lambda_K$; both functions must reflect the gravitational influence of the giant component size γ_t relatively to the degree level K_l, which evaluates also the component size of firms with degree K_l. As obvious, $S_t/N_1(t) = \gamma_t \geq \max\{K_l/N_1\}$ in order for a component to be the giant one.
As concerning the inflows, we consider the case of a $\psi_{1,l,t}$ which reacts inversely w.r.t. γ_t and directly to K_l, that is inversely to $k_l = K_l^{-1}$.
On the contrary, concerning the outflows, the greater the giant component the more a firm is tempted to match with the giant component, increasing the possibility of an outflow. But this effect should be more intense at low degree levels. Therefore we want $\psi_{0,l,t}$ to react directly w.r.t γ_t and inversely w.r.t. K_l, that is inversely to $k_l = K_l^{-1}$. Hence we guess the following externality functions

$$\psi_{1,l,t} = \exp(\pi_1^2(1-\gamma_t)(\phi_l^0/k_l)) \quad \wedge \quad \psi_{0,l,t} = \exp(\pi_1^2\gamma_t(\phi_l^0/k_l)) \tag{152}$$

where π_1 is the firm-bank matching probability[19]. These externality functions change with time but they apply equivalently to every component.
Now, according to the general set-up, we can define birth and death rates as done in eq. (13)

$$\lambda_{l,t} = \psi_{1,l,t}\zeta \text{ birth rate } \wedge \ \mu_{l,t} = \psi_{0,l,t}\iota \text{ death rate} \tag{153}$$

The last factor that is needed in order to specify the transition rates is the volume of agents in states. As regarding birth transition rate we consider that it depends on how many agents are not in the state of interest, that is $1 - n_l(t)$, while the death transition rate depends on how many agent we have in the state of interest, $n_l(t)$. Then, by using eq. (18) we now have that the implicit form of the birth transition rate is

$$\beta_t(n_l(t) - \vartheta\rho_{l,t}) = \begin{cases} \beta_t(n_l(t)) = w_t(n_l(t) + \rho_{l,t}|n_l(t)) \ : \vartheta = 0 \text{ outflow} \\ \beta_t(n_l(t) - \rho_{l,t}) = w_t(n_l(t)|n_l(t) - \rho_{l,t}) \ : \vartheta = 1 \text{ inflow} \end{cases} \tag{154}$$

By using the birth rate function we have

$$\beta_t(n_l(t)-\vartheta\rho_{l,t}) = \lambda_{l,t}[1-(n_l(t)-\vartheta\rho_{l,t})] \ : \ \lambda_{l,t} = \zeta\exp(\pi_1^2(1-\gamma_t)\phi_l^0/k_l) \tag{155}$$

[19]The probability π_1 is indicated by p_1 in the paper. The function ϕ_l^0 evaluates the initial condition for the macroscopic equation (173): indeed, not only high/low degree levels matter but also how much they are populated. Moreover, we used π_1^2 instead of π_1 this is the matching probability between a firm and a bank, therefore π_1^2 is the expected probability for two firms to be linked (i.e indirectly interact - mean field interaction) one another.

In the same way we provide the implicit death transition rate form

$$\delta_t(n_l(t)+\vartheta\rho_{l,t}) = \begin{cases} \delta_t(n_l(t)) = w_t(n_l(t)-\rho_{l,t}|n_l(t)) & : \vartheta = 0 \text{ outflow} \\ \delta_t(n_l(t)+\rho_{l,t}) = w_t(n_l(t)|n_l(t)+\rho_{l,t}) & : \vartheta = 1 \text{ inflow} \end{cases} \tag{156}$$

By using the death rate function we have

$$\delta_t(n_l(t)+\vartheta\rho_{l,t}) = \mu_{l,t}[n_l(t)+\vartheta\rho_{l,t})] \ : \ \mu_{l,t} = \iota\exp(\pi_1^2\gamma_t\phi_l^0/k_l) \tag{157}$$

We now have a complete specification of transition rates so we can involve them in methods developed in section 1.3 and 1.4 to solve the K-ME.

By using eq. (26) and (27) in eq. (148), setting $X_h = m_l$ as a fixed realization of the stochastic process $n_l(t)$, $m = \phi_l$ and $s = \epsilon_l$ we can write the l.h.s. of the K-ME as follows

$$\frac{1}{K_l}\frac{dP_t(n_l)}{dt} = \frac{1}{K_l}\frac{\partial Q_t(\epsilon_l)}{\partial t} - \frac{1}{\sqrt{K_l}}\frac{d\phi_l}{dt}\frac{\partial Q_t(\epsilon_l)}{\partial\epsilon_l} \tag{158}$$

as was done in eq. (35) since, according to eq. (148), we can always write $P_t(n_l(t)) = Q_t(\epsilon_l(t))$, as done in eq. (28) or (38). By using eq. (155) and (157) we see that transition rates are homogeneous functions w.r.t. to the (sub) system size parameter K_l, therefore eq.s (36)-(38) still hold true and can be rewritten as

$$\beta_t(n_l-\vartheta\rho_{l,t}) = K_l\beta_t(\phi_{l,(-\vartheta)}) \tag{159}$$

$$\delta_t(n_l+\vartheta\rho_{l,t}) = K_l\delta_t(\phi_{l,(+\vartheta)}) \tag{160}$$

$$P_t(n_l\pm\vartheta\rho_{l,t}) = Q_t(\epsilon_{l,(\pm\vartheta)}) \tag{161}$$

where $\phi_{l,(\pm\vartheta)}$ and $\epsilon_{l,(\pm\vartheta)}$ follow eq. (27) and read as

$$\phi_{l,(\pm\vartheta)} = \phi_l + \sqrt{k_l}\epsilon_{l,(\pm\vartheta)} \ : \ \epsilon_{l,(\pm\vartheta)} = \epsilon_l \pm \vartheta\rho_l(t)k_l \tag{162}$$

since

$$\frac{n_l(t)\pm\vartheta\rho_{l,t}}{K_l} = \phi_l + \sqrt{k_l}\epsilon_l \pm \vartheta\rho_{l,t}k_l \ : \ k_lK_l = 1 \tag{163}$$

Therefore, the K-ME to be solved follows eq. (42) and reads as

$$\begin{cases} \frac{1}{K_l}\frac{\partial Q_t(\epsilon_l)}{\partial t} - \frac{1}{\sqrt{K_l}}\frac{d\phi_l}{dt}\frac{\partial Q_t(\epsilon_l)}{\partial\epsilon_l} = \\ \left[\beta_t(\phi_{l,(-)})Q_t(\epsilon_{l,(-)}) + \delta_t(\phi_{l,(+)})Q_t(\epsilon_{l,(+)})\right] + \\ -\left[\Sigma_t(\phi_{l,(0)})Q_t(\epsilon_{l,(0)})\right] \end{cases} \tag{164}$$

where $\Sigma_t(\phi_{l,(0)})$ is defined in eq. (40) and its companion $\Delta_t(\phi_{l,(0)})$ follows eq. (41).
As explained in section 1.4, by developing expansions of transition rates about ϕ_l and of the density about ϵ_l we can have the usual system of coupled equations

$$\begin{cases} \dot{\phi}_l = \rho_{l,t}\Delta_t(\phi_l) = \rho l, t\left(\beta_t(\phi_l) - \delta_t(\phi_l)\right) \\ \partial_t Q_t(\epsilon_l) = -\rho_{l,t}\Delta'_t(\phi_l)\partial_{\epsilon_l}(\epsilon_l Q_t(\epsilon_l)) + \frac{\rho^2_{l,t}}{2}\Sigma_t(\phi_l)\partial^2_{\epsilon_l}Q_t(\epsilon_l) \end{cases} \tag{165}$$

with the macroscopic equation (first line) for the drifting trajectory and the Fokker-Planck equation (second line) for the density of fluctuations about this trajectory. The solution of the system (165) provides the solution to our problem, as stated in (81), by involving also the solution of the NSF-ME given in eq. (130). In the following sections we provide the solution of this system of coupled equation as shown in section 1.4.

3.3 Macroscopic equation: equilibrium and general solution

This subsection presents the solution of the macroscopic equation for the drifting trajectory following steps of section 1.4.1. Instead of r we here use $\rho_{l,t} = \rho(K_l, N_1(t))$ since it is not dependent on time but on $N_1(t)$, already determined by the NSF-ME solution which does not depend on the K-ME. Hence we have

$$\dot{\phi}_l = \rho_{l,t}\Delta_t(\phi_l) = \rho_{l,t}[\beta_t(\phi_l) - \delta_t(\phi_l)] \quad : \quad \phi_l(0) = \phi^0_l \tag{166}$$

By using eq. (155) and (159) we have

$$K_l\beta_t\left(\frac{n_l - \vartheta\rho_{l,t}}{K_l}\right) = K_l\zeta e^{\pi^2_1(1-\gamma_t)\phi^0_l/k_l}\frac{1-(n_l - \vartheta\rho_{l,t})}{K_l} = K_l\beta_t(\phi_{b,(-)}) \tag{167}$$

By using eq. (157) and (160) we have

$$K_l\delta_t\left(\frac{n_l + \vartheta\rho_{l,t}}{K_l}\right) = K_l\iota e^{\pi^2_1\gamma_t\phi^0_l/k_l}\frac{n_l + \vartheta\rho_{l,t}}{K_l} = K_l\delta_t(\phi_{l,(+)}) \tag{168}$$

Set $\vartheta = 0$, $k_l = K_l^{-1}$ and since $n_l = K_l\phi_l + \sqrt{K_l}\epsilon_l$ we have

$$\begin{cases} \beta_t(\phi_l + \sqrt{k_l}\epsilon_l) = \zeta e^{\pi^2_1(1-\gamma_t)\phi^0_l/k_l}[1-(\phi_l + \sqrt{k_l}\epsilon_l)] \Rightarrow \\ \beta_t(\phi_l) = \zeta e^{\pi^2_1(1-\gamma_t)\phi^0_l/k_l}(1-\phi_l) \end{cases} \tag{169}$$

and

$$\begin{cases} \delta_t(\phi_l + \sqrt{\rho_l}\epsilon_l) = \iota e^{\pi_1^2 \gamma_t \phi_l^0 / k_l}(\phi_l + \sqrt{k_l}\epsilon_l) \Rightarrow \\ \delta_t(\phi_l) = \iota e^{\pi_1^2 \gamma_t \phi_l^0 / k_l}\phi_l \quad : \quad \rho_l \to 0^+ \end{cases} \tag{170}$$

hence

$$\Sigma_t(\phi_l) = \beta_t(\phi_l) + \delta_t(\phi_l) = \zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \left(\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \iota e^{\pi_1 \gamma_t}\right)\phi_l \tag{171}$$

$$\Delta_t(\phi_l) = \beta_t(\phi_l) - \delta_t(\phi_l) = \zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \left(\zeta e^{\pi_1(1-\gamma_t)\phi_l^0/k_l} + \iota e^{\pi_1^2 \gamma_t \phi_l^0/k_l}\right)\phi_l \tag{172}$$

Therefore the macroscopic equation (166) reads as

$$\dot{\phi}_l = \rho_{l,t}\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \rho_{l,t}\left(\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} + \iota e^{\pi_1^2 \gamma_t \phi_l^0/k_l}\right)\phi_l \quad : \quad \phi_l(0) = \phi_l^0 \tag{173}$$

which is a first order linear ODE with exogenously and stochastically perturbed coefficients. Indeed γ_t is given by

$$\gamma_t = \frac{S_t}{N_1(t)} \in [0,1] \ : \begin{cases} S_t \overset{i.i.d}{\to} U([0, N_1(t)]) \\ N_1(t) = N\phi(t) + \sqrt{N}\epsilon(t) \text{ NSF-ME solution} \end{cases} \tag{174}$$

As already discussed in section 2.2, the stochastic coefficients are not involved in time derivatives since, as shown by eq. (174), S_t is exogenous while $N_1(t)$ is known at each point in time: time is just an indexing parameter. Therefore, by using eq. (153) the general solution of eq. (173) reads as

$$\phi_l(t) = (\phi_l^0 - \phi_l^*)\exp\left\{-\rho_{l,t}\left[\lambda_{l,t} + \mu_{l,t}\right]t\right\} + \phi_l^* \tag{175}$$

where

$$\dot{\phi}_l = 0 \Rightarrow \phi_l^* = \left[1 + \frac{\mu_{l,t}}{\lambda_{l,t}}\right]^{-1} \tag{176}$$

is the equilibrium solution.

3.4 Fokker-Planck equation: stationary and general solution

The second equation of system (165) is

$$\partial_t Q_t(\epsilon_l) = -\rho_{l,t}\Delta_t'(\phi_l)\partial_{\epsilon_l}(\epsilon_l Q_t(\epsilon_l)) + \frac{\rho_{l,t}^2}{2}\Sigma_t(\phi_l)\partial_{\epsilon_l}^2 Q_t(\epsilon_l) \tag{177}$$

which is a FP equation for the density $Q_t(\epsilon_l)$ of spreading fluctuations about the drifting trajectory ϕ_l of eq. (175). The FP equation (177) is a second order stochastic partial differential equation of parabolic type and, according to eq. (172), it involves a drift coefficient which reads as

$$\Delta_t'(\phi_l) = -\left(\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} + \iota e^{\pi_1^2\gamma_t\phi_l^0/k_l}\right) \tag{178}$$

and a diffusion coefficient which, according to eq. (171), reads as

$$\Sigma_t(\phi_l) = \zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} + \iota e^{\pi_1^2\gamma_t\phi_l^0/k_l} - \left(\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \iota e^{\pi_1^2\gamma_t\phi_l^0/k_l}\right)\phi_l \tag{179}$$

The drift coefficient is therefore constant with respect to ϕ_l while the diffusion coefficient depends on ϕ_l.
As in section 1.4.2, the stationary solution takes the form of eq. (62)

$$\begin{cases} Q(\epsilon_l) = K\exp\left\{-\frac{\epsilon_l^2}{\rho_{l,t}}R(\phi_l)\right\} \; s.t. \\ K = \sqrt{\frac{R(\phi_l)}{\pi}} \quad : \quad R(\phi_l) = -\frac{\Delta'(\phi_l)}{\rho_{l,t}\Sigma(\phi_l)} > 0 \end{cases} \tag{180}$$

where $\pi \cong 3.14$, π_1 is the firm-bank matching probability. The general solution follows the procedure detailed in section 1.4.3. As known, in order for the model to make sense, the condition $R(\phi_l) > 0$ is sufficient, therefore we need to determine the values of the involved quantities for which it holds true. Since

$$\begin{cases} R(\phi_l^*) = -\frac{\Delta_t'(\phi_l)}{\rho_{l,t}\Sigma_t(\phi_l)} \\ \frac{\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} + \iota e^{\pi_1^2\gamma_t\phi_l^0/k_l}}{\rho_{l,t}\left\{\zeta e^{\pi_1(1-\gamma_t)} - \left(\zeta e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} - \iota e^{\pi_1^2\gamma_t\phi_l^0/k_l}\right)\left(1+\frac{\iota}{\zeta}e^{-\pi_1^2(1-2\gamma_t)\phi_l^0/k_l}\right)^{-1}\right\}} > 0 \end{cases} \tag{181}$$

The condition is satisfied if and only if the denominator is positive: after some algebra, it is possible to show that this condition is always satisfied

$$R(\phi_l) > 0 \;\forall\; \zeta, \iota, \pi_1, \gamma_t, \rho_{l,t} \in (0, 1] \tag{182}$$

According to eq. (182), the model makes sense if a giant component of at least two nodes exists and link creation/destruction rates are positive, indeed π_1 and $\rho_{l,t}$ are strictly positive.
In order to develop the general solution we follow section 1.4.3. According to eq.s (63)-(66), the FP equation (177) reads as

$$\partial_t Q_t(\epsilon_l) = -\rho_{l,t}\Delta_t'(\phi_l)Q_t(\epsilon_l) - \Delta_t'\epsilon_l Q_t'(\epsilon_l) + \frac{\rho_{l,t}^2}{2}\Sigma_t(\phi_l)Q_t''(\epsilon_l) \tag{183}$$

By using eq. (63) with $m = \phi_l$ and $s = \epsilon_l$ we get

$$\left\{ \begin{array}{l} \partial_t \langle \epsilon_l \rangle = d_0 \langle \epsilon_l \rangle \quad : \quad d_0 = \rho_{l,t} \Delta_t'(\phi_l) \\ \partial_t \langle \epsilon_l^2 \rangle = 2d_0 \langle \epsilon_l^2 \rangle + d_1 \quad : \quad d_1 = \rho_{l,t}^2 \Sigma(\phi_l) \end{array} \right. \tag{184}$$

as done in eq. (75) and which provides the dynamic system of the first two moments of random fluctuations about the drifting trajectory. Therefore we have

$$\left\{ \begin{array}{l} \mu_{\epsilon_l}(t) = \langle \epsilon_l^0 \rangle \exp(-\rho_{l,t} \Delta_t'(\phi_l) t) \\ \sigma_{\epsilon_l}^2(t) = \langle \epsilon_l^{*2} \rangle \left[1 - \exp(2\rho_{l,t} \Delta_t'(\phi_l))t\right] \quad : \quad \langle \epsilon_l^{*2} \rangle = -\frac{\rho_{l,t} \Sigma_t(\phi_l)}{2\Delta_t'(\phi_l)} \end{array} \right. \tag{185}$$

which leads us to the general solution

$$Q_t(\epsilon_l) = \frac{1}{\sqrt{2\pi \sigma_{\epsilon_l}^2(t)}} \exp\left[-\frac{(\epsilon_l - \mu_{\epsilon_l}(t))^2}{2\sigma_{\epsilon_l}^2(t)} \right] \; : \; \mu_{\epsilon_l}(t) = 0; \text{due to eq. (150)} \tag{186}$$

as implied by (80).

3.5 K stochastic dynamics

Now we have all we need to write the final inferential result. Following section 1.4.4 we can easily provide the final result for the quantity $n_l(t)$ in eq. (147), according to the phenomenological model in eq. (148), by specifying all the needed component of eq. (81) including the results found concerning the last macroscopic and FP equations.

$$
\begin{cases}
n_l(t) = \frac{N^l(t)}{N_1(t)} = K_l\phi_l(t) + \sqrt{K_l}\epsilon_l(t) \in (0,1) \;\; s.t. \\
N_1(t) \;\; \text{solves the NSF-ME} \\
\phi_l(t) = (\phi_l^0 - \phi_l^*) \exp\left[-\rho_{l,t}\left(\lambda_{l,t} + \mu_{l,t}\right)t\right] + \phi_l^* \in (0,k_l) \;\; : k_l K_l = 1 \text{ with} \\
\quad \phi_l^0 \in (0,k_l) \quad \phi_l^* = \left[1 + \frac{\mu_{l,t}}{\lambda_{l,t}}\right]^{-1} \in (0,k_l) \quad \text{and} \\
\epsilon_l(t) \overset{i.i.d}{\to} \mathcal{N}(\mu_{\epsilon_l}(t), \sigma^2_{\epsilon_l}(t)) \quad \text{where} \\
\\
\begin{cases}
\mu_{\epsilon_l}(t) = \langle \epsilon_l^0 \rangle \exp(-\rho_{l,t}\Delta_t'(\phi_l)t) \;\; : \;\; \langle \epsilon_l^0 \rangle = 0 \\
\sigma^2_{\epsilon_l}(t) = \langle \epsilon_l^{*2} \rangle \left[1 - \exp(2\rho_{l,t}\Delta_t'(\phi_l))t\right] \quad : \quad \langle \epsilon_l^{*2} \rangle = -\frac{\rho_{l,t}}{2}\frac{\Sigma_t(\phi_l)}{\Delta_t'(\phi_l)} \\
Q_t(\epsilon_l) = \frac{1}{\sqrt{2\pi\sigma^2_{\epsilon_l}(t)}} \exp\left[-\frac{(\epsilon_l - \mu_{\epsilon_l}(t))^2}{2\sigma_{\epsilon_l}(t)}\right] \quad \text{and}
\end{cases} \\
\\
\begin{cases}
\Delta_t'(\phi_l) = \partial_{\phi_l}\left(\beta_t(\phi_l) - \delta_t(\phi_l)\right) \\
\Sigma_t(\phi_l) = \beta_t(\phi_l) + \delta_t(\phi_l) \;\; \text{being}
\end{cases} \\
\\
\begin{cases}
\beta_t(\phi_l) = \lambda_{l,t}(1-\phi_l) \\
\delta_t(\phi_l) = \mu_{l,t}\phi_l \;\; \text{where}
\end{cases} \\
\\
\begin{cases}
\lambda_{l,t} = \zeta\psi_{1,l,t} \;\; : \;\; \psi_{1,l,t} = e^{\pi_1^2(1-\gamma_t)\phi_l^0/k_l} \\
\mu_{l,t} = \iota\psi_{0,l,t} \;\; : \;\; \psi_{0,l,t} = e^{\pi_1^2\gamma_t\phi_l^0/k_l} \;\; \text{where}
\end{cases} \\
\\
\gamma_t = \frac{S_t}{N_1(t)} \in [0,1] \;\; : \begin{cases}
S_t \overset{i.i.d}{\to} U([0, N_1(t)]) \\
N_1(t) = N\phi(t) + \sqrt{N}\epsilon(t) \text{ NSF-ME solution}
\end{cases} \\
\\
\text{being} \;\; \rho_{l,t} = \sqrt{\left(\frac{k_l}{N_1(t)}\right)^3}.
\end{cases}
\tag{187}
$$

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