

Market Structure, Scale Economies and Industry Performance*

Rabah Amir[†]

This version: Sept. 2002

Abstract

We provide an extensive and general investigation of the effects on industry performance – profits, social welfare and price-cost margins – of exogenously changing the number of firms in Cournot markets. This amounts to an in-depth exploration of the well-known trade-off between competition and production efficiency. Most conventional beliefs actually require some qualifications to be valid. Under scale economies, welfare is maximized by a finite number of firms. Our results shed light on several policy debates in industrial organization, including the relationship between the Herfindahl index and social welfare, destructive competition and natural monopoly. Our analytical approach combines simplicity with generality.

Key words and phrases: Cournot oligopoly, returns to scale, entry, equilibrium comparative statics.

JEL codes: D43, D60, L13, L40.

*The author is grateful to Val Lambson for several helpful conversations on the topic of this paper. This work has also benefitted from comments from Stephen Martin, William Novshek, John Roberts, Jean Tirole, as well as feedback from seminar audiences at CORE, WZB-Berlin, and the Universities of Aarhus, Arizona, Bergen, U.L.Brussels, Cergy-Pontoise, Copenhagen, Helsinki, Manchester, Odense, Strasbourg and Stony-Brook.

[†]CORE, 34 voie du Roman Pays, 1348 Louvain-la-neuve, Belgium. (E-mail: amir@core.ucl.ac.be)

1. Introduction

The bulk of the theory of markets, both in its general and its partial equilibrium formulations, has been developed under the widespread presumption of decreasing or constant returns to scale in production. Yet few economists would dispute the prevalence of scale economies in many real-life industries, encompassing old economy sectors such as transportation, manufacturing and public utilities all the way to many new economy sectors such as software and internet-related business. In fact, most economists would probably support the view that the importance of scale economies in the overall economy is likely to continue to grow in the years to come. The failure of economic theory to integrate this key dimension of production processes to an adequate extent is thus probably due to methodological bias or limitations, rather than to a perceived lack of real-life relevance.

While partial equilibrium theory has been more successful in integrating increasing returns, the associated literature nevertheless reflects quite a few limitations. Most studies dealing with increasing returns belong to the literature on natural monopoly and contestable markets, a theory with rather limited real-life relevance (Spence, 1983). This literature preceded the game-theoretic paradigm in industrial organization. Within the latter, most of the research on Cournot and Bertrand competition assumes nonincreasing returns to scale. In both these two paradigms, scale economies typically appear in the form of a fixed-cost in production¹, and not in the form of declining marginal cost. In short, the present paper is an attempt to partially fill this gap, by providing a thorough and general theoretical analysis of Cournot markets under a varying number of firms and different types of returns to scale in production, including declining marginal cost. The prevalence of the latter in manufacturing industries has been documented in several empirical studies.²

There has been a rich and insightful debate in industrial organization about the welfare (Bain, 1956) and profitability (Mueller, 1986) effects of increasing the number of firms in an industry. While many empirical aspects of this fundamental issue remain partly unsettled, the basic compromise at work is well-known, though still a source of major controversy both among academics and antitrust practitioners. On the one hand, conventional intuition –

¹ This is probably due to the relative ease of dealing with this form of scale economies via the standard methodology using first-order conditions.

² See e.g. Friedlaender et al (1983) for automobile production and Ramey (1989) for different industries (metals, machinery and electrical equipment). Diewert and Wales (1987) estimate a cost function for manufacturing that turns out to be concave at all sample points. Ramey (1991) invokes declining marginal cost as a possible explanation for the excess volatility of production relative to sales, observed in many industries.

sometimes wrongly – holds that increasing the number of firms reduces monopoly power and allows closer approximation of the competitive ideal. On the other hand, increasing the number of firms may result in reduced ability to take advantage of scale economies.³

The welfare implications of market structure have been prominent in the early beginning of the field. The unquestioned view then was that barriers to entry are responsible for the presence of imperfect competition, which in turn results in sizable welfare losses. The belief that public policy must correct for this imperfection by removing barriers to entry and possibly subsidizing entry had dominated the profession and persisted until quite recently. Perceptive work by von Weiszacker (1980), Perry (1984), Mankiw and Whinston (1986) and Suzumura and Kiyono (1987) demonstrated that this view was fundamentally ill-founded by proving that free entry results in an excessive number of firms relative to a social optimum.

The present paper offers a thorough theoretical investigation of the effects on industry profits, social welfare and price-cost margins of *exogenously* increasing the number of firms in a Cournot industry composed of identical firms under different assumptions on the cost function. The analysis is both simple and rigorous, and fully takes into account the issues of existence and multiplicity of (pure-strategy) Cournot equilibria under the same sets of tight assumptions that allow for a fairly complete characterization of the properties of these equilibria. An extensive older literature deals with the effects of the number of firms on industry price and output levels, e.g. Ruffin (1971), Novshek (1980) and Seade (1980)⁴. Yet, to the best of our knowledge, no systematic theoretical analysis of industry profits, social welfare or price-cost margins has been conducted with an exogenous number of firms.

An important aspect of the paper, from a methodological viewpoint, is its reliance on the new lattice-theoretic comparative statics approach. We build directly on Amir and Lambson (2000) who use this same framework to analyse price and output effects. They derive two main results. The first one is that industry price falls (increases) with the number of firms if a firm's residual inverse demand declines faster (slower) than its marginal cost, globally.⁵

This is the so-called property of quasi-competitiveness (quasi-anticompetitiveness). Strong

³ Once the existence of the trade-off between the benefits of competition and the cost-savings of production scale is established, the main issues become largely empirical. Indeed, the empirical literature on the subject is extensive, though outdated by now. Several studies in Goldschmid, Mann and Weston (1974) provide empirical evidence and a general debate on this controversial issue.

⁴ See also Hahn (1962), McManus (1962), Frank (1973), and Okuguchi (1973) among others.

⁵ This sharp intuition behind our conclusions is only possible with the lattice-theoretic approach. In previous work, other method-imposed assumptions, such as concave profits and decreasing marginal revenue, prevented such simple and clear-cut economic interpretations based only on critical assumptions.

scale economies are required for demand to decline slower than marginal cost, and lead to the counterintuitive outcome on price. The second result is that per-firm profit falls with the number of firms in both cases.

For industry performance comparisons (Section 4), the quasi-competitive case has two clear-cut welfare results that are independent of the nature of returns to scale: welfare increases in the number of firms whenever per-firm output does, and in case of multiple equilibria, the maximal output equilibrium is the social-welfare maximizing equilibrium. The latter result vindicates the supremacy of consumer welfare over producer welfare. Otherwise, this case gives rise to two subcases, depending on the returns to scale in production. In the presence of economies of scale, industry profits are shown to globally decline with the number of firms, while social welfare is generally not monotonic. More precisely, we argue that the slightest amount of scale economies leads to welfare being decreasing at sufficiently high numbers of firms. Inversely, under diseconomies of scale, social welfare is globally increasing in the number of firms, while industry profits exhibit a tendency to initially increase in the number of firms (treated as a real number), starting at monopoly level. (Whether this tendency leads to duopoly, say, having higher total profit than monopoly depends on the magnitude of the returns). As an important corollary of the two monotonicity results above, under constant returns to scale, both conventional beliefs indeed hold: Industry profits fall and social welfare increases with the number of firms. As to price-cost margins, they always decrease in the number of firms⁶ except when demand is strictly concave and costs strictly convex, in which case it is shown by counter-example that price-cost margins may actually increase with more firms in the market.

For the quasi-anticompetitive case, monopoly always leads to the highest possible industry profits, with this being the only clear result on industry profits. On the other hand, social welfare and price-cost margins unambiguously decrease in the number of firms. In this case, there are strong enough scale economies to overcome all other considerations.

A complementary methodological feature is our reliance on tight illustrative examples. These serve a dual purpose. First, they confirm that the given sufficient conditions are, in some sense, critical to the resulting conclusions. Second, they illustrate in a more accessible manner the interaction between the various effects at work in the comparative statics at hand.

⁶ There is extensive empirical evidence spanning a long period of time, essentially confirming the intuitive behavior of price-cost margins: See Bresnahan (1989) for an account.

In particular, we present a Main Example (Section 3) which is a blueprint for the entire paper, in that most effects of interest can be captured by varying the Example parameters. This example can serve as a pedagogical tool to convey the main ideas of the paper in a simple and intuitive framework to undergraduate students or policy practitioners.

Our conclusions provide a precise theoretical foundation for intuitive beliefs about the need for a trade-off between the benefits of fostering increased competition and the ability of firms to exploit scale economies. The conclusions shed some light on some classical results in partial equilibrium analysis. It is well-known (Ruffin, 1973) that, under quasi-competitiveness, Cournot equilibria converge to perfectly competitive equilibria when average cost is nondecreasing, and that this same convergence fails when average cost is nonincreasing. Our results show that welfare is monotonically increasing in the former case (thus converging to first-best welfare), but not in the latter.

The results presented here may also be invoked to illuminate a number of important public policy debates: See Section 5 where selected important applications of our results are presented. The first of these addresses the extensive use made by antitrust authorities of the Herfindahl-Hirschman index of concentration under the presumption that the Index is a good inverse measure of social welfare. Recent theoretical work showed that with a constant number of firms, any output transfers across firms that leave price unchanged must cause the Index and social welfare to move in the same direction, Farrell and Shapiro (1990) and Salant and Shaffer (1999). We complement this insight by the observation that with scale economies – no matter how small – both the Index and welfare decrease with the number of firms, when the latter is larger than a threshold level, which may be equal to one.

The next point identifies the Cournot model in the quasi-anticompetitive case as an appropriate framework for modelling the old concept of destructive competition. Indeed, there is an excellent match between the theoretical predictions of the model in that case and the stylised facts commonly associated with destructive competition (Sharkey, 1982).

The final point proposes to define natural (unregulated) monopoly as an industry where the socially optimal number of firms is one, as opposed to the old definition of (regulated) natural monopoly based on the inability to improve on costs by subdividing production, a purely production-based criterion. This new definition clearly balances the market and production sides, and is more appropriate in the absence of regulation and contestability.

All these applications emphasize the role of scale economies in engendering a trade-off

between the market effect and the production efficiency effect. They convey the sense that our simple results form a pre-requisite for a thorough understanding of the issues at hand.

2. The Model

The fundamental industry performance issues under investigation can be described as follows: How do industry profit, social welfare and price-cost margins vary with the exogenously given number of firms in the industry? We consider these fundamental questions in the framework of equilibrium comparative statics, the exogenous parameter being the integer number of firms. We begin with the basic notation.

Let $P : R^+ \rightarrow R^+$ be the inverse demand function, $C : R^+ \rightarrow R^+$ the (common) cost function, $A : R^+ \rightarrow R^+$ the average cost function, and n the number of firms in the industry. Let x denote the output of the firm under consideration, y the total output for the other $(n - 1)$ firms, and z the cumulative industry output, i.e., $z = x + y$. At equilibrium, these quantities will be indexed by the underlying number of firms n . We explicitly deal with the (possible) nonuniqueness of Cournot equilibria by considering extremal equilibria. Denote the maximal and minimal points of any equilibrium set by an upper and a lower bar, respectively. Thus, for instance, \bar{z}_n and \underline{z}_n are the highest and lowest total equilibrium outputs, with corresponding equilibrium prices \underline{p}_n and \bar{p}_n , respectively. Performing comparative statics on equilibrium sets will consist of predicting the direction of change of these extremal elements as the parameter n varies.

The profit function of the firm under consideration is

$$\Pi(x, y) = xP(x + y) - C(x).$$

Alternatively, one may think of the firm as choosing total output $z = x + y$, given the other firms' cumulative output y , in which case its profit is given by

$$\tilde{\Pi}(z, y) = \Pi(z - y, y) = (z - y)P(z) - C(z - y).$$

Let $\Delta(z, y)$ denote the cross-partial derivative of $\tilde{\Pi}$ with respect to z and y , i.e.,

$$\Delta(z, y) \triangleq -P'(z) + C''(z - y).$$

Both $\tilde{\Pi}$ and Δ are defined on (the lattice) $\varphi \triangleq \{(z, y) : y \geq 0, z \geq y\}$. A firm's reaction correspondence is defined by $r(y) \triangleq \arg \max \{\Pi(x, y) : x \geq 0\}$. The Marshallian social welfare

when each of n firms produces x is $V_n(x) \triangleq \int_0^{nx} P(t)dt - nC(x)$. Equilibrium per-firm profit and social welfare will be denoted respectively by (the sets) π_n and W_n .

The following **Standard Assumptions** are in effect throughout the paper:

(A1) $P(\cdot)$ is continuously differentiable and $P'(\cdot) < 0$.

(A2) $C(\cdot)$ is twice continuously differentiable on $(0, \infty)$ and $C'(\cdot) > 0$.

(A3) There exists $\hat{x} > 0$ such that $P(x) \leq A(x)$ for all $x \geq \hat{x}$.

Although convenient, the smoothness assumptions are by no means necessary for our main results. (A3) simply guarantees that that a firm's reaction curve eventually coincides with the horizontal axis, so that a firm's effective outputs, and thus all Cournot equilibrium outputs, are bounded by some constant, say K , for all n .

The qualitative nature of most of our results hinges entirely on the global sign of Δ , so that we will distinguish two main cases: $\Delta > 0$ and $\Delta < 0$. When $\Delta > 0$ globally, there will be two subcases of interest depending on the returns to scale. This division is already apparent in the upcoming example, which may serve as a blueprint for the entire paper.

3. Returns to Scale, Concentration and Industry Performance: The Main Example

We now consider a simple example that provides an excellent and thorough overview of most of the results derived in this paper.⁷ As a parameter capturing the returns to scale is varied, the example can fit the two major cases of analysis of the general model: $\Delta < 0$ and $\Delta > 0$. In the latter case, the example can also capture the two subcases of interest: economies or diseconomies of scale. In addition, this example will also be invoked later on to gain further insight into the tightness of the conditions behind our general results.

Let the inverse demand be linear and the cost function be quadratic, i.e.,

$$P(z) = a - bz \quad \text{and} \quad C(x) = cx + dx^2$$

with the assumptions throughout that $a > c > 0$, $b > 0$, $b + d > 0$ and $ad + bc > 0$ (these guarantee marginal cost and cost are positive at all per-firm Cournot equilibrium outputs.)

Since $A(x) = c + dx$, returns to scale are increasing (decreasing) if $d < (>)0$. Thus d is

⁷ This example would be very appropriate for the purpose of presenting in a very elementary framework the essentials of the analysis to undergraduate students or to economic practitioners.

our returns to scale parameter, key to many results below. Furthermore

$$\Delta = -P'(z) + C''(z - y) = b + 2d \gtrless 0 \text{ if } d \gtrless -b/2$$

The (linear) reaction function is $r(y) = \frac{a-c-by}{2(b+d)}$. Thus for any number of firms n there is always a unique symmetric Cournot equilibrium. Omitting some lengthy calculations (including solving for the symmetric Cournot equilibrium via $r[(n-1)x_n] = x_n$), this equilibrium has per-firm output and profit, and social welfare given respectively by

$$x_n = \frac{a-c}{b(n+1)+2d}, \pi_n = \frac{(b+d)(a-c)^2}{[b(n+1)+2d]^2}, W_n = \frac{n[b(n+2)+2d](a-c)^2}{2[b(n+1)+2d]^2}$$

Furthermore, if $d > -b/2$ (or $\Delta > 0$), the slope of the reaction curve is larger than -1 and the symmetric equilibrium is the unique equilibrium. It is also globally stable in the sense that best-reply Cournot dynamics converges to this equilibrium, from any initial outputs.

On the other hand, if $d < -b/2$ (or $\Delta < 0$), the slope of the reaction curve is smaller than -1 , so that $r(y)$ decreases steeply and is equal to 0 when $y \geq (a-c)/b$. Consequently, there are several other Cournot equilibria, all of which can be characterized as follows. With n being the total number of firms in the industry, if any m firms (with $m < n$) produce the output x_m each, and the remaining $n-m$ firms produce nothing, the resulting output configuration is clearly a Cournot equilibrium⁸. For the n -firm oligopoly, the unique symmetric equilibrium (with all firms active) is unstable in the sense that best-reply Cournot dynamics diverge away from it (Seade, 1980).

It can be verified via simple calculation that

- (i) per-firm output x_n is always decreasing in n (cf. Proposition 1b).
- (ii) industry output $z_n = nx_n$ is increasing in n if $d > -b/2$ (or $\Delta > 0$) and decreasing in n if $d < -b/2$ (or $\Delta < 0$) (cf. Propositions 1a and 2a)⁹.
- (iii) per-firm profit π_n is always decreasing in n (cf. Propositions 1d and 2a).
- (iv) price-cost margins $m_n \triangleq P(z_n) - C'(x_n)$ decrease in n (cf. Propositions 9 and 13.)

It remains to analyse the effects of n on industry profits and social welfare. It is convenient

⁸ To see this, observe that $r(mx_m) = 0$, since $mx_m \geq (a-c)/b$, as can be easily checked. In particular, if any one firm produces the optimal monopoly output $x_1 = (a-c)/(b+2d)$, and all the others produce nothing, we have a Cournot equilibrium. This follows from $r(x_1) = 0$ since $x_1 \geq (a-c)/b$ and $x_1 = r(0)$. Given the linearity of the reaction curve, this is easy to see graphically (see d'Aspremont et al, 2000)

⁹ The intuition behind the counter-intuitive case $\Delta < 0$ is that with more competition, each firm lowers output drastically since $r'(y) < -1$, thereby moving up its steeply declining average cost curve. The resulting efficiency loss is large enough to overcome the downward pressure on price engendered by the increase in competition. The increase in average cost is passed on to consumers via a higher price.

here to treat n as a real variable. For industry profits, we have (with details left out)

$$\frac{\partial(n\pi_n)}{\partial n} \begin{matrix} \geq \\ < \end{matrix} 0 \text{ if and only if } n \begin{matrix} \leq \\ \geq \end{matrix} 1 + 2d/b. \quad (1)$$

Here, there are two separate cases of interest:

(i) $d < 0$: Then industry profits always decrease with the number of firms, with in particular monopoly having the largest industry profit (cf. Proposition 3.)

(ii) $d > 0$: Then $\tilde{n} = 1 + 2d/b$ maximizes industry profits, which thus increase in n when $n < \tilde{n}$ and decrease in n when $n > \tilde{n}$, starting from any given n . Observe that if $d/b < 1/2$, then $1 < \tilde{n} < 2$. Hence, in particular, if $\tilde{n} = 1$ (i.e. monopoly is the market structure that maximizes total profits), industry profits would be globally decreasing in n . But if d is large enough, i.e., if there are sufficiently high diseconomies of scale, industry profits will be rising in n initially, all the way to \tilde{n} which may be a large number of firms, but industry profits always eventually decrease in n , i.e. for $n > \tilde{n}$ (cf. Proposition 4).

For social welfare, one can easily verify that

$$\frac{\partial W_n}{\partial n} \begin{matrix} \geq \\ < \end{matrix} 0 \text{ if and only if } nbd \begin{matrix} \geq \\ < \end{matrix} -(b+d)(b+2d). \quad (2)$$

Again, there are two separate cases of interest:

(a) $\Delta < 0$ iff $d < -b/2$: Welfare always decreases with n (cf. Proposition 11.)

(b) $\Delta > 0$ iff $d > -b/2$: Here, there are two different subcases of interest.

(i) $d > 0$: Welfare always increases in n (cf. Proposition 6.)

(ii) $-b/2 < d < 0$: Welfare is maximized at $n^* = -(b+d)(b+2d)/bd$, increases in n for $n < n^*$ and decreases in n for $n > n^*$ (cf. Proposition 8). Observe that this statement is true no matter how close d is to 0 (from below)! In other words, the slightest presence of scale economies causes welfare to be eventually declining in n (i.e. for sufficiently large values of n). The parameters of this example can be chosen to make n^* equal any desired value from 1 on, while satisfying all the underlying constraints here. Thus, in particular, monopoly is the welfare-maximizing market structure if $bd \leq -(b+d)(b+2d)$, with the latter condition always holding when $d < -b/2$ (or $\Delta < 0$), and never holding if $d > 0$.

The overall economic intuition can now be stated concisely and precisely since the main results hinge mainly on the sign of $\Delta = -P' + C''$, and sometimes also on the returns to scale. For industry price, there are two effects at work, a market or competition effect captured by the term $-P'$, and a production or scale effect captured by C'' . The market

effect always pushes in the intuitive direction that price falls with the number of firms. The scale effect goes in the same direction if and only if costs are convex. When costs are concave, the overall outcome on price is determined by the relative strengths of the two conflicting effects. Per-firm profit always behave in the intuitive way.

For industry profits, the market effect pushes in the intuitive direction if and only if industry price is well-behaved (i.e. $\Delta > 0$). The production effect works in the intuitive direction if scale economies are present. When the two effects are antagonistic, the outcome depends on the relative strengths again.

Viewed as the sum of consumer and producer surpluses, social welfare can be discussed on the basis of the previous assessments. Thus, with $\Delta > 0$ and diseconomies of scale, consumer surplus increases with n , overcoming a possible decrease in producer surplus (the latter effect being ambiguous). With $\Delta < 0$, strong scale economies are necessarily present, and both surpluses decrease with n . Finally, with $\Delta > 0$ and economies of scale, consumer surplus moves up and producer surplus down, with an ambiguous net effect.

In conclusion, this example provides a microcosm for the entire paper. In the remainder, we present a generalization of the insights illustrated so far, preserving another key role for this example in testing the tightness of the sufficient conditions given for our various results.

4. A General Cournot Analysis of Industry Performance

This section contains the general analysis of the interplay between market structure and returns to scale in determining industry performance as reflected in total profits, social welfare and price-cost margins. This amounts to comparing Cournot equilibria along these characteristics as the number of firms varies. In an attempt to gain the broadest possible understanding of the issues involved, we provide a series of minimally sufficient conditions for our conclusions, combined with tight complementary examples to shed further light on the relationship between assumptions and conclusions. The proofs combine analytical simplicity with generality. Methodologically, we make use of the lattice-theoretic comparative statics approach.¹⁰ This allows for very general conclusions relying only on critical assumptions, thereby leading to clean and tight economic interpretations of the conclusions, as well as analytical rigor. In the present context, the usual arguments in favor of this approach

¹⁰We invoke the general results of Topkis (1978), Vives (1990) and Milgrom and Roberts (1990, 1994). For the stochastic case, see Athey (2002). More specific to Cournot oligopoly, we build on the results of Amir (1996) and Amir and Lambson (2000). See also McManus (1962) and Roberts and Sonnenschein (1976).

become even more pertinent, as the parameter of interest, the number of firms, is an integer: See Example 4 for a key point on this.¹¹

While the condition $\Delta > 0$ is familiar in Cournot theory at least since Hahn (1962), it has typically been used in conjunction with many other assumptions, such as some form of concavity of each firm's profit in own output, decreasing marginal revenue, etc...The latter assumptions interfere with a good intuitive understanding of the economic forces at work, as they are also made for the case $\Delta < 0$. As shown below, there is a very natural division of the results here, and it depends *only* on the global sign of Δ . The latter has a very simple and appealing interpretation: $\Delta > 0$ ($\Delta < 0$) means that price, or residual inverse demand, decreases faster (slower) than marginal cost. Since $P' < 0$, it is clear that the convexity of C implies $\Delta > 0$ on φ . Likewise, strong concavity of C is required for $\Delta < 0$. As seen in the Main Example (when $-b/2 < d < 0$), $\Delta > 0$ can hold globally even when the cost function is everywhere concave, an important subcase of analysis in this paper.

4.1 Equilibrium Price and Outputs

The results of this subsection have been proved in Amir and Lambson (2000). They are stated here without proof, and used in the sequel in looking at industry profit, social welfare and price-cost margins. In the Appendix, a graphical illustration of the need for the new comparative statics is presented, with the conclusion that only extremal equilibria can be unambiguously compared as the number of firms varies.¹² For any variable of interest, the maximal (minimal) value will always be denoted by an upper (lower) bar.

Proposition 1 *Let $\Delta(z, y) > 0$ on φ . For any $y_1 \geq y_2, r_1 \in r(y_1), r_2 \in r(y_2)$, there holds $r_1 - r_2 \geq y_1 - y_2$. For each n , there exists at least one symmetric equilibrium and no asymmetric equilibria. For the extremal equilibria,*

- (a) *Industry output z_n is nondecreasing in n , and hence price p_n is nonincreasing in n .*
- (b) *x_n is nonincreasing [nondecreasing] in n if $\log P$ is concave [convex and $C(\cdot) \equiv 0$].*
- (c) *The total rivals' output $y_n \triangleq (n - 1)x_n$ is nondecreasing in n .*
- (d) *The corresponding equilibrium profit π_n is nonincreasing in n .*

Thus the Cournot model is quasi-competitive here (Part (a)). The fundamentally needed assumption is the supermodularity of $\tilde{\Pi}$ on φ , which is equivalent to $\partial^2 \tilde{\Pi} / \partial z \partial y = \Delta > 0$.

¹¹The traditional method based on the Implicit Function Theorem can provide insight for special cases, but is ill-suited for a general analysis. The main reason is that it rests on superfluous assumptions (such as concavity and equilibrium uniqueness) that are needed in all the otherwise mutually exclusive cases of analysis, thus preventing a clear intuition behind the results. Also, in the context at hand, as the parameter of interest is an integer, there are other serious drawbacks: See Example 4 here and de Meza (1985).

¹²For unstable equilibria (in the sense of best-reply Cournot dynamics), the price comparative statics is counter-intuitive (as seen in the Appendix), and this will carry through to other results.

This implies that the line segment joining any two points on the graph of the reaction correspondence r of a firm must have a slope ≥ -1 , so that, in response to an increase in rivals' output, a firm can never contract its output by more than this increase. In particular, this precludes downward jumps for r (while allowing for upward jumps).¹³

The other case is characterized by the assumption $\Delta(z, y) < 0$, implying that r has all its slopes ≤ -1 : As rivals' joint output is increased, a firm optimally reacts by contracting its output so much that the resulting total output decreases. Hence, r is strongly decreasing, so that the results of Novshek (1985) and Amir (1996) guarantee the existence of a Cournot equilibrium.¹⁴ However, existence of a symmetric equilibrium is not guaranteed for all n without the additional (and otherwise unnecessary) assumption of quasi-concavity of $\Pi(\cdot, y)$.

Proposition 2 *Assume $\Delta(z, y) < 0$ on φ and quasi-concavity of $\Pi(\cdot, y)$ for every y . Then, for any $y_1 \geq y_2, r_1 \in r(y_1), r_2 \in r(y_2)$, there holds $r_1 - r_2 \geq y_1 - y_2$. Furthermore,*

- (a) *There is a unique symmetric equilibrium, and it satisfies: x_n, z_n and π_n are nonincreasing in n . Hence p_n is nondecreasing in n . Rivals' output $y_n \triangleq (n - 1)x_n$ is nondecreasing in n .*
- (b) *For any m such that $1 \leq m < n$, the following is an equilibrium for the n -firm oligopoly: Each of any m firms produces x_m while the remaining $(n - m)$ firms produce nothing. All these Cournot equilibria are invariant in n , in that all entering firms would produce zero.*
- (c) *No other Cournot equilibrium (than those of Parts (a) and (b)) can exist.*

Due to the fact that asymmetric equilibria involve a symmetric equilibrium among the active subset of firms and to their invariance in n , attention will be limited throughout w.l.o.g. to symmetric equilibria for the case $\Delta < 0$.

When quasi-concavity of the profit function in own output is needed, it is desirable to derive it from properties of the primitives of the model. We show in the Appendix that a sufficient condition is that $[P(x + y) - A(x)]^{-1}$ is convex in x for each fixed $y \geq 0$.

4.2 Industry Profits, Social Welfare and Price-Cost Margins

Here, the effects of an exogenous change in the number of firms on industry profits, social welfare and price-cost margins are investigated. While not standard in industrial organization, our proofs combine *generality* with *simplicity*. As before, we continue to focus on the two extremal Cournot equilibria for all our results, and to separate the analysis of our comparative-equilibria results into two cases, according to the global sign of Δ .

¹³This property was used by McManus (1962) and Roberts and Sonnenschein (1976) to establish the existence of symmetric Cournot equilibrium under convex costs. Also see Amir (1996) and Amir and Lambson (2000).

¹⁴However, there is no guarantee a priori that this includes more than the monopoly equilibrium (which always exists), as described in Part (b) of Proposition 2.

4.2.1 The Case $\Delta > 0$

Recall that $\Delta > 0$ globally is consistent with both globally increasing and decreasing returns. We begin with the effects on industry profits. Our first result essentially says that industry profit is globally nonincreasing in n under scale economies, defined by nonincreasing $A(\cdot)$.

Proposition 3 *Let $\Delta > 0$ on φ . For the extremal equilibria, $n\pi_n \geq (n+1)\pi_{n+1}$ for any given n if $A(\frac{n+1}{n}x_{n+1}) \leq A(x_{n+1})$.*

Proof. Let x_n be an extremal Cournot equilibrium per-firm output, and consider

$$\begin{aligned} \pi_n &= x_n \{P(nx_n) - A(x_n)\} \\ &\geq \frac{n+1}{n}x_{n+1} \{P[\frac{n+1}{n}x_{n+1} + (n-1)x_n] - A(\frac{n+1}{n}x_{n+1})\} \\ &\geq \frac{n+1}{n}x_{n+1} \{P[\frac{n+1}{n}x_{n+1} + (n-1)\frac{n+1}{n}x_{n+1}] - A(\frac{n+1}{n}x_{n+1})\} \\ &= \frac{n+1}{n}x_{n+1} \{P[(n+1)x_{n+1}] - A(\frac{n+1}{n}x_{n+1})\} \\ &\geq \frac{n+1}{n}x_{n+1} \{P[(n+1)x_{n+1}] - A(x_{n+1})\} \\ &= \frac{n+1}{n}\pi_{n+1} \end{aligned}$$

where the first inequality follows from the Cournot equilibrium property, the second from the facts that P is decreasing and $nx_n \leq (n+1)x_{n+1}$ since $\Delta > 0$, and the third from the assumption $A(\frac{n+1}{n}x_{n+1}) \leq A(x_{n+1})$. Multiplying across by n gives the conclusion. ■

Since per-firm profit π_n always falls with n , Proposition 3 asks whether π_n falls fast enough to have $n\pi_n \geq (n+1)\pi_{n+1}$. In interpreting the proposition, it is convenient to separate the overall effect of an increase in the number of firms on industry profits into two distinct parts, as suggested by the above proof. The market or total revenue effect, which may be isolated by setting $A' = 0$, always pushes in the intuitive direction that industry profits must fall. On the other hand, the production or efficiency effect goes in the same direction if and only if scale economies are present.¹⁵

The proof of the result also makes it clear that the conclusion follows when both the market and the production efficiency effects push in the same direction, which suggests the condition on average cost is sufficient but not necessary. It is natural then to ask how critical this assumption is for this conclusion. Treating the number of firms as a real number, the

¹⁵By contrast, an n -firm cartel always has higher optimal profit than the total n -firm oligopoly profit, since the cartel, with access to n plants, always has the option of producing nx_n at a cost at most equal to the total cost of the n -firm oligopoly. There is thus an obvious difference between a monopoly (with access to one plant) and a cartel composed of n identical firms. For a discussion of why a monopolist may not simply reproduce plants to get around diseconomies of scale, see e.g. Baumol, Panzar and Willig (1982).

following argument provides a simple but interesting insight: Monopoly is never the profit maximizing market structure under increasing average cost.

Proposition 4 *if $A'(\cdot) > 0$, industry profit increases from monopoly level as the number of firms is increased slightly beyond $n = 1$.*

Proof. With z denoting industry output, industry profit is given by $\Pi_n(z) = z[P(z) - A(z/n)]$. Since for fixed z , $\Pi_n(z)$ is increasing in n if $A' > 0$, the result follows from the envelope theorem¹⁶, as monopoly profit $\pi_1 = \max_z \Pi_1(z)$. ■

Observe that this need not mean that duopoly has higher profit than monopoly, as industry profit may peak between $n = 1$ and $n = 2$, with either π_1 or $2\pi_2$ as the highest value. This point is illustrated in the Main Example where, for $d < b/2$, industry profits may well be globally decreasing in the number of firms, and are certainly decreasing in n for $n \geq 2$ (see (1)). Nonetheless, the point made here is important as it shows that the slightest amount of decreasing returns pushes toward industry profits that are increasing in the number of firms. Whether this effect actually succeeds in preventing industry profit from being globally decreasing in the *integer* number of firms depends on the strength of the increasing returns, as suggested by the Main Example. Indeed, from (1), a sufficient condition for industry profit not to be globally decreasing in n is $d > -b/2$.

We now turn to the welfare analysis. It can easily be shown that \underline{x}_n is the Pareto-dominant equilibrium for the firms (i.e. leads to the largest producer surplus) while \bar{x}_n is the Pareto-preferred equilibrium for the consumers (i.e. leads to the largest consumer surplus). It is then of interest to know whether the Cournot equilibria are ranked according to the Marshallian measure of social welfare. In other words, is one of the two surpluses always dominant? The next result settles this question in favor of consumer surplus.

Proposition 5 *Let $\Delta > 0$, and x_n and x'_n denote two distinct equilibrium per-firm outputs with corresponding social welfare levels W_n and W'_n . If $x_n < x'_n$, then $W_n < W'_n$. Hence, \bar{x}_n is the social welfare maximizer among all equilibrium per-firm outputs.*

Proof. Since $\Delta > 0$ or $P'(z) - C''(z - y) < 0$ on $\varphi \triangleq \{(z, y) : y \geq 0, z \geq y\}$, the function $U_n(z) \triangleq \int_0^z P(t)dt - nC(z/n)$ is strictly concave in z , since $U_n''(z) = P'(z) - \frac{1}{n}C''(z/n) < 0$.

¹⁶Heuristically, treating n as a real variable, it is easily shown that $\left[\frac{d(n\pi_n)}{dn}\right]_{n=1} = x_1 A'(x_1) > 0$.

Now, consider

$$\begin{aligned}
W'_n - W_n &= \int_0^{z'_n} P(t)dt - nC(z'_n/n) - [\int_0^{z_n} P(t)dt - nC(z_n/n)] \\
&= U_n(z'_n) - U_n(z_n) \\
&> U'(z'_n)(z'_n - z_n), \text{ since } U \text{ is strictly concave and } z'_n > z_n. \\
&= [P(z'_n) - C'(z'_n/n)]n(x'_n - x_n) \\
&\geq 0, \text{ since } x'_n > x_n \text{ and } P(z'_n) \geq C'(z'_n/n).
\end{aligned}$$

The second part of the lemma follows from the first as $\bar{x}_n =$ largest equilibrium output. ■

As Proposition 1 shows, the case $\Delta > 0$ is consistent with both x_n being decreasing or increasing. The implications of these two possibilities on social welfare are quite different, as reflected in the next result. Also, if the demand function does not satisfy either condition (log-concavity or log-convexity) from Proposition 1(b), then x_n will generally not be monotonic in n . The next result gives sufficient conditions for nondecreasing social welfare.

Proposition 6 *Let $\Delta > 0$ on φ . For any n , at an extremal equilibrium, $W_{n+1} \geq W_n$ for a given n if either one of the following holds: (i) $A(x_{n+1}) \leq A(x_n)$, or (ii) $x_n \leq x_{n+1}$.*

Proof. For an extremal equilibrium, to prove Part (i), consider:

$$\begin{aligned}
W_{n+1} - W_n &= \left\{ \int_0^{z_{n+1}} P(t)dt - z_{n+1}A(x_{n+1}) \right\} - \left\{ \int_0^{z_n} P(t)dt - z_nA(x_n) \right\} \\
&= \int_{z_n}^{z_{n+1}} P(t)dt - z_{n+1}A(x_{n+1}) + z_nA(x_n) \\
&\geq (z_{n+1} - z_n)P(z_{n+1}) - z_{n+1}A(x_{n+1}) + z_nA(x_n) \\
&= z_{n+1} [P(z_{n+1}) - A(x_{n+1})] - z_n [P(z_{n+1}) - A(x_n)] \\
&\geq z_{n+1} [P(z_{n+1}) - A(x_{n+1})] - z_n [P(z_{n+1}) - A(x_{n+1})] \\
&= (z_{n+1} - z_n) [P(z_{n+1}) - A(x_{n+1})] \geq 0,
\end{aligned}$$

where the first inequality follows from the fact that $P(\cdot)$ is decreasing, the second from the assumption $A(x_n) \geq A(x_{n+1})$, while the last follows from the facts that $z_{n+1} \geq z_n$ (since $\Delta > 0$) and $P(z_{n+1}) \geq A(x_{n+1})$.

To prove Part (ii), we begin with two preliminary observations. First, the function $V_n(x) \triangleq \int_0^{nx} P(t)dt - nC(x)$ is concave in x for each n since $V_n''(x) = n[nP'(nx) - C''(x)] < 0$, as a result of $\Delta > 0$. Second, since $z_{n+1} = (n+1)x_{n+1}$ and P is decreasing,

$$\int_0^{z_{n+1}} P(t)dt = \int_0^{nx_{n+1}} P(t)dt + \int_{nx_{n+1}}^{z_{n+1}} P(t)dt \geq \int_0^{nx_{n+1}} P(t)dt + x_{n+1}P(z_{n+1}). \quad (3)$$

Now, consider,

$$\begin{aligned}
W_{n+1} - W_n &= \left\{ \int_0^{z_{n+1}} P(t)dt - (n+1)C(x_{n+1}) \right\} - \left\{ \int_0^{z_n} P(t)dt - nC(x_n) \right\} \\
&\geq x_{n+1}P(z_{n+1}) - C(x_{n+1}) + \left\{ \int_0^{nx_{n+1}} P(t)dt - nC(x_{n+1}) \right\} - \left\{ \int_0^{nx_n} P(t)dt - nC(x_n) \right\} \\
&= \pi_{n+1} + V_n(x_{n+1}) - V_n(x_n) \\
&\geq \pi_{n+1} + V'_n(x_{n+1})(x_{n+1} - x_n) \\
&= \pi_{n+1} + n[P(nx_{n+1}) - C'(x_{n+1})](x_{n+1} - x_n) \geq 0,
\end{aligned}$$

where the first inequality follows from (3), the second from the concavity of V_n in x , and the last from the facts that $P(nx_{n+1}) \geq P[(n+1)x_{n+1}] \geq C'(x_{n+1})$ and $x_{n+1} \geq x_n$. ■

Since price falls with the number of firms here, consumer surplus always increases. However, producer surplus may a priori move either way. So the proposition identifies two sufficient conditions (diseconomies of scale and decreasing per-firm output, or increasing per-firm output) implying that total profit will never decrease enough to overcome the increase in consumer welfare and result in lower social welfare.¹⁷

Propositions 3 and 6, taken together, imply that conventional wisdom fully prevails for the case of constant returns to scale, which is widely invoked in industrial organization.

Corollary 7 *With linear cost, $C(x) = cx$, at an extremal equilibrium, $n\pi_n$ is nonincreasing in n and social welfare W_n is nondecreasing in n , for all n .*

Proof. This follows directly from Propositions 3 and 6, as average cost is constant. ■

In the presence of scale economies, it is well-known that Cournot equilibria do *not converge* to perfectly competitive outcomes, with the latter being ill-defined then. It is thus of interest to shed light on the behavior of the social welfare function when $n < \infty$.

Proposition 8 *Assume $\Delta > 0$ and uniform scale economies prevail (i.e. $A'(\cdot) \leq -\alpha < 0$, for some $\alpha > 0$). Then social welfare is maximized by some finite number of firms, or $n^* \triangleq \arg \max\{W_n : n \geq 1\} < \infty$.*

Proof. Since the reaction correspondence r is ultimately decreasing under our standard assumptions here (in particular A3; see Amir, 1996), it follows that x_n is decreasing in n for large enough n . For such values of n , consider (with $z_n \leq z_{n+1}$ here, since $\Delta > 0$):

$$W_n - W_{n+1} = \left\{ \int_0^{z_n} P(t)dt - z_n A(x_n) \right\} - \left\{ \int_0^{z_{n+1}} P(t)dt - z_{n+1} A(x_{n+1}) \right\}$$

¹⁷An alternative way to think of this result is as follows. Due to the increase in industry output, the sum of consumer surplus and industry revenue (i.e. total benefit or the total area under the inverse demand up to the equilibrium output) always increases with the number of firms. On the other hand, industry costs may go either way. In this perspective, Proposition 6 identifies conditions ensuring that industry costs will never increase enough to cause social welfare to overall decrease, in spite of the increase in total benefit.

$$\begin{aligned}
&= \int_{z_{n+1}}^{z_n} P(t)dt - z_n A(x_n) + z_{n+1} A(x_{n+1}) \\
&> (z_n - z_{n+1})P(z_n) - z_n A(x_n) + z_{n+1} A(x_{n+1}) + [z_{n+1} A(x_n) - z_n A(x_n)] \\
&= z_n [P(z_n) - A(x_n)] - z_{n+1} [P(z_n) - A(x_n)] + z_{n+1} [A(x_{n+1}) - A(x_n)] \\
&\geq (z_n - z_{n+1}) [P(z_n) - A(x_n)] + \alpha z_{n+1} (x_n - x_{n+1}) \\
&= z_{n+1} (x_n - x_{n+1}) \left\{ \alpha - \frac{z_{n+1} - z_n}{z_{n+1} (x_n - x_{n+1})} [P(z_n) - A(x_n)] \right\} \tag{4}
\end{aligned}$$

where the first inequality follows from the fact that $P(\cdot)$ is strictly decreasing, the second from the assumptions of uniform scale economies and $x_n \geq x_{n+1}$.

We now show that the bracketed term in (4) converges to α (or the second term converges to 0) as $n \rightarrow \infty$. A well-known argument based on the first-order condition (see Ruffin, 1971) shows that $P(z_n) - A(x_n) \rightarrow 0$, $z_n \rightarrow Q$ (some constant), and $x_n \rightarrow 0$. Since $x_n \rightarrow 0$ and $n x_n \rightarrow Q$, we know that $x_n \rightarrow 0$ at the rate of $1/n$. It follows that $\frac{z_{n+1} - z_n}{x_n - x_{n+1}} \rightarrow 1$, so that the second term in (4) converges to 0. Hence, for n sufficiently large, the bracketed term in (4) is > 0 (i.e. arbitrarily close to α), so that $W_n > W_{n+1}$, implying the conclusion. ■

While the fact that the socially optimal number of firms is typically finite in the presence of fixed costs is well-known, Proposition 8 is nonetheless somewhat surprising as it relies only on the *slightest level* of decreasing marginal cost. Recall, as seen in the Main Example, that n^* may well be equal to 1, or any other number.

The next example shows that if $A' \leq 0$ but $A'(0) = 0$, W_n may be globally increasing in n , so that $n^* = \infty$. Hence, the assumption $A'(0) > 0$ in Proposition 8 is crucially needed:¹⁸

Example 2. Let $P(z) = 2 - z$ and $C(x) = x - 0.1x^3/3$.

Since $C'(x) \geq 0$ if and only if $x \leq \sqrt{10}$, we will restrict consideration to output levels in $[0, \sqrt{10}]$. We have $A(x) = 1 - .1x^2/3$, so that $A'(0) = 0$ and $A'(x) < 0$ for all $x < \sqrt{30}$. There is a unique Cournot equilibrium with $x_n = 5[n + 1 - \sqrt{(n + 1)^2 - .4}]$. In particular, $x_1 = 7$, $x_2 = .337$, so that with one or two firms, we are outside the range $[0, \sqrt{10}]$ of valid outputs (note that this poses no problem as we are only interested in large n here.) On the other hand, outputs are within $[0, \sqrt{10}]$ from $n = 3$ onwards. Indeed, $x_3 = .252$, $x_4 = .2$, ...

It can be numerically verified that W_n is globally increasing in n .¹⁹ ■

¹⁸The Main Example also shows that any (uniform) level of scale economies, i.e. the smallest (in absolute value) $d < 0$, the conclusion that social welfare *globally* increases with the number of firms would fail as shown by (8): See Point (b)(ii) just below (8).

¹⁹It is tedious but straightforward to verify that

$$W_n = 5n(n + 1 - \sqrt{(n + 1)^2 - .4}) - 12.5n^2(n + 1 - \sqrt{(n + 1)^2 - .4})^2 + 4.1667n(n + 1 - \sqrt{(n + 1)^2 - .4})^3.$$

We next turn to the comparative statics of price-cost margins $m_n \triangleq P(z_n) - C'(x_n)$, often regarded as the most adequate measure of industry competitiveness. Inspection of the first-order condition for a Cournot equilibrium reveals that $m_n \rightarrow 0$ as $n \rightarrow \infty$, irrespective of whether the equilibrium converges to perfect competition. The next result provides alternative sharp sufficient conditions under which this convergence is monotone.

Proposition 9 *Let $\Delta > 0$. Then for the extremal equilibria, $m_{n+1} \leq m_n$ for all n , if any one of these conditions holds:*

- (a) $x_{n+1} \geq x_n$.
- (b) C is concave on $[x_{n+1}, x_n]$.
- (c) P is convex on $[z_n, z_{n+1}]$ and $x_{n+1} \leq x_n$.

Proof. The first-order condition for a Cournot equilibrium can be written as

$$m_n = P(z_n) - C'(x_n) = -x_n P'(z_n)$$

By the mean value theorem, there exist $\tilde{z} \in [z_n, z_{n+1}]$ and \tilde{x} between x_{n+1} and x_n such that

$$\begin{aligned} m_{n+1} - m_n &= P(z_{n+1}) - P(z_n) - [C'(x_{n+1}) - C'(x_n)] \\ &= P'(\tilde{z})(z_{n+1} - z_n) - C''(\tilde{x})(x_{n+1} - x_n). \end{aligned} \quad (5)$$

From Propositions 1(c), $y_{n+1} - y_n = z_{n+1} - x_{n+1} - z_n + x_n \geq 0$. Hence,

$$z_{n+1} - z_n \geq x_{n+1} - x_n. \quad (6)$$

The rest of the proof proceeds separately for the three cases.

(a) Using (6) in (5), we have

$$\begin{aligned} m_{n+1} - m_n &\leq [P'(\tilde{z}) - C''(\tilde{x})](x_{n+1} - x_n) \\ &\leq 0 \text{ since } x_{n+1} \geq x_n \text{ and } P'(\tilde{z}) - C''(\tilde{x}) < 0 \text{ (since } \Delta > 0). \end{aligned}$$

(b) Since C is concave on $[x_{n+1}, x_n]$, $C''(\tilde{x}) \leq 0$. Then using (6) in (5), we have

$$\begin{aligned} m_{n+1} - m_n &\leq [P'(\tilde{z}) - C''(\tilde{x})](z_{n+1} - z_n) \\ &\leq 0 \text{ since } \Delta > 0 \text{ and } z_{n+1} \geq z_n. \end{aligned} \quad (7)$$

(c) From the first-order conditions, we have

$$\begin{aligned} m_{n+1} - m_n &= x_n P'(z_n) - x_{n+1} P'(z_{n+1}) \\ &= x_n P'(z_n) - x_n P'(z_{n+1}) + x_n P'(z_{n+1}) - x_{n+1} P'(z_{n+1}) \\ &= x_n [P'(z_n) - P'(z_{n+1})] + P'(z_{n+1})(x_n - x_{n+1}) \\ &\leq 0 \text{ since } x_n \geq x_{n+1} \text{ and } P'(z_n) \leq P'(z_{n+1}) \text{ by convexity of } P. \end{aligned}$$

This completes the proof²⁰ of Proposition 9. ■

This result leaves the issue open when C is strictly convex on $[x_{n+1}, x_n]$ and P is strictly concave on $[z_n, z_{n+1}]$. The following counter-example, where the latter properties hold globally, shows that the price-cost margin is then not globally decreasing in n .²¹

Example 3. Let $P(z) = a - z^2$ and $C(x) = 2x^2$.

Then a firm's profit function is $\Pi(x, y) = x[1 - (x + y)^2] - 2x^2$. The first-order condition can be reduced to $n(n + 2)x_n^2 + 4x_n - 1 = 0$. The unique (and symmetric) equilibrium is $x_n = \frac{1}{n(n+2)}[-2 + \sqrt{4 + n(n+2)}]$, and the price-cost margin is $m_n = 1 - x_n^2 - 4x_n$.

It is easy to check that m_n increases in n for $n < 5$ and then decreases in n for $n \geq 5$ (specifically, $m_1 = .09267, m_2 = .13397, m_3 = .14838, m_4 = .15047, m_5 = .1471, m_6 < m_5$, etc...) This example is robust to changes in demand and cost parameters. ■

The intuition behind this counter-intuitive outcome is easy to grasp: As n increases from a small value and thus a relatively low total output but high per-firm output, price falls slowly due to the concavity of $P(\cdot)$ while marginal cost falls (convex C) relatively slower. The net effect then may well be an increasing price-cost margin for small values of n .

4.2.2 The case $\Delta < 0$

Strong economies of scale are necessary for Δ to be globally negative. One feature that is known to give rise to economies of scale is the presence of (avoidable) fixed-costs. Without these, one needs a strongly concave cost function for $\Delta < 0$ to be possible.

As existence of a symmetric equilibrium with all firms active is not guaranteed for all n , there are two meaningful ways to proceed. One is to assume quasi-concavity of each profit function in own output and restore existence. The other is to view the comparative statics results as holding for those n 's for which a symmetric equilibrium exists. We elect the former and stress that the quasi-concavity assumption is needed only for existence.

The only general result on industry profit we can offer here vindicates the conventional wisdom only about monopoly.

Proposition 10 *Let $\Delta < 0$ on φ and $\Pi(x, y)$ be quasi-concave in x for each y . Industry profit is highest under a monopoly than under any other market structure: $\pi_1 \geq n\pi_n, \forall n$.*

²⁰The proof makes clear once more that signing Δ is the appropriate assumption for investigating changes in price-cost margins, in addition to separating the issues of existence and uniqueness of equilibrium and the price comparative statics into natural mutually exclusive cases.

²¹This case is the only one for which price-cost margins behave in a counter-intuitive way, as will be seen in the next subsection. Recall that in the Main Example, price-cost margins are decreasing in n in all cases.

Proof. Since the cost function is concave (hence subadditive), a single firm has the *option* of producing the n -firm total Cournot output $z_n = nx_n$ at a cost no higher than that of the n -firm oligopoly (i.e. $C(nx_n) \leq nC(x_n)$) for any n . The conclusion then follows. ■

While no counterexample could be found to establish that $n\pi_n$ is not always decreasing in n , the following argument suggests the conjecture might be false. Total cost is easily seen to increase in n here, but the revenue part moves in the "counterintuitive direction".²²

The welfare comparative statics is unambiguous here, due to the strong scale economies: With more firms, output per firm is strongly reduced, resulting in a drastic increase in average cost. This efficiency loss overcomes any other countervailing considerations.

Proposition 11 *Assume $\Delta < 0$ on φ and $\Pi(x, y)$ is quasi-concave in x for each y . Then at the unique symmetric equilibrium, social welfare W_n is nonincreasing in n , for all n .*

Proof. Consider (with $z_n \geq z_{n+1}$ here, since $\Delta < 0$):

$$\begin{aligned} W_n - W_{n+1} &= \left\{ \int_0^{z_n} P(t)dt - z_n A(x_n) \right\} - \left\{ \int_0^{z_{n+1}} P(t)dt - z_{n+1} A(x_{n+1}) \right\} \\ &= \int_{z_{n+1}}^{z_n} P(t)dt - z_n A(x_n) + z_{n+1} A(x_{n+1}) \\ &\geq (z_n - z_{n+1})P(z_n) - z_n A(x_n) + z_{n+1} A(x_{n+1}) \\ &= z_n [P(z_n) - A(x_n)] - z_{n+1} [P(z_n) - A(x_{n+1})] \\ &\geq z_n [P(z_n) - A(x_n)] - z_{n+1} [P(z_n) - A(x_n)] \\ &= (z_n - z_{n+1}) [P(z_n) - A(x_n)] \geq 0 \end{aligned}$$

where the first inequality follows from the fact that $P(\cdot)$ is decreasing, the second from the facts that $x_n \geq x_{n+1}$ and $A(\cdot)$ is nonincreasing (the latter follows since $\Delta < 0$ requires concavity of C), and the last from the fact that $z_n \geq z_{n+1}$ (since $\Delta < 0$). ■

Since price increases with the number of firms here (Proposition 2a), consumer surplus decreases. Also, as average cost and equilibrium per-firm output both decline rapidly, equilibrium total production costs increase rapidly with the number of firms here. Hence, even if total profits go up, the increase will never be sufficient (recall also that per-firm profit goes down) to overcome the fall in consumer surplus.

Corollary 12 *Assume $\Delta < 0$ on φ and $\Pi(x, y)$ is quasi-concave in x for each y . Monopoly leads to the highest producer surplus and to the highest consumer surplus levels.*

²²Indeed, if it were possible to have $nx_n \geq (n+1)x_{n+1}$ while $C \equiv 0$ (which we know is impossible since $C \equiv 0$ clearly implies $\Delta > 0$), we would have $\Pi_{n+1} = x_{n+1}P[(n+1)x_{n+1}] \geq \frac{nx_n}{n+1}P\left(\frac{nx_n}{n+1} + nx_{n+1}\right) \geq \frac{nx_n}{n+1}P\left(\frac{nx_n}{n+1} + n\frac{nx_n}{n+1}\right) = \frac{nx_n}{n+1}P(nx_n) = \frac{n\Pi_n}{n+1}$, where the first inequality is from the Cournot equilibrium property and the second from the facts that P is decreasing and $nx_n \geq (n+1)x_{n+1}$. It would then follow that $(n+1)\Pi_{n+1} \geq n\Pi_n$: Industry profit would be increasing in the number of firms!

Proof. The two statements follow respectively from Propositions 10 and 2a. ■

As to price-cost margins, they always decline with competition in the case at hand.

Proposition 13 *Let $\Delta < 0$. Then for any n , we have $m_{n+1} \leq m_n$.*

Proof. Since $C''(\tilde{x}) \leq 0$ is implied by $\Delta < 0$, (7) holds here, so

$$\begin{aligned} m_{n+1} - m_n &\leq [P'(\tilde{z}) - C''(\tilde{x})](z_{n+1} - z_n) \\ &\leq 0 \text{ since } \Delta < 0 \text{ and } z_{n+1} \leq z_n. \blacksquare \end{aligned}$$

The intuition is simple. With strong scale economies, having more firms in the market raises both a firm's marginal cost and the market price, with the former being the more dominant effect as a firm's output is drastically cut down. The overall effect is that margins fall.

In view of the counterintuitive nature of some results in the case $\Delta < 0$, it is natural to ask whether they have any predictive value in describing imperfect competition in real-world markets.²³ Experimental evidence suggests that unique (stable) Cournot equilibria are good predictors of actual behavior (Holt, 1986). By contrast, Cox and Walker (1998) report that in a symmetric Cournot game with three equilibria, a symmetric unstable one and two boundary or monopoly equilibria (cf. Proposition 2), laboratory behavior reflected no regular patterns of play that would support any equilibrium. Rather, play seemed to proceed along irregular cycles around the three equilibria, with the players continuously exhibiting large swings in output levels, conveying a clear sense of unstable behavior.

None of the Nash equilibrium refinements for one-shot games, such as perfection, properness, Kohlberg-Mertens's (1986) strategic stability, can discard Cournot-unstable equilibria, although some evolutionary learning processes might. Furthermore, regardless of stability properties, symmetric Cournot equilibria often emerge as focal (Schelling, 1960.)

4.2.3 Hybrid Cases

In view of the level of generality of our conclusions, the fact that the entire analysis rests essentially on one easily checked condition on the global sign of Δ is a remarkable feature. On the other hand, there are many demand-cost combinations of interest for which Δ changes signs on its domain: Hybrid cases. For these, Cournot equilibria will generally not behave in the globally monotonic ways uncovered here. The issue of existence also needs separate

²³Further discussion about this case is provided in Section 5 where the characteristics of the Cournot equilibria here are identified with the old concept of destructive competition, among other applications.

attention then. De Meza (1985) provides an interesting counterexample highlighting the differences between local and global comparative statics, and also showing that treating n as a real variable in the present context and signing dp_n/dn can lead to truly misleading results. His example shows that it is crucial that Δ have a uniform sign globally for the present analysis, thus further vindicating the lattice-theoretic approach.

In spite of the nonmonotonic behavior of the equilibria, some of the insights we developed can still be useful here. We offer two related examples. The first is a simpler alternative to De Meza's example that does not require a piece-wise definition of the profit function and thus satisfies the smoothness conditions made in the literature.

Example 4. Let $P(z) = \frac{1}{z+0.9}$ and $C(x) = \frac{1}{2} \log(x+1)$.

Then $\Delta(z, y) = \frac{1}{(z+0.9)^2} - \frac{1}{2(z-y+1)^2} > 0$ iff $y < .414x + .514$. Hence, Δ changes signs on φ so that our results do not apply. The profit function $\Pi(x, y) = \frac{x}{x+y+0.9} - \frac{1}{2} \log(x+1)$ is quasi-concave in x , $\forall y$. The first-order condition is $(x+1)(y+.9) = \frac{1}{2}(x+y+.9)^2$ and the reaction curve is $r(y) = \frac{1}{10} \sqrt{20y+99-100y^2}$. The equilibrium output is $x_n = \frac{1}{10(n^2-2n+2)} (n-1 + \sqrt{100n^2-200n+199})$. In particular, $x_2 = .75534$ and $x_3 = .48677$, so that $z_2 = 1.5107$ and $z_3 = 1.4603$. Hence $p_2 < p_3$.

Also $r'(y) = (1-10y)/\sqrt{20y+99-100y^2}$ and $r'(x_2) = -.86762 > -1$, so the duopoly point lies within the quasi-competitive part (the border between the two regions is reached at $y = .80711$, i.e. $r'(.80711) = -1$). The triopoly point lies outside the quasi-competitive part since $y_3 = 2x_3 = .97354 > .80711$.²⁴ A local analysis based on signing dp_n/dn as adopted in the classical literature on the subject would then lead to the conclusion that $[dp_n/dn]_{n=2} < 0$. This may be thought of as correct as n goes from 2 to 2.1414 firms, but not for $n > 2.1414$ (this is the fictitious number of firms \tilde{n} for which $y_{\tilde{n}} = .80711$.) Hence, such an analysis²⁵ cannot be the basis for a comparison of z_2 and z_3 (or p_2 and p_3 .) ■

This example illustrates the inadequacy of a local analysis to address the issues at hand. It clearly shows that in order to attain an unambiguous comparative statics conclusion between n and $n+1$, the intersections of the reaction curve with the lines $y/(n-1)$ and $y/(n-2)$ must *both* lie within a region where Δ keeps the same sign.

The last example has social welfare maximized by monopoly when Δ is not globally < 0 .

²⁴In other words, it may be checked that $\Delta(z_2, y_2) = .0098 > 0$ while $\Delta(z_3, y_3) = -.0467 < 0$.

²⁵Our findings may also be phrased in terms of Cournot stability as in Seade (1980). The duopoly point is stable so that a local analysis would predict, incorrectly, that $p_2 > p_3$. The triopoly point is not stable. The reasoning here is the same as in De Meza (1985), which compares monopoly with duopoly.

Example 5. Let $P(z) = \frac{1}{z+1}$ and $C(x) = \frac{1}{2} \log(x+1)$.

Here $\Delta(z, y)$ changes signs on φ and $\Pi(x, y) = \frac{x}{x+y+1} - \frac{1}{2} \log(x+1)$ is quasi-concave in x , for fixed y . The reaction curve is $r(y) = \sqrt{1-y^2}$, and the unique Cournot equilibrium is given by $x_n = \frac{1}{\sqrt{n^2-2n+2}}$. Simple calculations show that while x_n and $n\pi_n$ are decreasing in n for all n , z_n and thus p_n are not monotonic in n .

Social welfare is given by $W_n = \log(\frac{n}{\sqrt{n^2-2n+2}} + 1) - \frac{n}{2} \log(\frac{1}{\sqrt{n^2-2n+2}} + 1)$. In particular, $W_1 = \frac{1}{2} \ln 2 \simeq .34657$, and $W_2 = \log(\sqrt{2} + 1) - \log(\frac{1}{2}\sqrt{2} + 1) \simeq .34657$, so that $W_2 = W_1$. Then W_n decreases monotonically from $n = 2$ onwards. Thus²⁶, $n^* = \{1, 2\}$: A social planner would be indifferent between monopoly and duopoly as the optimal choice! ■

5. On some Theoretical and Policy Implications

The results presented here lie at the heart of the modern theory of industrial organization and can, to some extent, illuminate a number of past as well as present theoretical issues and public policy debates. In particular, we relate our findings to the relationship between Cournot outcomes and perfect competition, the welfare content of the Herfindahl index, destructive competition and natural monopoly. Surprisingly, the latter two notions have not been linked with Cournot theory in the past. We attempt to fill this gap below.²⁷

5.1 Relationship to Perfect Competition

Ruffin (1971) showed that if the number of firms is increased with fixed demand,²⁸ Cournot equilibria converge to the perfectly competitive equilibrium under global diseconomies of scale, but not under global economies of scale. Our conclusions shed some light on this result by indicating that (i) in the former case, equilibrium welfare converges monotonically to first-best welfare, and (ii) in the latter case, although industry profits and per-firm output both monotonically converge to zero, welfare does not increase to first-best welfare, due to firms producing at increasing (and in the limit, maximal) average cost. Here first-best welfare would involve one firm producing the entire output and pricing at marginal cost.²⁹ Thus our

²⁶Viewing n as a real variable, W_n is single-peaked in n and achieves its maximum at $n \simeq 1.36$.

²⁷Other applications can be discussed as well, including efficiency aspects of merger policy, the concentration/profitability debate, and entry regulation. With an exogenous number of firms, the latter topic is covered in detail Mankiw and Whinston (1986) and Suzumura and Kiyono (1987). Our results also shed light on their main result, that free entry leads to excessive entry relative to a planner's solution.

²⁸See Novshek (1980) for the other approach, where demand is replicated.

²⁹The planner's objective is then $\max\{\int_0^{nx} P(t)dt - nC(x) : n \geq 1, x \geq 0\}$. The first-order conditions are $P(nx) = C'(x)$ and $xP(nx) = C(x)$. These imply that marginal and average cost are equal, as is well-known

results fill an important gap in the foundations of partial-equilibrium perfect competition.

5.2 The Herfindahl Concentration Index and Welfare

The Herfindahl-Hirschman Index (or HHI) of industry concentration, defined as a (normalized) sum of the squares of the firms' market shares, is the most often used quantitative assessment of industry concentration. In particular, the value of the HHI constitutes the primary indicator for antitrust authorities of market power and of the likelihood of overt or tacit collusion in a given market. The HHI is also one of the main elements of the 1982 Merger Guidelines in determining whether a proposed merger is to be allowed.³⁰

Underlying the extensive reliance of economic law on this measure is a fundamental belief that social welfare and the HHI are always inversely related (see e.g. Dansby and Willig, 1979). Yet, this belief has recently been challenged by theoretical studies based on the Cournot model. Farrell and Shapiro (1990) establish that, with a fixed number of (nonidentical) firms, whenever industry output is unchanged following individual firm output changes, social welfare and the HHI must change in the same direction. Salant and Shaffer (1999) provide further insight into this result in the case of constant unit costs by showing that both welfare and the HHI increase if and only if the variance of the unit costs increases in a mean-preserving way. Also see Daughety (1990).

The present paper sheds some further light on this issue by considering the effects of changing the number of firms instead. Given the symmetry assumption, the HHI with n firms here is clearly given by (a constant factor of) $1/n$. Hence, the HHI decreases if and only if the number of firms increases. On the other hand, our results indicate that in the presence of scale economies (with $A' < 0$), social welfare decreases if the number of firms exceeds some socially optimal level n^* . Thus, both the HHI and welfare decrease whenever n increases beyond n^* . In particular, in industries where $n^* = 1$, the two measures would *always* produce conflicting prescriptions as the number of firms increases.

This conclusion clearly suggests that the HHI should be augmented by some measure of economies of scale in the industry that would allow appropriate balancing between the legitimate fears of market power and the desire for production efficiency.

for a first-best solution. In particular, the first-best number of firms is then 1 if $A' < 0$ and ∞ if $A' > 0$.

³⁰For some historical background on these Guidelines and an exchange of views among experts, see the Symposium in the Journal of Economic Perspectives, vol. 1, 1987.

5.3 Destructive Competition

Destructive competition was a recurrent theme in older case and empirical studies of regulated industries, particularly those in the transportation sector such as railroad and trucking (see Sharkey (1982) for a historical account). It is typically associated with a combination of industry characteristics, such as strong economies of scale (often due to large fixed costs), large productive capacity, relatively easy entry, and ill-guided government subsidies. The symptoms of destructive competition in such industries include high levels of market instability, excessive capacity and widespread price discrimination, often leading to frequent changes in regulatory regimes, including entry regulation.

Sharkey (1982) develops a cooperative game-theoretic approach to model destructive competition, defining industry stability by the nonemptiness of the core. The results here suggest a simple and natural alternative within the noncooperative paradigm: In the absence of any regulatory interference, destructive competition can be fruitfully modelled by Cournot competition under the assumption that Δ is globally negative. Indeed, increases in competition from any pre-existing level, including in particular monopoly, result in lower consumer welfare, per-firm profit and social welfare. Thus higher competition is unambiguously detrimental to all economic agents, with even unregulated monopoly emerging as the best among market outcomes. Furthermore, and more strikingly, some aspects of reported market instability in industries thought to have undergone phases of destructive competition may be instructively linked to the *indeterminacy in the number of active firms* and the *unstable nature of the Cournot equilibria* (in the sense of divergence of best-reply dynamics), both of which are characteristics of the case Δ globally negative (see Section 4.4.)

5.4 Natural Monopoly

Following various attempts, Baumol, Panzar and Willig (1982) provided the final definition of natural monopoly: An industry with a subadditive cost function. This is the least restrictive property of a cost function that captures the notion that any amount of final output is cheaper to produce by one firm, or, in other words, subdividing production cannot possibly save on costs. This definition completely ignores the demand side of the market, which is justified in light of two special features that were dominant in the economic scene two decades ago. The first, reflecting the prevalent public policy view of the times, is that monopolies are to be regulated anyway, so that market conduct is not really an issue, leaving production

efficiency as the primary concern. The second, a theoretical belief, is that if an industry has a downward-sloping average cost curve and the market is contestable, the only stable configuration will involve a single firm pricing at average cost, resulting in zero profits.

Subsequently, a near-consensus emerged, recognizing the limited real-life validity of contestable markets³¹, and a wave of deregulation originating in the US and the UK swept through the industrialized world. In view of the need to incorporate the demand side of the market now in a revised definition of natural monopoly, the analysis of the present paper suggests an obvious alternative: An unregulated monopoly is natural if social welfare W_n is maximized by $n^* = 1$. According to our results, this would require scale economies of sufficient magnitude over the relevant range, but not necessarily that Δ be globally < 0 . Recall that the Main Example shows that n^* can be equal to 1 for an industry for which $\Delta > 0$ globally. This definition is clearly more restrictive than the old one³², as it incorporates the market or demand side of the industry. In other words, it strikes a socially optimal balance between the detrimental effects of concentration and the cost-saving effects of size.

A natural n^* -firm oligopoly can be analogously defined by $n^* = \arg \max_n W_n$. If $\Delta < 0$, Proposition 11 implies that n^* must necessarily be equal to 1. However $n^* > 1$ is compatible with $\Delta > 0$ globally, and with Δ not having a uniform sign on all its domain (cf. Example 4).■

REFERENCES

Amir, R. (1996), "Cournot Oligopoly and the Theory of Supermodular Games", *Games and Economic Behavior*, 15, 132-148.

Amir R., and V. E. Lambson (2000), "On the Effects of Entry in Cournot Markets," *Review of Economic Studies*, 67, 235-254.

d'Aspremont, C., R. Dos Santos Ferreira, L.-A. Gerard-Varet (2000), Contestability and the Indeterminacy of Free-Entry Equilibria, *Japanese Economic Review*, 51, 72-83.

Athey, S. (2002), "Monotone Comparative Statics Under Uncertainty", *Quarterly Journal of Economics*, 187-223.

Bain, J. (1956), *Barriers to New Competition*, Harvard University Press, Cambridge.

Baumol, W., J. C. Panzar and R. Willig (1982), "Contestable Markets and the Theory of Industry Structure", Harcourt Brace Jovanovich, New York.

³¹In a book review of Baumol, Panzar and Willig (1982), Spence (1983) offered an eloquent account of the currently prevailing view on contestable markets. He concludes that the benefits of this theory lie in its thorough analysis of cost functions, and not in its empirical relevance as a theory of market structure.

³²Scale economies of any magnitude imply the subadditivity of the cost function, but not vice-versa.

- Bresnahan, T. (1989), "Empirical Studies in Industries with Market Power", in *Handbook of Industrial Organization*, Vol II, 1011-58.
- Caplin, A. and B. Nalebuff (1991), Aggregation and Imperfect Competition: On the Existence of Equilibrium, *Econometrica*, 59, 25-69.
- Cox, J. and M. Walker (1998), "Learning to Play Cournot Duopoly Strategies", *Journal of Economic Behavior and Organization*, 36, 141-161.
- Daughety, A. (1990), Beneficial Concentration, *American Economic Review*, 80, 1231-37.
- Diewert, W. and T. Wales (1987), "Flexible Functional Forms and Global Curvature Conditions", *Econometrica*, 5, 43-68.
- Farrell, J. and C. Shapiro (1990), Horizontal Mergers: An Equilibrium Analysis, *American Economic Review*, 80, 107-26.
- Friedlaender, A., C. Winston and K. Wang (1983), "Costs, Technology and Productivity in the US Automobile Industry", *Bell Journal of Economics*, 14, 1-20.
- Frank, C. (1965), "Entry in a Cournot Market", *Review of Economic Studies*, 32, 245-50.
- Goldschmid, H., H. M. Mann and J.F. Weston (1974), eds., *Industrial Concentration: The New Learning*, Boston: Little, Brown.
- Hahn, F. (1962), The Stability of the Cournot Oligopoly Solution, *Review of Economic Studies*, 29, 329-221.
- Holt, C. (1995), "Industrial Organization: A Survey of the Results of Laboratory Experiments", in *Handbook of Experimental Economics*, A. Roth and J. Kagel eds., Princeton University Press.
- Kohlberg, E. and J.-F. Mertens (1986), Strategic Stability of Equilibria, *Econometrica*, 54, 1003-37.
- MacManus, M. (1962), "Number and Size in Cournot Equilibrium", *Yorkshire Bulletin of Economic and Social Research*, 14, 14-22.
- Mankiw, N.G. and Whinston, M. D. (1986), "Free Entry and Social Inefficiency", *Rand Journal of Economics*, 17, 48-58.
- de Meza, D. (1985), "A Stable Cournot-Nash Industry Need not be Quasi-Competitive", *Bulletin of Economic Research*, 37, 153-156.
- Milgrom, P. and J. Roberts (1990), "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities", *Econometrica*, 58, 1255-1278.
- Milgrom, P. and J. Roberts (1994), "Comparing Equilibria", *American Economic Review*, 84, 441-459.

- Mueller, D. (1986), *Profits in the Long Run*, Cambridge University Press, New-York.
- Novshek, W. (1980), "Cournot Equilibrium with free entry", *Review of Economic Studies*, 47, 473-486.
- Novshek, W. (1985), "On the Existence of Cournot Equilibrium", *Review of Economic Studies*, L II, 85-98.
- Okuguchi, K. (1973), "Quasi-Competitiveness and Cournot Oligopoly", *Review of Economic Studies*, 40, 145-148.
- Perry, M. (1984), "Scale Economies, Imperfect Competition and Public Policy", *Journal of Industrial Economics*, 32, 313-330.
- Ramey, V. (1989), "Inventories as Factors of Production and Economic Fluctuations", *American Economic Review*, 79, 338-54.
- Ramey, V. (1989), "Nonconvex Costs and the Behavior of Inventories", *Journal of Political Economy*, 99, 306-34.
- Roberts, J. and H. Sonnenschein (1976), "On the Existence of Cournot Equilibrium without Concave Profit Functions", *Journal of Economic Theory*, 13, 112-117.
- Ruffin, R. J. (1971), "Cournot Oligopoly and Competitive Behavior", *Review of Economic Studies*, 38, 493-502.
- Salant, S. and G. Shaffer (1999): "Unequal Treatment of Identical Agents in Cournot Equilibrium", *American Economic Review*, 89, 585-604.
- Schelling, T. (1960), *The Strategy of Conflict*, Harvard University Press, Cambridge.
- Seade, J. (1980), "On the Effects of Entry", *Econometrica*, 48, 479-489.
- Sharkey (1982), *The Theory of Natural Monopoly*, Cambridge University Press, Cambridge, U.K.
- Spence, M. (1983), "Contestable Markets and the Theory of Industry Structure: A Review Article", *Journal of Economic Literature*, 21(3), 981-90.
- Suzumura, K. and K. Kiyono (1987), "Entry Barriers and Economic Welfare", *Review of Economic Studies*, L IV, 157-167.
- Topkis, D. (1978), "Minimizing a Submodular Function on a Lattice", *Operations Research*, 26, 305-321.
- von Weiszacker, C. (1980), "A Welfare Analysis of Barriers to Entry", *Rand Journal of Economics*, 11, 399-420.

6. Appendix

Here, we provide a sufficient condition for quasi-concave profits.³³

Let $\bar{y} \triangleq P^{-1}[C'(0)]$ or $P(\bar{y}) = [C'(0)] = A(0)$ by l'Hospital's rule. Thus \bar{y} is the rivals' total output that equates price and average cost when the responding firm produces 0.

Proposition A. *Let $[P(x + y) - A(x)]^{-1}$ be convex in $x \in U(y) \triangleq \{x : P(x + y) > A(x)\}$ for each $y \in [0, \bar{y}]$. Then (i) $\Pi(x, y)$ is strictly quasi-concave in $x \in U(y)$ for fixed $y \in [0, \bar{y}]$, and (ii) r is a continuous function satisfying $r(y) > 0$ for $y \in [0, \bar{y})$ and $r(y) = 0$ for $y \geq \bar{y}$.*

Proof. We prove Part (i) by contradiction. If strict quasi-concavity fails, there exists $\lambda \in (0, 1)$, $y \in [0, \bar{y}]$ and $x, x' \in U(y)$ such that

$$x[P(x + y) - A(x)] \geq [\lambda x + (1 - \lambda)x']\{P[\lambda x + (1 - \lambda)x' + y] - A[\lambda x + (1 - \lambda)x']\} \text{ and}$$

$$x'[P(x' + y) - A(x')] \geq [\lambda x + (1 - \lambda)x']\{P[\lambda x + (1 - \lambda)x' + y] - A[\lambda x + (1 - \lambda)x']\}$$

Rewriting the two inequalities (one of which is actually strict),

$$\frac{x}{P[\lambda x + (1 - \lambda)x' + y] - A[\lambda x + (1 - \lambda)x']} \geq \frac{\lambda x + (1 - \lambda)x'}{P(x + y) - A(x)} \text{ and}$$

$$\frac{x'}{P[\lambda x + (1 - \lambda)x' + y] - A[\lambda x + (1 - \lambda)x']} \geq \frac{\lambda x + (1 - \lambda)x'}{P(x' + y) - A(x')}.$$

Multiplying the two inequalities by λ and $1 - \lambda$ respectively, adding them up, and simplifying yields (since one of them is strict)

$$\frac{1}{P[\lambda x + (1 - \lambda)x' + y] - A[\lambda x + (1 - \lambda)x']} > \frac{\lambda}{P(x + y) - A(x)} + \frac{1 - \lambda}{P(x' + y) - A(x')}$$

which contradicts the assumption of convexity of $1/[P(x + y) - A(x)]$ in x . Hence, (i) follows.

For Part (ii), it follows directly from Part (i) that $r(y)$ is a continuous function for $y \in [0, \bar{y}]$. Also $r(\bar{y}) = 0$ by definition of \bar{y} , and thus $\Pi(r(\bar{y}), \bar{y}) = 0$. We now show that $r(y) = 0$ for $y \geq \bar{y}$ by contradiction. Assume that $r(y') > 0$ for some $y' > \bar{y}$. Then we must have $\Pi(r(y'), y') \geq 0$, since a response of 0 guarantees zero profit. Then $\Pi(r(y'), \bar{y}) > \Pi(r(y'), y') \geq 0$ since Π is strictly decreasing in y (from Assumption A1) and $y' > \bar{y}$. But $\Pi(r(y'), \bar{y}) > 0$ is a contradiction to the fact that $r(\bar{y})$ is a best-response to \bar{y} yielding $\Pi(r(\bar{y}), \bar{y}) = 0$. Since \bar{y} is uniquely defined, the conclusion follows directly. ■

³³For a similar argument in price competition, see Caplin and Nalebuff (1991).