A Nonparametric Test of Granger Causality in Continuous Time*

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Abstract

This paper develops a nonparametric Granger causality test for continuous time point process data. Unlike popular Granger causality tests with strong parametric assumptions on discrete time series, the test applies directly to strictly increasing raw event time sequences sampled from a bivariate temporal point process satisfying mild stationarity and moment conditions. This eliminates the sensitivity of the test to model assumptions and data sampling frequency. Taking the form of an $\mathcal{L}^2$-norm, the test statistic delivers a consistent test against all alternatives with pairwise causal feedback from one component process to another, and can simultaneously detect multiple causal relationships over variable time spans up to the sample length. The test enjoys asymptotic normality under the null of no Granger causality and exhibits reasonable empirical size and power performance. Its usefulness is illustrated in three applications: tests of trade-to-quote causal dynamics in market microstructure study, credit contagion of U.S. corporate bankruptcies over different industrial sectors, and financial contagion across international stock exchanges.

1 Introduction

The concept of Granger causality was first introduced to econometrics in the groundbreaking work of Granger (1969) and Sims (1972). Since then it has generated an

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extensive line of research and quickly became a standard topic in econometrics and time series analysis textbooks. The idea is straightforward: a process X_t does not strongly (weakly) Granger cause another process Y_t if, at all time t, the conditional distribution (expectation) of Y_t given its own history is the same as that given the histories of both X_t and Y_t almost surely. Intuitively, it means that the history of process X_t does not affect the prediction of process Y_t.

Granger causality tests are abundant in economics and finance. Instead of giving a general overview on Granger causality tests, I will focus on some of the shortfalls of popular causality tests. Currently, most Granger causality tests in empirical applications rely on parametric assumptions, most notably the discrete time vector autoregressive (VAR) models. Although it is convenient to base the tests on discrete time parametric models, there are a couple of issues that can potentially invalidate this approach:

(1) Model uncertainty. If the data generating process (DGP) is far from the parametric model, the econometrician will run the risk of model misspecification. The conclusion of a Granger causality test drawn from a wrong model can be misleading. A series of studies attempts to reduce the effect of model uncertainty by relaxing or eliminating the reliance on strong parametric assumptions.[1]

(2) Sampling frequency uncertainty. Existing tests of Granger causality in discrete time often assume that the time difference between consecutive observations is constant and prespecified. However, it is important to realize that the conclusion of a Granger causality test can be sensitive to the sampling frequency of the time series. As implied by the results of Sims (1971) and argued by Engle and Liu (1972), the test would potentially be biased if we estimated a discretized time series model with temporally aggregated data which are from a continuous time DGP (see section 1.1).

To address the above shortcomings, I consider a nonparametric Granger causality test in continuous time. The test is independent of any parametric model and thus the first problem is eliminated. Unlike discrete time Granger causality tests, the test applies to data sampled in continuous time - the highest sampling frequency possible - and can simultaneously and consistently detect causal relationships of various durations spanning up to the sample length. The DGP is taken to be a pure-jump process known as *bivariate temporal point process*.

A *temporal point process* is one of the simplest kinds of stochastic process and is the central object of this paper. It is a pure-jump process consisting of a sequence of *events* represented by jumps that occur over a continuum, and the observations are event occurrence times (called *event times*).[2] Apart from their simplicity, point processes are indispensable building blocks of other more complicated stochastic processes (e.g. Lévy processes, subordinated diffusion processes). In this paper, I study the testing

[1] One line of research extends the test to nonlinear Granger causality test. To relax the strong linear assumption in VAR models, Hiemstra and Jones (1994) developed a nonparametric Granger causality tests on discrete time series without imposing any parametric structures on the DGP except some mild ones such as stationarity and Markovian dynamics. In the application of their test, they found that volume Granger causes stock return.

[2] The trajectory of a *counting process*, an equivalent representation constructed from point process observations, is a stepwise increasing and right-continuous function with a jump at each event time. An important example is the Poisson process in which events occur independently of each other.

of Granger causality in the context of a *simple*[3] *bivariate point process*, which consists of a strictly monotonic sequence of event times originated from two event types with possible interactions among them. The problem of testing Granger causality consistently and nonparametrically in a continuous time set-up for a simple bivariate point process is non-trivial: all interactive relationship of event times over the continuum of the sample period needs to be summarized in a test statistic, and continuous time martingale theory is necessary to analyze its asymptotic properties. It is hoped that the results reported in this paper will shed light on a similar test for more general types of stochastic processes.

To examine the causal relation between two point processes, I first construct event counts (as a function of time) of the two types of events from the observed event times. The functions of event counts, also known as *counting processes*, are monotone increasing functions by construction. To remove the increasing trends, I consider the differentials of the two counting processes. After subtracting their respective conditional means (estimated nonparametrically), I obtain the *innovation processes* that contain the surprise components of the point processes. It is possible to check, from the cross-covariance between the innovation processes, if there is a significant feedback from one counting process to another. As detailed in section 2, such a feedback relationship is linked to the Granger causality concept that was defined for general continuous time processes (including counting processes as a particular case) in the extant literature. More surprisingly, if the raw event times are strictly monotonic, then *all pairwise cross-dependence* can be sufficiently captured by the *cross-covariance* between the innovation processes. This insight comes from the Bernoulli nature of the jump increments of the associated counting processes, and will greatly facilitate the development and implementation of the test.

The paper is organized as follows. Empirical applications of point processes are described in sections 1.3 and 1.4. The relevant concepts and properties of a simple bivariate point process is introduced in section 2, while the concept of Granger causality is discussed and adapted to the context of point processes in section 3. The test statistic is constructed in section 4 as a weighted integral of the squared cross-covariance between the innovation processes. and the key results on its asymptotic behaviors are presented in section 5. Variants of the test statistic under different bandwidth choices are discussed in section 6. In the simulation experiments in section 7, I show that the nonparametric test has reasonable size performance under the null hypothesis of no Granger causality and nontrivial power against different alternatives. In section 8, I demonstrate the usefulness of the nonparametric Granger causality test in a series of three empirical applications. In the first application on the study of market microstructure hypotheses (section 8.1), we see that the test confirms the existence of a significant causal relationship from trades to quote revisions in high frequency financial datasets. Next, I turn to the application in credit contagion (section 8.2) and provide the first empirical evidence that bankruptcies in financial-related sectors tend to Granger-cause those in manufacturing-related sectors during crises and recessions. In the last application on international financial contagion (section 8.3), I examine the extent to which

[3]The *simple* property will be formally defined in assumption (A1) in section 2.

an extreme negative shock of a major stock index is transmitted across international financial markets. The test reveals the presence of financial contagion, with U.S. and European stock indices being the sources of contagion. Finally, the paper concludes in section 9. Proofs and derivations are collected in the Appendix.

1.1 The Need for Continuous Time Causality Test

The original definition of Granger causality is not only confined to discrete time series but also applicable to continuous time stochastic processes. However, an overwhelming majority of research work on Granger causality tests, be it theoretical or empirical, has focused on a discrete time framework. One key reason for this is the limited availability of (near) continuous time data. However, with much improved computing power and storage capacity, economic and financial data sampled at increasingly high frequencies have become more accessible.[4] This calls for more sophisticated techniques for analyzing these datasets. To this end, continuous time models provide a better approximation to frequently observed data than discrete time series models with very short time lags and many time steps. Indeed, even though the data are observed and recorded in discrete time, it is sometimes more natural to think of the DGP as evolving in continuous time, because economic agents do not necessarily make decisions at the same time when the data are sampled. The advantages of continuous time analyses are more pronounced when the observations are sampled (or available) at random time points. Imposing a fixed discrete time grid on highly irregularly spaced time data may lead to too many observations in frequently sampled periods and/or excessive null intervals with no observations in sparsely sampled periods.[5]

Furthermore, discretization in time dimension can result in the loss of time point data and *spurious (non)causality*. The latter problem often arises when "the finite time delay between cause and effect is small compared to the time interval over which data is collected", as pointed out by Granger (1988, p.205). A Granger causality test applied to coarsely sampled data can deliver very misleading results: while the DGP implies a unidirectional causality from process X_t to process Y_t, the test may indicate (i) a significant *bidirectional causality* between X_t and Y_t, or (ii) insignificant causality between X_t and Y_t in either one or both directions.[6] The intuitive reason is that the causality of the discretized series is the aggregate result of the causal effects in each sampling intervals, amplified or diminished by the autocorrelations of the marginal processes. The severity of these problems depends on prespecified sampling intervals: the wider they are relative to the *causal durations* (the actual time durations in which

[4] For example, trade and quote data now include records of trade and quote timestamps in unit of milliseconds.

[5] Continuous time models are more parsimonious for modeling high frequency observations and are more capable of endogenizing irregular and possibly random observation times. See, for instance, Duffie and Glynn (2004), Aït-Sahalia and Mykland (2003), Li, Mykland, Renault, Zhang, Zheng (2010).

[6] Sims (1971) provided the first theoretical explanation in the context of distributed lag model (a continuous time analog of autoregressive model). See also Geweke (1978), Christiano and Eichenbaum (1987), Marcet (1991) and, for a more recent survey, McCrorie and Chambers (2006).

causality effect transmits), the more serious the problems.[7] With increasingly accessible high frequency and irregularly spaced data, it is necessary to develop theories and techniques tailored to the continuous time framework to uncover any interactive patterns between stochastic processes. Analyzing (near) continuous time data with inappropriate discrete time techniques is often the culprit of misleading conclusions.

To remedy the above problems, there have been theoretical attempts to extend discrete time causality analyses to fully continuous time settings. For example, Florens and Fougere (1996) examined the relationship between different definitions of Granger non-causality for general continuous time models. Comte and Renault (1996) studied a continuous time version of ARMA model and provided conditions on parameters that characterize when there is no Granger causality, while Renault, Sekkat and Szafarz (1998) gave corresponding characterizations for parametric Markov processes. All of the above work, however, did not elaborate further on the implementation of the tests, let alone any formal test statistic and empirical applications.

Due to a lack of continuous time testing tools for high-frequency data, practitioners generally rely on parametric discrete time series methodology or multivariate parametric point process models. Traditionally, time series econometricians have little choice but to adhere to a fixed sampling frequency of the available dataset, even though they have been making an effort to obtain more accurate inference by using the highest sampling frequency that the data allow (Engle, 2000). The need to relax the rigid sampling frequency is addressed by the literature on mixed frequency time series analyses.[8] On the other hand, inferring causal relationships from parametric point process models may address some of these problems as this approach respects the irregular nature of event times.

It is important to reiterate that correct inference about the directions of Granger causality stems from an appropriate choice of sampling grid. The actual causal durations, however, are often unknown or even random over time (as is the case for high-frequency financial data). In light of this reality, it is more appealing to carry out Granger causality tests on continuous time processes in a way that is independent of the choice of sampling intervals and allows for simultaneous testing of causal relationships with variable ranges.

1.2 The Need for Nonparametric Causality Test

Often times, the modelers adopt a pragmatic approach when choosing a parametric model in order to match the model features to the observed stylized facts of the data. In the study of empirical market microstructure, there exist parametric bivariate point

[7]For instance, suppose the DGP implies a causal relationship between two economic variables which typically lasts for less than a month. A Granger causality test applied to the two variables sampled weekly can potentially reveal a significant causal relationship, but the test result may turn insignificant if applied to the same variables sampled monthly.

[8]Ghysels (2012) extends the previous mixed frequency regression to VAR models with a mixture of two sampling frequencies. Chiu, Eraker, Foerster, Kim and Seoane (2011) proposed a Bayesian mixed frequency VAR models which are suitable for irregularly sampled data. This kind of models has power for DGPs in which Granger causality acts over varying horizons.

process models that explain trade and quote sequences. For example, Russell (1999) proposed the flexible multivariate autoregressive conditional intensity (MACI) model, in which the intensity takes a log-ARMA structure that explains the clustered nature of tick events (trades and quotes). More recently, Bowsher (2007) generalized the Hawkes model (gHawkes), formerly applied to clustered processes such as earthquakes and neural spikes, to accommodate intraday seasonality and interday dependence features of high frequency TAQ data. Even though structural models exist that predict the existence of Granger causality between trade and quote sequences, the functional forms of the intensity functions are hardly justified by economic theories. Apart from their lack of theoretical foundation, MACI and gHawkes models were often inadequate for explaining all the observed clustering in high frequency data, as evidenced by unsatisfatory goodness-of-fit test results. Model misspecification may potentially bias the result of causal inference. Hence, it would be ideal to have a nonparametric test that provides robust and model-free results on the causal dynamics of the data.

In this paper, I pursue an alternative approach by considering a nonparametric test of Granger causality that does not rely on any parametric assumption and thus is free from the risk of model misspecification. Since I assume no parametric assumptions and only impose standard requirements on kernel functions and smoothing parameters, the conclusion of the test is expected to be more robust than existing techniques. In addition, the nonparametric test in this paper can be regarded as a measure of the strength of Granger causality over different spans as the bandwidth of the weight function varies. Such "impulse response" profile is an indispensible tool in the quest for suitable parametric models.

More importantly, the conclusions from any statistical inference exercise are model specific and have to be interpreted with care. In other words, all interpretations from an estimated model are valid only under the assumption that the parametric model represents the true DGP. For example, in the credit risk literature, there has been ongoing discussion on whether the conditional independence model or the self-exciting clustering model provides a better description of the stylized facts of default data. This is certainly a valid goodness-of-fit problem from the statistical point of view, but it is dangerous to infer that the preferred model represents the true DGP. There may be more than one point process model that can generate the same dataset.[9] The conclusion can entail substantial economic consequences: under a doubly stochastic model, credit contagion is believed to spread through information channels (Bayesian learning on common factors); while under a clustering model, credit contagion is transmitted through direct business links (i.e. counterparty risk exposure). The two families of DGPs are very different in both model forms and economic contents, but they can generate virtually indistinguishable data (Barlett, 1964). Without further assumptions, we are unable to differentiate the two schools of belief solely based on empirical analyses of Granger causality. It is precisely the untestability and non-uniqueness of model assumptions that necessitate a model-free way of uncovering the causal dynamics of a

[9]An example is provided by Barlett (1964), which showed that it is mathematically impossible to distinguish a linear doubly stochastic model and a clustering model with a Poisson parent process and one generation of offsprings (each of which is independently and identically distributed around each parent), as their characteristic functions are identical.

point process.

1.3 Point Processes in High Frequency Finance

Point process models are prevalent in modeling trade and quote tick sequences in high frequency finance. The theoretical motivation comes from the seminal work by Easley and O'hara (1992), who suggested that transaction time is endogenous to stock return dynamics and plays a crucial role in the formation of a dealer's belief in the fundamental stock price. Extending Glosten and Milgrom's (1985) static sequential trade model, their dynamic Bayesian model yields testable implications regarding the relation between trade frequency and the amount of information disseminated to the market, as reflected in the spread and bid/ask quotes set by dealers.

In one of the first empirical analyses, Hasbrouck (1991) applied a discrete vector autoregressive (VAR) model to examine the interaction between trades and quote revisions. Dufour and Engle (2000) extended Hasbrouck's work by considering time duration between consecutive trades as an additional regressor of quote change. They found a negative correlation between a trade-to-trade duration and the next trade-to-quote duration, thus confirming that trade intensity has an impact on the updating of beliefs on fundamental prices.[10]

Given the conjecture of Easley and O'hara (1992) and the empirical evidence of Dufour and Engle (2000), it is important to have a way to extract and model transaction time, which may contain valuable information about the dynamics of quote prices. To this end, Engle and Russell (1998) proposed the Autoregressive Conditional Duration (ACD) model, which became popular for modeling tick data in high frequency finance. It is well known that stock transactions on the tick level tend to cluster over time, and time durations between consecutive trades exhibit strong and persistent autocorrelations. The ACD model is capable of capturing these stylized facts by imposing an autoregressive structure on the time series of trade durations.[11]

A problem with duration models is the lack of a natural multivariate extension due to the unsynchronized nature of trade and quote durations by construction (i.e. a trade duration always starts and ends in the middle of some other quote durations). At the time a trade occurs, the econometrician's information set would be updated to reflect the new trade arrival, but it is difficult to transmit the updated information to the dynamic equation for quote durations, because the current quote duration has not ended yet. The same difficulty arises when information from a new quote arrival needs to be transmitted in the opposite direction to the trade dynamics. Indeed, as argued by Granger (1988, p.206), the problem stems from the fact that durations are flow variables. As a result, it is impossible to identify clearly the causal direction between two flow variable sequences when the flow variables overlap one another in

[10] See Hasbrouck (2007, p.53) for more details.

[11] Existing applications of ACD model to trade and quote data are widespread, including (but not limited to) estimation of price volatility from tick data, testing of market microstructure hypotheses regarding spread and volume and intraday value-at-risk estimation. See Pacurar (2008) for a survey on ACD models.

time dimension.[12] Nevertheless, there exist a number of methods that attempt to get around this problem, such as transforming the tick data to event counts over a prespecified time grid (Heinen and Rengifo, 2007) and redefining trade/quote durations in an asymmetric manner to avoid overlapping of durations (Engle and Lunde, 2003). They are not perfect solutions either.[13]

It is possible to mitigate the information transmission problem in a systematic manner, but this requires a change of viewpoint: we may characterize a multivariate point process from the point of view of *intensities* rather than duration sequences. The *intensity function* of a point process, which is better known as *hazard function* or *hazard rate* for more specific types of point processes in biostatistics, quantifies the event arrival rate at every time instant. Technically, it is the probability that at least one event occurs. While duration is a *flow* concept, event arrival rate is a *stock* concept and thus not susceptible to the information transmission problem. To specify a complete dynamic model for event times, it is necessary to introduce the concept of *conditional intensity function*: the conditional probability of having at least one event at the next instant given the history of the entire multivariate point process up to the present. The dynamics of different type events can be fully characterized by the corresponding conditional intensity functions. Russell (1999), Hautsch and Bauwens (2006), and Bowsher (2007) proposed some prominent examples of intensity models.[14] The objective is to infer the direction and strength of the lead-lag dependence among the marginal point processes from the proposed parametric model.

1.4 Point Processes in Counterparty Risk Modeling

The Granger causality test can be useful to test for the existence of counterparty risk in credit risk analysis. Counterparty risk was first analyzed in a bivariate reduced form model in Jarrow and Yu (2001) and was then extended to multivariate setting by Yu (2007). Under this model, the default likelihood of a firm is directly affected by the default status of other firms. See Appendix A.13 for a summary of the counterparty risk model.

In a related empirical study, Chava and Jarrow (2004) examined if industry effect plays a role in predicting the probability of a firm's bankruptcy. They divided the

[12] As another example, Renault and Werker (2011) tested for a causal relationship between quote durations and price volatility. They assume that tick-by-tick stock returns are sampled from a continuous time Lévy process. Based on the moment conditions implied from the assumptions, they uncovered instantaneous causality from quote update dynamics to price volatility calculated from tick-by-tick returns. Similar criticism on Engle and Lunde (2003) applies to this work as well because trade durations over which volatility is computed overlap with quote durations.

[13] Information about durations is lost under the event count model of Heinen and Rengifo. Data loss problem occurs in the Engle and Lunde model when there are multiple consecutive quote revisions, as only the quote revision immediately after a trade is used. Moreover, the asymmetry of the Engle and Lunde model only allows the detection of trade-to-quote causality but not vice versa.

[14] Russell (1999) estimated a bivariate ACI model to uncover the causal relationship between transaction and limit order arrivals of FedEx from November 1990 to January 1991. With the gHawkes model, Bowsher (2007) provided empirical evidence of significant two-way Granger causality between trade arrivals and mid-quote updates of GM traded on the NYSE over a 40 day span from July 5 to August 29, 2000.

firms into four industrial groups according to SIC codes, and ran a logistic regression on each group of firms. Apart from a better in-sample fit, introducing the industrial factor significantly improves their out-of-sample forecast of bankruptcy events.

A robust line of research uses panel data techniques to study the default risk of firms. The default probabilities of firms are modeled by individual conditional intensity functions. A common way to model dependence of defaults among firms is to include exogenous factors that enter the default intensities of all firms. This type of *conditional independence models*, also known as *Cox models* or *doubly stochastic models*, is straightforward to estimate because the defaults of firms are independent of each other after controlling for exogenous factors. In a log-linear regression, Das, Duffie, Kapadia and Saita (2006, DDKS hereafter) estimate the default probabilities of U.S. firms over a 25 year time span (January 1979 to October 2004) with exogenous factors[15]. However, a series of diagnostic checks unanimously rejects the estimated DDKS model. A potential reason is an incorrect conditional independence assumption, but it could also be due to missing covariates. Their work stimulated future research effort in the pursuit of a more adequate default risk model. As a follow-up, Duffie, Eckners, Horel and Saita (2009) attempt to extend the DDKS model by including additional latent variables. Lando and Nielsen (2010) validate the conditional independence assumption by identifying another exogenous variable (industrial productivity index) and showing that the DDKS model with this additional covariate cannot be rejected.

In view of the inadequacy of conditional independence models, Azizpour, Giesecke and Schwenkler (2008) advocate a top-down approach to modeling corporate bankruptcies: rather than focusing on firm-specific default intensities, they directly model the aggregate default intensity for all firms over time. This approach offers a macroscopic view of default pattern of a portfolio of 6,048 issuers of corporate debts in the U.S.. A key advantage of this approach is that it provides a parsimonious way to model *self-exciting dynamics* which is hard to incorporate in the DDKS model. The authors showed that the self-exciting mechanism effectively explains a larger portion of default clustering. Idiosyncratic components such as firm-specific variables may indirectly drive the dynamics of the default process through the self-exciting mechanism.

1.5 Test of Dependence between two stochastic processes

Various techniques that test for the dependence between two stochastic processes are available. They are particularly well studied when the processes are time series in discrete time. Inspired by the seminal work of Box and Pierce (1970), Haugh (1976) derives the asymptotic distribution of the residual cross-correlations between two independent covariance-stationary ARMA models. A chi-squared test of no cross-correlation up to a fixed lag is constructed in the form of a sum of squared cross-correlations over a finite number of lags. Hong (1996b) generalizes Haugh's test by considering a weighted sum of squared cross-correlations over all possible lags, thereby ensuring consistency against all linear alternatives with significant cross-correlation at any lag. A similar

[15]They include macroeconomic variables such as three-year Treasury yields and trailing one year return of S&P500 index, and firm-specific variables such as distance to default and trailing one year stock return.

test of serial dependence was developed for dynamic regression models with unknown forms of serial correlations (Hong, 1996a).

In the point process literature, there exist similar tests of no cross-correlation. Cox (1965) proposes an estimator of the second-order intensity function of a univariate stationary point process and derived the first two moments of the estimator when the process is a Poisson process. Cox and Lewis (1972) extend the estimator to a bivariate stationary point process framework. Brillinger (1976) derives the pointwise asymptotic distribution of the second-order intensity function estimator when the bivariate process exhibits no cross-correlation and satisfies certain mixing conditions. Based on these theoretical results, one can construct a test statistic in the form of a (weighted) summation of the second-order intensity estimator over a countable number of lags. Under the null of no cross-correlations, the test statistic has an asymptotic standard normal distribution. Doss (1991) considers the same testing problem but proposes using the distribution function analog to the second-order intensity function as a test statistic. Under a different set of moment and mixing conditions, he shows that this test is more efficient than Brillinger's test while retaining asymptotic normality. Similar to the work of Brillinger, Doss' asymptotic normality result holds in a pointwise sense only. The users of these tests are left with the task of determining the grid of lags to evaluate the intensity function estimator. The grid of lags must be sparse enough to ensure independence so that central limit theorem is applicable, but not too sparse as to leave out too many alternatives. For the test considered in this paper, such concern is removed because the test statistic is in the form of a weighted integration over a continuum of lags up to the sample length.

2 Bivariate Point Process

The bivariate point process $\mathbf{\Pi}$ consists of two sequences of event time $0 < \tau_1^k \leq \tau_2^k \leq \ldots < \infty$ $(k = a, b)$ on the positive real line $\mathbb{R}_+$, where τ_i^k represents the time at which the i^{th} event of type k occurs. Another representation of the event time sequences is the bivariate counting process $\mathbf{N} = (N^a, N^b)'$, with the marginal counting process for type k events defined by $N^k(B) = \sum_{i=1}^{\infty} 1\{\tau_i^k \in B\}$, $k = a, b$, for any set B on $\mathbb{R}_+$. Let $N_t^k = N^k((0,t])$ for all $t > 0$ and $N_0^k = 0$, $k = a, b$. It is clear that both representations are equivalent - from a trajectory of $\mathbf{N}$ one can recover that of $\mathbf{\Pi}$ and vice versa; hence, for notational simplicity, the probability space for both $\mathbf{\Pi}$ and $\mathbf{N}$ is denoted by (Ω, P).

First, I suppose that the bivariate counting process $\mathbf{N}$ satisfies the following assumption:

Assumption (A1) The *pooled* counting process $N \equiv N^a + N^b$ is simple, that is $P(N(\{t\}) = 0 \text{ or } 1 \text{ for all } t) = 1$.

Essentially, assumption (A1) means that, almost surely, there is at most one event happening at any time point, and if an event happens, it can either be a type a or type b event, but not both. In other words, the pooled counting process N, which counts the number of events over time regardless of event types, is a monotonic increasing piecewise constant random function which jumps by exactly one at countable number of time points or otherwise stays constant at integer values. As it turns out, this simple property imposed on the pooled counting process plays a crucial role in simplifying the computation of moments of the test statistic. More importantly, the Bernoulli nature of the increments dN_t (which is either zero or one almost surely) of N at time t implies that if two increments dN_s and dN_t ($s \neq t$) are uncorrelated, then they must be *independent*.[16] Therefore, a statistical test that checks for zero cross correlation between any pair of increments of N^a and N^b is sufficient for testing for pairwise independence between the increments.

In theory, assumption (A1) is mild enough to include a wide range of bivariate point process models. It is certainly satisfied if events happen randomly and independently of each other over a continuum (i.e. when the pooled point process is a Poisson process). Also, the assumption is often imposed on the pooled process of many other bivariate point process models that are capable of generating dependent events (e.g. doubly stochastic models, bivariate Hawkes models, bivariate autoregressive conditional intensity models). In practice, however, it is not uncommon to have events happening at exactly the same time point. In many cases, this is the artifact of recording or collecting point process data over a discrete time grid that is too coarse.[17] In some other cases, multiple events really happen at the same time. Given a fixed time resolution, it is impossible to tell the difference between the two cases.[18] There are two ways to get around this conundrum: I may either drop assumption (A1) and include a bigger family of models (e.g. compound Poisson processes), or keep the assumption but lump multiple events at the same time point into a single event. In this paper, I would adopt the latter approach by keeping assumption (A1) and treating multiple events at the same time point as a single event, so that an occurrence of a type k event is interpreted as an occurrence of at least one type k event at that time point. In the datasets of empirical applications, the proportions that events of different types occur simultaneously turn out to be small or even zero by construction.[19]

I can as well replace assumption (A1) by the assumption:

Assumption (A1b) the *pooled* counting process $N \equiv N^a + N^b$ is orderly, that is $P(N((0, s]) \geq 2) = o(s)$ as $s \downarrow 0$.

[16] If two random variables X and Y are uncorrelated, it does not follow in general that they are statistically independent. However, there are two exceptions: one is when (X, Y) follows a bivariate normal distribution, another is when X and Y are Bernoulli distributed.

[17] For instance, in a typical TAQ dataset, timestamps for trades and quote revisions are accurate up to a second. There is a considerable chance that more than two transactions or quote revisions happen within a second. This is at odds with assumption (A1).

[18] TAQ datasets recorded with millisecond timestamps are available more recently. The improvement in resolution of timestamps mitigates the conflict with assumption (A1) by a large extent. A comparison with the TAQ datasets with timestamps in seconds can reveal whether a lump of events in the latter datasets is indeed the case or due to discrete time recording.

[19] Among all trades and quote revisions of PG (GM) from 1997/8/4 to 1997/9/30 in the TAQ data, 3.6% (2.6%) of them occur within the same second. In the bankruptcy data ranging from January 1980 to June 2010, the proportion of cases in which bankruptcies of a manufacturing related firm and a financial related firm occur on the same date is 4.9% (out of a total of 892 cases). In the international financial contagion data, the proportions are all 0% because I intentionally pair up the leading indices of different stock markets which are in different time zones.

It can be shown that with the second-order stationarity of N (see assumption (A2) to be stated later), assumptions (A1) and (A1b) are equivalent (Daley and Vere-Jones, 2003).

It is worth noting that assumptions (A1) and (A1b) are imposed on the pooled counting process N, and thus stronger than if they were imposed on the marginal processes N^a and N^b instead, because simple (or orderly) property of marginal counting processes does not carry over to the pooled counting process. For instance, if N^a is simple (or orderly) and $N^b \equiv N^a$ for each trajectory, then $N = N^a + N^b = 2N^a$ is not.

To make statistical inference possible, some sort of time homogeneity (i.e. stationarity) condition is necessary. Before discussing stationarity, let us define the *second-order factorial moment measure* as

$$G^{ij}(B_1 \times B_2) = E\left[\int_{B_2}\int_{B_1} 1_{\{t_1 \neq t_2\}} dN^i_{t_1} dN^j_{t_2}\right],$$

for $i, j = a, b$ (see Daley and Vere-Jones, 2003, section 8.1). Note that the indicator $1_{\{t_1 \neq t_2\}}$ is redundant if the pooled process of N is simple (assumption (A1)). The concept of second-order stationarity can then be expressed in terms of the second-order factorial moment measure $G^{ij}(\cdot, \cdot)$.

Definition 1 *A bivariate counting process* $\mathbf{N} = (N^a, N^b)'$ *is second-order stationary if*

(i) $G^{ij}((0,1]^2) = E\left[N^i((0,1])N^j((0,1])\right] < \infty$ *for all* $i, j = a, b$*; and*

(ii) $G^{ij}((B_1 + t) \times (B_2 + t)) = G^{ij}(B_1 \times B_2)$ *for all bounded Borel sets* B_1*,* B_2 *in* $\mathbb{R}_+$ *and* $t \in \mathbb{R}_+$*.*

The analogy to the stationarity concept in time series is clear from the above definition, which requires that the second-order (auto- and cross-) moments exist and that the second-order factorial moment measure is shift-invariant. By the shift-invariance property, the measure $G^{ij}(\cdot, \cdot)$ can be reduced to a function of one argument, say $\breve{G}^{ij}(\cdot)$, as it depends only on the time difference of the component point process increments. If $\ell(\cdot)$ denotes the Lebesgue measure, then second-order stationarity of $\mathbf{N}$ implies that, for any bounded measurable functions f with bounded support, the following decomposition is valid:

$$\int_{\mathbb{R}^2} f(s,t) G^{ij}(ds, dt) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x, x+u)\ell(dx)\, \breve{G}^{ij}(du).$$

From the moment condition in Definition 1 (i), second-order stationarity implies that the first-order moments exist by Cauchy-Schwarz inequality, so that

$$\lambda^k \equiv E\left[N^k((0,1])\right] < \infty \tag{1}$$

for $k = a, b$. This is an integrability condition on N^k which ensures that events are not too closely packed together. Often known as *hazard rate* or *unconditional intensity*, the quantity λ^k gives the mean number of events from the component process N^k over a unit interval. Given stationarity, the unconditional intensity defined in (1) also satisfies $\lambda^k = \lim_{\Delta t \downarrow 0} (\Delta t)^{-1} P(N^k((t, t+\Delta t]) > 0)$. If I further assume that N^k is simple, then $\lambda = \lim_{\Delta t \downarrow 0} (\Delta t)^{-1} P(N^k((t, t+\Delta t]) = 1) = E(dN^k_t/dt)$, which is the mean occurrence rate of events at any time instant t, thus justifying the name *intensity*.

Furthermore, if the reduced measure $\breve{G}^{ij}(\cdot)$ is absolutely continuous, then the reduced form factorial product densities $\varphi^{ij}(\cdot)$ $(i, j = a, b)$ exist, so that, in differential form, $\breve{G}^{ij}(d\ell) = \varphi^{ij}(\ell)\, d\ell$. It is important to note that the factorial product density function $\varphi^{ij}(\ell)$ is not symmetric about zero unless $i = j$. Also, the reduced form auto-covariance (when $i = j$) and cross-covariance (when $i \neq j$) density functions of $\mathbf{N}$ are well-defined:

$$c^{ij}(\ell) \equiv \varphi^{ij}(\ell) - \lambda^i \lambda^j \tag{2}$$

for $i, j = a, b$.

The assumptions are summarized as follows:

Assumption (A2) The bivariate counting process $\mathbf{N} = (N^a, N^b)$ is *second-order stationary* and that the *second-order reduced product densities* $\varphi^{ij}(\cdot)$ $(i, j = a, b)$ exist.

Analogous to time series modeling, there is a strict stationarity concept: a bivariate process $\mathbf{N} = (N^a, N^b)$ is *strictly stationary* if the joint distribution of $\{\mathbf{N}(B_1 + u), \ldots, \mathbf{N}(B_r + u)\}$ does not depend on u, for all bounded Borel sets B_i on $\mathbb{R}^2$, $u \in \mathbb{R}^2$ and integers $r \geq 1$. Provided that the second-order moments exist, strict stationarity is stronger than second-order stationarity.

While the simple property is imposed on the pooled point process in assumption (A1), second-order stationarity is required for the bivariate process in assumption (A2). Suppose instead that only the pooled counting process is assumed second-order stationary. It does not follow that the marginal counting processes are second-order stationary too.[20]

The assumption of second-order stationarity on $\mathbf{N}$ ensures that the mean and variance of the test statistic (to be introduced in (13)) are finite under the null hypothesis of no causality (in (11)), but in order to show asymptotic normality I need to assume the existence of fourth-order moments for each component process, as follows:

Assumption (A6) $E\left[\{N^k(B_1)N^k(B_2)N^k(B_3)N^k(B_4)\}\right] < \infty$ for $k = a, b$ and for all bounded Borel sets B_i on $\mathbb{R}_+$, $i = 1, 2, 3, 4$.

Fourth-order moment condition is typical for invoking central limit theorems. In a related work, David (2008) imposes a much stronger assumption of Brillinger-mixing, which essentially requires the existence of all moments of the point process over bounded intervals.

Before proceeding, let me introduce another important concept: the *conditional intensity* of a counting process:

[20] For instance, if $N = N^a + N^b$ is second-order stationary, and if we define $N^a_t = N\left(\cup_{i \geq 0} (2i, 2i+1] \cap (0, t]\right)$ and $N^b_t = N_t - N^a_t$, then N^a and N^b are clearly not second-order stationary. The statement is still valid if second-order stationarity is replaced by strict stationarity.

Definition 2 *Given a filtration*[21] $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$, *the* $\mathcal{G}-$*conditional intensity* $\lambda(t|\mathcal{G}_{t^-})$ *of a univariate counting process* $\breve{N} = \left(\breve{N}_t\right)_{t\geq 0}$ *is any* $\mathcal{G}$*-measurable stochastic process such that for any Borel set* B *and any* $\mathcal{G}_t$*-measurable function* C_t*, the following condition is satisfied:*

$$E\left[\int_B C_t d\breve{N}_t\right] = E\left[\int_B C_t \lambda(t|\mathcal{G}_{t^-})dt\right]. \tag{3}$$

It can be shown (Brémaud, 1981) that the $\mathcal{G}$-conditional intensity $\lambda(t|\mathcal{G}_{t^-})$ is unique almost surely if those $\lambda(t|\mathcal{G}_{t^-})$ that satisfy (3) are required to be $\mathcal{G}$-predictable. In the rest of the paper, I will assume predictability for all conditional intensity functions (see assumption (A3) at the end of this section).

Similar to unconditional intensity, we can interpret the conditional intensity at time t of a simple counting process $\breve{N}$ as the mean occurrence rate of events given the history $\mathcal{G}$ just before time t, as $\lambda(t|\mathcal{G}_{t^-}) = \lim_{\Delta t\downarrow 0} (\Delta t)^{-1} P(\breve{N}((t, t+\Delta t]) > 0|\mathcal{G}_{t^-}) = \lim_{\Delta t\downarrow 0} (\Delta t)^{-1} P(\breve{N}((t, t+\Delta t]) = 1|\mathcal{G}_{t^-}) = E(d\breve{N}_t/dt|\mathcal{G}_{t^-})$, P-almost surely[22], where the second equality follows from (A1).

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the bivariate counting process $\mathbf{N}$, i.e. , and $\mathcal{F}^k = (\mathcal{F}^k_t)_{t\geq 0}$ $(k = a, b)$ be the natural filtration of N^k, so that $\mathcal{F}_t$ and $\mathcal{F}^k_t$ are the sigma fields generated by the processes $\mathbf{N}$ and N^k on $[0, t]$, i.e. $\mathcal{F}_t = \sigma\{(N^a_s, N^b_s), 0 \leq s \leq t\}$ and $\mathcal{F}^k_t = \sigma\{N^k_s : s \in [0, t]\}$. Clearly, $\mathcal{F} = \mathcal{F}^a \vee F^b$. Let $\lambda^k(t|\mathcal{F}_{t^-})$ be the $\mathcal{F}$-conditional intensity of N^k_t, and define the error process by

$$e^k_t := N^k_t - \int_0^t \lambda^k(s|\mathcal{F}_{s^-})ds \tag{4}$$

for $k = a, b$.

By Doob-Meyer decomposition, the error process e^k_t is an $\mathcal{F}$-martingale process, in the sense that $E\left(e^k_t|\mathcal{F}_s\right) = e^k_s$ for all $t > s \geq 0$. The integral $\Lambda_t = \int_0^t \lambda^k(s|\mathcal{F}_{s^-})ds$ as a process is called the $\mathcal{F}$*-compensator* of N^k_t which always exists by Doob-Meyer decomposition, but the existence of $\mathcal{F}$-conditional intensity $\lambda^k(t|\mathcal{F}_{t^-})$ is not guaranteed unless the compensator is *absolutely continuous*. For later analyses, I will assume the existence of $\lambda^k(t|\mathcal{F}_{t^-})$ (see assumption (A3) at the end of this section).

I can express (4) in differential form:

$$de^k_t = dN^k_t - \lambda^k(t|\mathcal{F}_{t^-})dt = dN^k_t - E(dN^k_t|\mathcal{F}_{t^-})$$

for $k = a, b$. From the martingale property of e^k_t, it is then clear that the differential de^k_t is a mean-zero martingale process. In particular, $E\left(de^k_t|\mathcal{F}_{t^-}\right) = 0$ for all $t > 0$. In other words, based on the bivariate process history $\mathcal{F}_{t^-}$ just before time t, an econometrician can obtain the $\mathcal{F}$-conditional intensities $\lambda^a(t|\mathcal{F}_{t^-})$ and $\lambda^b(t|\mathcal{F}_{t^-})$ which are computable just before time t (recall that $\lambda^k(t|\mathcal{F}_{t^-})$ is $\mathcal{F}$-predictable) and give the best prediction of the bivariate counting process $\mathbf{N}$ at time t. Since by (A1) the term $\lambda^k(t|\mathcal{F}_{t^-})dt$

[21] All filtrations in this paper satisfy the usual conditions in Protter (2004).

[22] In the rest of the paper, all equalities involving conditional expectations hold in an almost surely sense.

becomes the conditional mean of dN_t^k, the prediction is the best in the mean square sense.

One may wonder whether it is possible to achieve equally accurate prediction of $\mathbf{\Pi}$ with a reduced information set. For instance, can we predict dN_t^b equally well with its $\mathcal{F}^b$-conditional intensity $\lambda^b(t|\mathcal{F}_{t-}^b)$, where $\lambda^b(t|\mathcal{F}_{t-}^b)dt = E(dN_t^b|\mathcal{F}_{t-}^b)$, instead of its $\mathcal{F}$-conditional intensity $\lambda^b(t|\mathcal{F}_{t-})$? Through computing the $\mathcal{F}^b$-conditional intensity, we attempt to predict the value of N^b solely based on the history of N^b. Without using the history of N^a, the prediction $\lambda^b(t|\mathcal{F}_{t-}^b)dt$ ignores the feedback or causal effect that shocks to N^a in the past may have on the future dynamics of N^b. One would thus expect the answer to the previous question is no in general. Indeed, given that $\mathbf{\Pi}$ is in the filtered probability space $(\Omega, P, \mathcal{F})$, the error process

$$\epsilon_t^b := N_t^b - \int_0^t \lambda^b(s|\mathcal{F}_{s-}^b)ds \tag{5}$$

is no longer an $\mathcal{F}$-martingale. However, ϵ_t^b *is* an $\mathcal{F}$-martingale under one special circumstance: when the $\mathcal{F}^b$- and $\mathcal{F}$-conditional intensities

$$\lambda^b(t|\mathcal{F}_{t-}^b) = \lambda^b(t|\mathcal{F}_{t-})$$

are the same for all $t > 0$. I am going to discuss this circumstance in depth in the next section.

Let me summarize the assumptions in this section:

Assumption (A3) The $\mathcal{F}$-conditional intensity $\lambda^k(t|\mathcal{F}_{t-})$ and $\mathcal{F}^k$-conditional intensity $\lambda_t^k \equiv \lambda^k(t|\mathcal{F}_{t-}^k)$ of the counting process N_t^k exist and are predictable.

3 Granger Causality

In this section, I am going to discuss the concept of Granger causality in the bivariate counting process set-up described in the previous section. Assuming (A1), (A2) and (A3), and with the notations in the previous section, we say that N^a *does not Granger-cause* N^b if the $\mathcal{F}$-conditional intensity of N^b is identical to the $\mathcal{F}^b$-conditional intensity of N^b. That is, for all $t > s \geq 0$, P-almost surely,

$$E[dN_t^b|\mathcal{F}_s] = E[dN_t^b|\mathcal{F}_s^b] \tag{6}$$

A remarkable result, as proven by Florens and Fougere (1996, section 4, example I), is the following equivalence statement in the context of simple counting processes.

Theorem 3 *If N^a and N^b are simple counting processes, then the following four definitions of Granger noncausality are equivalent:*

1. *N^a does not weakly globally cause N^b, i.e. $E[dN_t^b|\mathcal{F}_s] = E[dN_t^b|\mathcal{F}_s^b]$, P-a.s. for all s, t.*

2. N^a *does not strongly globally cause* N^b, *i.e.* $\mathcal{F}_t^b \perp \mathcal{F}_s | \mathcal{F}_s^b$ *for all* s, t.

3. N^a *does not weakly instantaneously cause* N^b, *i.e.* N^b, *which is an* $\mathcal{F}^b$*-semi-martingale with decomposition* $dN_t^b = d\epsilon_t^b + E[dN_t^b | \mathcal{F}_{t-}^b]$, *remains an* $\mathcal{F}$*-semi-martingale with the same decomposition.*

4. N^a *does not strongly instantaneously cause* N^b, *i.e. any* $\mathcal{F}^b$*-semi-martingale with decomposition remains an* $\mathcal{F}$*-semi-martingale with the same decomposition.*

According to the theorem, weakly global noncausality is equivalent to weakly instantaneous noncausality, and hence testing for (6) is equivalent to checking ϵ_t^b defined in (5) is an $\mathcal{F}$-martingale process, or, checking $d\epsilon_t^b$ is an $\mathcal{F}$-martingale difference process:

$$E[d\epsilon_t^b | \mathcal{F}_s] = 0 \tag{7}$$

for all $0 \leq s < t$.

If one is interested in testing for pairwise dependence only, then (7) implies

$$E\left[f\left(d\epsilon_s^a\right) d\epsilon_t^b\right] = 0 \tag{8}$$

and

$$E\left[f\left(d\epsilon_s^b\right) d\epsilon_t^b\right] = 0 \tag{9}$$

for all $0 \leq s < t$ and any $\mathcal{F}^a$-measurable function $f(\cdot)$. However, since ϵ_t^b is an $\mathcal{F}^b$-martingale by construction, condition (9) is automatically satisfied and thus is not interesting from testing's point of view as long as the conditional intensity $\lambda^b(t | \mathcal{F}_{t-}^b)$ is computed correctly.

There is a loss of generality to base a statistical test on (8) instead of (7), as it would miss the alternatives in which a type b event is not Granger-caused by the occurrence (or non-occurrence) of any single type a event at a past instant, but is Granger-caused by the occurrence (or non-occurrence) of multiple type a events jointly at multiple past instants or over some past intervals.[23]

I can simplify the test condition (8) further. Due to the dichotomous nature of $d\epsilon_t^a$, it suffices to test

$$E\left[d\epsilon_s^a d\epsilon_t^b\right] = 0 \tag{10}$$

for all $0 \leq s < t$, as justified by the following lemma.

Lemma 4 *If* N^a *and* N^b *are simple counting processes, then (8) and (10) are equivalent.*

[23] One hypothetical example in default risk application is given as follows. Suppose I want to detect whether corporate bankruptcies in industry a Granger-cause bankruptcies in industry b. Suppose also that there were three consecutive bankruptcies in industry a at times s_1, s_2 and s_3, followed by a bankruptcy in industry b at time t ($s_1 < s_2 < s_3 < t$). Each bankruptcy in industry a alone would not be significant enough to influence the well-being of the companies in industry b, but three industry a bankruptcies may jointly trigger an industry b bankruptcy. It is possible that a test based on (8) can still pick up such a scenario, depending on the way the statistic summarizes the information of (8) for all $0 \leq s_i < t$.

Proof. The implication from (8) to (10) is trivial by taking $f(\cdot)$ to be the identity function. Now assuming that (10) holds, i.e. $Cov(d\epsilon_s^a, d\epsilon_t^b) = 0$. Given that N^a and N^b are simple, $dN_s^a|\mathcal{F}_{s-}^a$ and $dN_t^b|\mathcal{F}_{t-}^b$ are Bernoulli random variables (with means $\lambda^a(s|\mathcal{F}_{s-}^a)ds$ and $\lambda^b(t|\mathcal{F}_{t-}^b)dt$, respectively), and hence zero correlation implies independence, i.e. for all measurable functions $f(\cdot)$ and $g(\cdot)$, we have $Cov\left[f(d\epsilon_s^a), g(d\epsilon_t^b)\right] = 0$. We thus obtain (8) by taking $g(\cdot)$ to be the identity function. ■

Thanks to the simple property of point process assumed in (A1), two innovations $d\epsilon_s^a$ and $d\epsilon_t^b$ are pairwise cross-independent if they are not pairwise cross-correlated by Lemma 4. In other words, a suitable linear measure of cross-correlation between the residuals from two component processes would suffice to test for their pairwise cross-independence (both linear and nonlinear), as each infinitesimal increment takes one out of two values almost surely. From testing's point of view, a continuous time framework justifies the simple property of point processes (assumption (A1)) and hence allows for a simpler treatment on the nonlinearity issue, as assumption (A1) gets rid of the possibility of nonlinear dependence on the infinitesimal level. Indeed, if a point process $\breve{N}$ is simple, then $d\breve{N}_t$ can only take values zero (no jump at time t) or one (a jump at time t), and so $\left(d\breve{N}_t\right)^p = d\breve{N}_t$ for any positive integers p. Without assumption (A1), the test procedure would still be valid (to be introduced in section 4, with appropriate adjustments to the mean and variance of the test statistic), but it would just check for an implication of pairwise Granger noncausality, as the equivalence of (8) and (10) would be lost.

Making sense of condition (10) requires a thorough understanding of the conditional intensity concept and its relation to Granger causality. From Definition 2, it is crucial to specify the filtration with respect to which the conditional intensity is adapted. The $\mathcal{G}$-conditional intensity can be different depending on the choice of the filtration $\mathcal{G}$. If $\mathcal{G} = \mathcal{F} = \mathcal{F}^a \vee F^b$, then the $\mathcal{G}$-conditional intensity is evaluated with respect to the history of the whole bivariate counting process $\mathbf{N}$. If instead $\mathcal{G} = \mathcal{F}^k$, then it is evaluated with respect to the history of the marginal point process N^k only.

From the definition of weakly instantaneous noncausality in Theorem 3, Granger-noncausality for point processes is the property that the conditional intensity is invariant to an enlargement of the conditioning set from the natural filtration of the marginal process to that of the bivariate process. More specifically, if the counting process N^a does not Granger-cause N^b, then we have

$$E[dN_t^b|\mathcal{F}_{t-}] = E[dN_t^b|\mathcal{F}_{t-}^b]$$

for all $t > 0$, which conforms to the intuition of Granger causality that the predicted value of N_t^b given its history remains unchanged with or without the additional information of the history of N^a by time t. Condition (10), on the other hand, means that any past innovation $d\epsilon_s^a = dN_s^a - E[dN_s^a|\mathcal{F}_{s-}^a]$ of N^a is *independent of* (not merely uncorrelated with, due to the Bernoulli nature of jump sizes for simple point processes according to Lemma 4) the future innovation $d\epsilon_t^b = dN_t^b - E[dN_t^b|\mathcal{F}_{t-}^b]$ of N^b $(t > s)$. This is exactly the implication of Granger noncausality from N^a to N^b, and except for those loss-of-generality cases discussed underneath (9), the two statements are equivalent.

Assuming (A2) and (A3), the *reduced form cross covariance density function* of the innovations $d\epsilon_t^a$ and $d\epsilon_t^b$ is then well-defined, and is denoted by $\gamma(\ell)\,dtd\ell = E\left(d\epsilon_t^a d\epsilon_{t+\ell}^b\right)$. The null hypothesis of interest can thus be written down formally as follows:

$$\begin{aligned} \mathbf{H}_0 &: \gamma(\ell) = 0 \text{ for all } \ell > 0 \quad \text{vs} \\ \mathbf{H}_1 &: \gamma(\ell) \neq 0 \text{ for some } \ell > 0. \end{aligned} \tag{11}$$

It is important to distinguish the reduced form cross-covariance density function $\gamma(\ell)$ of the innovations $d\epsilon_t^a$ and $d\epsilon_t^b$ from the cross-covariance density function $c^{ab}(\ell)$ of the counting process $\mathbf{N} = (N^a, N^b)$, defined earlier in (2). The key difference rests on the way the jumps are demeaned: the increment dN_t^k at time t is compared against the conditional mean $\lambda^k(t|\mathcal{F}_{t-}^k)dt$ in $\gamma(\ell)$, but it is compared against the unconditional mean $\lambda^k dt$ in $c^{ab}(\ell)$. In this sense, the former $\gamma(\ell)$ captures the *dynamic* feedback effect as reflected in the shocks of the component processes, but the latter $c^{ab}(\ell)$ merely summarizes the *static* correlation relationship between the jumps of component processes. Indeed, valuable information of Granger causality between component processes is only contained in $\gamma(\ell)$ (as argued earlier in this section) but not in $c^{ab}(\ell)$. Previous research focused mostly on the large sample properties of estimators of the static auto-covariance density function $c^{kk}(\ell)$ or cross-covariance density function $c^{ab}(\ell)$. This paper, however, is devoted to the analysis of the dynamic cross-covariance density function $\gamma(\ell)$. As we will see, the approach in getting asymptotic properties of $\gamma(\ell)$ is quite different. I will apply the *martingale central limit theorem* - a dynamic version of the ordinary central limit theorem - to derive the sampling distribution of a test statistic involving estimators of $\gamma(\ell)$.

4 The statistic

The econometrician observes two event time sequences of a simple bivariate stationary point process $\mathbf{\Pi}$ over the time horizon $[0, T]$, namely, $0 < \tau_1^k < \tau_2^k < \cdots < \tau_{N^k(T)}^k$ for $k = a, b$. This is the dataset required to calculate the test statistic to be constructed in this section.

4.1 Nonparametric cross-covariance estimator

In this section, I am going to construct a statistic for testing condition (10) from the data. One candidate for the lag ℓ sample cross-covariance $\gamma(\ell)$ of the innovations $d\epsilon_t^a$ and $d\epsilon_t^b$ is given by

$$\hat{C}(\ell)d\ell = \frac{1}{T}\int_0^T d\hat{\epsilon}_t^a d\hat{\epsilon}_{t+\ell}^b$$

where $d\hat{\epsilon}_t^k = dN_t^k - \hat{\lambda}_t^k dt$ ($k = a, b$) is the residual and $\hat{\lambda}_t^k$ is some local estimator of the $\mathcal{F}^k$-conditional intensity λ_t^k in (A3) (to be discussed in section 4.4). The integration is done with respect to t. However, if the jumps of N^k are finite or countable (which is the case for point processes satisfying (A2)), the product of increments $dN_t^a dN_{t+\ell}^b$ is zero almost everywhere except over a set of P-measure zero, so that $\hat{C}(\ell)$ is inconsistent for $\gamma(\ell)$. This suggests that some form of local smoothing is necessary. The problem is analogous to the probability density function estimation in which the empirical density estimator would be zero almost everywhere over the support if there were no smoothing. This motivates the use of a kernel function $K(\cdot)$, with a bandwidth H which controls the degree of smoothing applied to the sample cross-covariance estimator $\hat{C}(\ell)$ above. To simplify notation, let $K_H(x) = K(x/H)/H$. The corresponding kernel estimator is given by

$$\begin{aligned} \hat{\gamma}_H(\ell) &= \frac{1}{T}\int_0^T \int_0^T K_H(t-s-\ell)\, d\hat{\epsilon}_s^a d\hat{\epsilon}_t^b \\ &= \frac{1}{T}\int_0^T \int_0^T K_H(t-s-\ell)\left(dN_s^a - \hat{\lambda}_s^a ds\right)\left(dN_t^b - \hat{\lambda}_t^b dt\right). \end{aligned} \tag{12}$$

The kernel estimator is the result of averaging the weighted products of innovations $d\hat{\epsilon}_s^a$ and $d\hat{\epsilon}_t^b$ over all possible pairs of time points (s,t). The kernel $K_H(\cdot)$ gives the heaviest weight to the product of innovations at the time difference $t-s=\ell$, and the weight becomes lighter as the time difference is further away from ℓ. The following integrability conditions are imposed on the kernel:

Assumption (A4a) The kernel function $K(\cdot)$ is symmetric around zero and satisfies $\kappa_1 \equiv \int_{-\infty}^{\infty} K(u)du = 1$, $\kappa_2 \equiv \int_{-\infty}^{\infty} K^2(u)du < \infty$, $\kappa_4 \equiv \iiint_{(-\infty,\infty)} K(u)K(v)K(u+w)K(v+w)dudvdw < \infty$ and $\int_{-\infty}^{\infty} u^2 K(u)du < \infty$.

4.2 The statistic as $\mathcal{L}^2$ norm

An ideal test statistic for testing (11) would summarize appropriately all the cross-covariances of residuals $d\hat{\epsilon}_s^a$ and $d\hat{\epsilon}_t^b$ over all $0 \le s < t$. This problem is similar to that of Haugh (1976) when he checked the independence of two time series, but there are two important departures: here I am working with two continuous time point processes instead of discrete time series, and I do not assume any parametric models on the conditional means. To this end, I propose a weighted integral of the squared sample cross-covariance function, defined as follows:

$$Q \equiv \|\hat{\gamma}_H\|_2 \equiv \int_I w(\ell)\hat{\gamma}_H^2(\ell)d\ell. \tag{13}$$

where $I \subseteq [-T,T]$. To test the null hypothesis in (11), the integration range is set to be $I=[0,T]$.

Applying an $\mathcal{L}^2$ norm rather than an $\mathcal{L}^1$ norm on the sample cross-covariance function $\hat{\gamma}_H(\ell)$ is standard in the literature of discrete time serial correlation test. If I decided to test (11) based on

$$\|\hat{\gamma}_H\|_1 \equiv \int_I w(\ell)\hat{\gamma}_H(\ell)d\ell$$

instead, it would lead to excessive type II error - the test would fail to reject those DGP's in which the true cross-covariance function $\gamma(\ell)$ is significantly away from zero for certain $\ell \in I$ but the weighted integral $\|\hat{\gamma}_H\|_1$ is close to zero due to cancellation.

A test based on the test statistic Q in (13) is on the conservative side as Q is an $\mathcal{L}^2$ norm. More specifically, the total causality effect from N^a to N^b is the aggregate of the weighted squared contribution from each individual type a-type b event pair (see Figure A.2). If $E(d\epsilon^a_{s_i} d\epsilon^b_t) = c_i$ then the aggregate causality effect is $\sum_{i=1}^3 c_i^2$ without kernel smoothing. However, less conservative test can be constructed with other choices of norms (e.g. Hellinger and Kullback-Leibler distance) as in Hong (1996a), and the methodology in this paper is still valid with appropriate adjustment.

4.3 Weighting function

I assume that

Assumption (A5) The weighting function $w(\ell)$ is integrable over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} w(\ell) d\ell < \infty.$$

The motivations behind the introduction of the weighting function $w(\ell)$ on lags are in a similar spirit as the test of serial correlation proposed by Hong (1996a) in the discrete time series context. The economic motivation is that the contagious effect from one process to another diminishes over time, as manifested by the property that the weighting function discounts more heavily the sample cross covariance as the time lag increases. From the econometric point of view, by choosing a weighting function whose support covers all possible lags in $I \subseteq [-T, T]$, the statistic Q can deliver a consistent test to (11) against all pairwise cross dependence of the two processes as it summarizes their cross covariances over all lags in an $\mathcal{L}^2$ norm, whereas the statistic with a truncated weighting function over a fixed lag window $I = [c_1, c_2]$ cannot. From the statistical point of view, a weighting function that satisfies (A5) is a crucial device for controlling the variation of the integrated squared cross-covariance function over an expanding lag interval $I = [0, T]$, so that Q enjoys asymptotic normality. It can be shown that the asymptotic normality property would break down without an appropriate weighting function $w(\ell)$ that satisfies (A5).

4.4 Conditional intensity estimator

In this section, I will discuss how to estimate the time-varying $\mathcal{F}^k$-conditional intensity nonparametrically. I employ the following *Nadaraya-Watson estimator* for the $\mathcal{F}^k$-conditional intensity $\lambda^k_t \equiv \lambda^k(t|\mathcal{F}^k_{t^-})$,

$$\hat{\lambda}^k_t = \int_0^T \mathring{K}_M (t-u)\, dN^k_u. \tag{14}$$

While the cross-covariance estimator $\hat{\gamma}_H(\ell)$ is smoothed by the kernel $K(\cdot)$ with bandwidth H, the conditional intensity estimator is smoothed by the kernel $\mathring{K}(\cdot)$ with bandwidth M. The kernel $\mathring{K}(\cdot)$ is assumed to satisfy the following:

Assumption (A4b) The kernel function $\mathring{K}(\cdot)$ is symmetric around zero and satisfies $\mathring{\kappa}_1 \equiv \int_{-\infty}^{\infty} \mathring{K}(u)du = 1$, $\mathring{\kappa}_2 \equiv \int_{-\infty}^{\infty} \mathring{K}^2(u)du < \infty$, $\mathring{\kappa}_4 \equiv \iiint_{(-\infty,\infty)} \mathring{K}(u)\mathring{K}(v)\mathring{K}(u+w)\mathring{K}(v+w)dudvdw < \infty$ and $\int_{-\infty}^{\infty} u^2\mathring{K}(u)du < \infty$.

The motivation of (14) comes from estimating the conditional mean of dN_t^k by a nonparametric local regression. Indeed, the Nadaraya-Watson estimator is the local constant least square estimator of $E(dN_t^k|\mathcal{F}_{t-}^k)$ around time t weighted by $\mathring{K}_M(\cdot)$. (As usual, I denote $\mathring{K}_M(\ell) = \mathring{K}(\ell/M)/M$.) By (A4b) it follows that $\int_0^T \mathring{K}_M(t-u)\,du = 1 + o(1)$ as $M/T \to 0$ and thus the Nadaraya-Watson estimator becomes (14). The estimator (14) implies that the conditional intensity takes a constant value over a local window, but one may readily extend it to a local linear or local polynomial estimator. Some candidates for regressors include the backward recurrence time $t - t_{N_t^k}^k$ of the marginal process N^k, and the backward recurrence time $t - t_{N_t}$ of the pooled process N.

Another way to estimate the $\mathcal{F}^k$-conditional intensity is by fitting a parametric conditional intensity model on each component point process. For $k = a, b$, let $\boldsymbol{\theta}^k \in \mathbb{R}^{d^k}$ be the vector of parameters of the $\mathcal{F}^k$-conditional intensity λ_t^k, which is modeled by

$$\lambda_t^k \equiv \lambda^k\left(t; \boldsymbol{\theta}^k\right)$$

for $t \in [0, \infty)$. Each component model is estimated by some parametric model estimation techniques (e.g. MLE, GMM). The estimator $\boldsymbol{\theta}^k$ converges to $\boldsymbol{\theta}^k$ at the typical parametric convergence rate of $T^{-1/2}$ (or equivalently $\left(n^k\right)^{-1/2} = \left(N_T^k\right)^{-1/2}$), which is faster than the nonparametric rate of $M^{-1/2}$.

4.5 Computation of $\hat{\gamma}_H(\ell)$

To implement the test, it is important to compute the test statistic Q efficiently. From the definition, there are three layers of integrations to be computed: the first layer is the weighted integration with respect to different lags ℓ, a second layer involves two integrations with respect to the component point processes in the cross-covariance function estimator $\hat{\gamma}_H(\ell)$, and a third layer is a single integration with respect to each component process inside the $\mathcal{F}^k$-conditional intensity estimator $\hat{\lambda}_t^k$. The first layer of integration will be evaluated numerically, but it is possible to reduce the second and third layers of integrations to summations over marked event times in the case of Gaussian kernels, thus simplifying a lot the computation of $\hat{\gamma}_H(\ell)$ and hence Q. Therefore, I make the following assumption:

Assumption (A4d) The kernels $K(x)$, $\mathring{K}(x)$ and $\ddot{K}(x)$ are all standard Gaussian kernels. That is: $K(x) = \mathring{K}(x) = \ddot{K}(x) = (2\pi)^{-1/2}\exp(-x^2/2)$.

Theorem 5 *Under assumptions (A1-3, 4a, 4b, 4d), the cross-covariance function estimator $\hat{\gamma}_H(\ell)$ defined in (12) and (14) is given by*

$$\hat{\gamma}_H(\ell) = \frac{1}{T}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}\left[\frac{1}{H}K\left(\frac{t_j^b - t_i^a - \ell}{H}\right) - \frac{2}{\sqrt{H^2+M^2}}K\left(\frac{t_j^b - t_i^a - \ell}{\sqrt{H^2+M^2}}\right) + \frac{1}{\sqrt{H^2+2M^2}}K\left(\frac{t_j^b - t_i^a - \ell}{\sqrt{H^2+2M^2}}\right)\right].$$

4.6 Consistency of conditional intensity estimator

Unlike traditional time series asymptotic theories in which data points are separated by a fixed (but possibly irregular) time lag in an expanding observation window $[0, T]$ (scheme 1), consistent estimation of moments of point processes requires a fixed observation window $[0, T_0]$ in which events grow in number and are increasingly packed (scheme 2). The details of the two schemes are laid out in Table 1.

As we will see shortly, the asymptotic mechanism of scheme 2 is crucial for consistent estimation of the first and second order moments, including the $\mathcal{F}^k$-conditional intensity functions λ_t^k for $k = a, b$, the auto- and cross-covariance density functions $c^{ij}(\cdot)$ of $\mathbf{N}$ (for $i, j = a, b$), as well as the cross-covariance density function $\gamma(\cdot)$ of the innovation processes $d\epsilon_t^k$ for $k = a, b$. However, the limiting processes of scheme 2 would inadvertently distort various moments of $\mathbf{N}$. For instance, the $\mathcal{F}^k$-conditional intensity λ_t^k will diverge to infinity as the number of observed events $n^k = N^k(T_0)$ in a finite observation window $[0, T_0]$ goes to infinity. In contrast, under traditional time series asymptotics (scheme 1) as $T \to \infty$, the moment features of $\mathbf{N}$ are maintained as the event times are fixed with respect to T, but all moment estimators are doomed to be pointwise inconsistent since new information is only added to the right of the process (rather than everywhere over the observation window) as $T \to \infty$.

Let us take the estimation of $\mathcal{F}^k$-conditional intensity function λ_t^k as an example. At first sight, scheme 1 is preferable because the spacing between events is fixed relative to the sample size and we want the conditional intensity λ_t^k at time t to be invariant to the sample size in the limit. However, the estimated $\mathcal{F}^k$-conditional intensity is not pointwise consistent under scheme 1's asymptotics since there are only a fixed and finite number of observations around time t. On the other hand, under scheme 2's asymptotics, the number of observations around any time t increases as the sample grows, thus ensuring consistent estimation of λ_t^k, but as events get more and more crowded in a local window around time t, the $\mathcal{F}^k$-conditional intensity λ_t^k diverges to infinity. [24]

How can we solve the above dilemma? Knowing that there is no hope to estimate λ_t^k consistently at each time t, let us stick to scheme 2, and estimate the moment properties of a rescaled counting process $\tilde{\mathbf{N}}_v = (\tilde{N}_v^a, \tilde{N}_v^b)$,where

$$\tilde{N}_v^k := \frac{N_{Tv}^k}{T} \tag{15}$$

for $k = a, b$ and $v \in [0, 1]$ (a fixed interval, with $T_0 = 1$). The stationarity property

[24] Note the similarity of the problem to probability density function estimation on a bounded support.

of $\tilde{\mathbf{N}}$ and the Bernoulli nature of the increments of the pooled process $\tilde{N} = \tilde{N}^a + \tilde{N}^b$ are preserved.[25] The time change acts as a bridge between the two schemes - the asymptotics of original process $\mathbf{N}$ is governed by scheme 1, while that of the rescaled process $\tilde{\mathbf{N}}$ is governed by scheme 2; and the two schemes are equivalent to one another after rescaling by $1/T$. Indeed, it is easily seen, by a change of variable $t = Tv$, that the conditional intensities of $\tilde{N}^k_v$ and N^k_{Tv} are identical:

$$\begin{aligned}
\lambda^k_{Tv} &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} E\left(N^k_{Tv+\Delta t} - N^k_{Tv} | \mathcal{F}^k_{Tv^-}\right) \\
&= \lim_{\Delta v \downarrow 0} \frac{1}{T\Delta v} E\left(N^k_{T(v+\Delta v)} - N^k_{Tv} | \mathcal{F}^k_{Tv^-}\right) \\
&= \lim_{\Delta v \downarrow 0} \frac{1}{\Delta v} E\left(\tilde{N}^k_{v+\Delta v} - \tilde{N}^k_v | \tilde{\mathcal{F}}^k_{v^-}\right) =: \tilde{\lambda}^k_v, \qquad (16)
\end{aligned}$$

where I denoted the natural filtration of $\tilde{N}^k$ by $\tilde{\mathcal{F}}^k$ and the $\tilde{\mathcal{F}}^k$-conditional intensity function of $\tilde{N}^k_v$ by $\tilde{\lambda}^k_v$ on the last line.

If the conditional intensity $\tilde{\lambda}^k_v$ of the rescaled point process $\tilde{N}^k_v$ is continuous and is an unknown but deterministic function, then it can be consistently estimated for each $v \in [0, 1]$. In the same vein, other second-order moments of $\tilde{\mathbf{N}}$ are well-defined and can be consistently estimated, including the (auto- and cross-) covariance density functions $\tilde{c}^{ij}(\cdot)$ of $\tilde{\mathbf{N}}$ (for $i, j = a, b$) and the cross-covariance density function $\tilde{\gamma}(\cdot)$ of the innovation processes $d\tilde{\epsilon}^k_v := d\tilde{N}^k_v - \tilde{\lambda}^k_v dv$ for $k = a, b$. Specifically, it can be shown that

$$\tilde{\gamma}(\sigma) = \gamma(T\sigma) \qquad (17)$$

and $\tilde{c}^{ij}(\sigma) = c^{ij}(T\sigma)$ for $i, j = a, b$, and consistent estimation is possible for fixed $\sigma \in [0, 1]$.

To show the consistency and asymptotic normality of the conditional intensity kernel estimator $\hat{\lambda}^k_{Tv}$, the following assumption is imposed:

Assumption (A7) The rescaled counting process $\tilde{N}^k_u \equiv N^k_{Tu}/T$ (with natural filtration $\tilde{\mathcal{F}}^k$) has an $\tilde{\mathcal{F}}^k$-conditional intensity function $\tilde{\lambda}^k_u$, which is twice continuously differentiable with respect to u, and is unobservable but deterministic.

Theorem 6 *Given that a bivariate counting process* $\mathbf{N}$ *satisfies assumptions (A1-3,4a,4b,7) and is observed over* $[0, T]$. *Let* $\hat{\lambda}^k_t$ $(k = a, b)$ *be the* $\mathcal{F}^k$*-conditional intensity kernel estimator of the component process* N^k *defined in (14). Assume that* $M^5/T^4 \to 0$ *as* $T \to \infty$, $M \to \infty$ *and* $M/T \to 0$. *Then, for any fixed* $v \in [0, 1]$, *the kernel estimator* $\hat{\lambda}^k_{Tv}$ *converges in mean squares to the conditional intensity* λ^k_{Tv}, *i.e.*

$$E[\left(\hat{\lambda}^k_{Tv} - \lambda^k_{Tv}\right)^2] \to 0,$$

[25] Strictly speaking, the pooled process $\tilde{N}$ of $\tilde{\mathbf{N}}$ is no longer simple because the increment $d\tilde{N}_t$ takes values of either zero or $1/T$ (instead of 1) almost surely, but the asymptotic theory of the test statistic on $\tilde{\mathbf{N}}$ only requires that the increments $d\tilde{N}^k_t$ are Bernoulli distributed with mean $\lambda^k_t dt$.

and the normalized difference

$$\xi_v^k := \sqrt{M}\left(\frac{\hat{\lambda}_{Tv}^k - \lambda_{Tv}^k}{\sqrt{\lambda_{Tv}^k}}\right) \tag{18}$$

converges to a normal distribution with mean 0 *and variance* $\mathring{\kappa}_2 = \int_{-\infty}^{\infty} \mathring{K}(x)dx$*, as* $T \to \infty$*,* $M \to \infty$ *and* $M/T \to 0$*.*

By Theorem 6, it follows that $\hat{\lambda}_{Tv}^k$ is mean-squared consistent and that in the limit $\hat{\lambda}_{Tv}^k - \lambda_{Tv}^k = O_P(M^{-1/2})$ for $k = a, b$.

There is a corresponding kernel estimator of the cross-covariance function $\tilde{\gamma}_h(\cdot)$ of the innovations of the rescaled point process defined in (15). With an appropriate adjustment to the bandwidth, by setting the new bandwidth after rescaling H to $h = H/T$, I can reduce it to $\hat{\gamma}_H(\ell)$. For a fixed $\sigma \in [0,1]$,

$$\begin{aligned}
\hat{\gamma}_H(T\sigma) &= \frac{1}{T}\int_0^T\int_0^T K_H(t - s - T\sigma)\left(dN_s^a - \hat{\lambda}_s^a ds\right)\left(dN_t^b - \hat{\lambda}_t^b dt\right) \\
&= \frac{1}{T}\int_0^1\int_0^1 K_H(T(v - u - \sigma))\left(dN_{Tu}^a - \hat{\lambda}_{Tu}^a T du\right)\left(dN_{Tv}^b - \hat{\lambda}_{Tv}^b T dv\right) \\
&= \frac{T^2}{T}\int_0^1\int_0^1 \frac{1}{H}K\left(\frac{v - u - \sigma}{H/T}\right)\left(d\tilde{N}_u^a - \hat{\tilde{\lambda}}_u^a du\right)\left(d\tilde{N}_v^b - \hat{\tilde{\lambda}}_v^b dv\right) \\
&= \int_0^1\int_0^1 K_h(v - u - \sigma)\, d\hat{\tilde{\epsilon}}_u^a d\hat{\tilde{\epsilon}}_v^b =: \hat{\tilde{\gamma}}_h(\sigma).
\end{aligned}$$

For a fixed lag $\sigma \in [0,1]$, the kernel cross-covariance estimator $\hat{\tilde{\gamma}}_h(\sigma)$ consistently estimates $\tilde{\gamma}(\sigma)$ as $n^k = \tilde{N}^k(1) \to \infty$, $h \to 0$ and $n^k h \to \infty$ for $k = a, b$.

The statistic Q can thus be expressed in terms of the squared sample cross-covariance function of the rescaled point process defined in (15) with rescaled bandwidths. Assuming that the weighting function is another kernel with bandwidth B, i.e. $w(\ell) = w_B(\ell)$, I can rewrite Q into

$$\begin{aligned}
Q &= \int_I w_B(\ell)\hat{\gamma}_H^2(\ell)d\ell \\
&= T\int_{I/T} w_B(T\sigma)\hat{\gamma}_H^2(T\sigma)d\sigma \\
&= \int_{I/T} w_b(\sigma)\hat{\tilde{\gamma}}_h^2(\sigma)d\sigma,
\end{aligned}$$

where $b = B/T$ and $h = H/T$.

4.7 Simplified Statistic

Another statistic that deserves our study is

$$Q^s = \frac{1}{T^2} \int_I \int_J w_B(\ell)\, dN_s^a dN_{s+\ell}^b.$$

where $I \subseteq [-T, T]$ and $J = [-\ell, T-\ell] \cap [0, T]$ are the ranges of integration with respect to ℓ and s, respectively. In fact, this statistic is the continuous version of the statistic of Cox and Lewis (1972), whose asymptotic distribution was derived by Brillinger (1976). Both statistics find their root in the serial correlation statistic for univariate stationary point process (Cox, 1965). Instead of the continuous weighting function $w(\ell)$, they essentially considered a discrete set of weights on the product increments of the counting processes at a prespecified grid of lags, which are separated wide enough to guarantee the independence of the product increments when summed together.

To quantify how much we lose with the simplified statistic, let us do a comparison between Q^s and Q. If the pooled point process is simple (assumption (A1)), then the statistic Q^s is equal to, almost surely,

$$Q^s = \frac{1}{T^2} \int_I \int_J w_B(\ell) \left(d\hat{\epsilon}_s^a\right)^2 \left(d\hat{\epsilon}_{s+\ell}^b\right)^2,$$

which is the weighted integral of the squared product of residuals.[26] On the other hand, observe that there are two levels of smoothing in Q: the sample cross covariance $\hat{\gamma}_H(\ell)$ with kernel function $K_H(\cdot)$ which smooths the cross product increments $d\hat{\epsilon}_s^a d\hat{\epsilon}_t^b$ around the time difference $t-s=\ell$, as well as the weighting function $w_B(\ell)$ which smooths the squared sample cross-covariance function around lag $\ell = 0$. Suppose that B is large relative to H in the limit, such that $H = o(B)$ as $B \to \infty$. Then, the smoothing effect is dominated by $w_B(\ell)$. Indeed, as $B \to \infty$, the following approximation holds

$$w_B(\ell) K_H(t_1 - s_1 - \ell)\, K_H(t_2 - s_2 - \ell) = w_B(\ell)\delta_\ell(t_1 - s_1)\delta_\ell(t_2 - s_2) + o(1)$$

where $\delta_\ell(\cdot)$ is the Dirac delta function at ℓ. Hence, the difference $Q - Q^s$ becomes

$$\begin{aligned} Q - Q^s &= \int_I w_B(\ell)\hat{\gamma}_H^2(\ell) d\ell - Q^s \\ &= \frac{1}{T^2H^2} \int_I \iiiint_{(0,T]^4} w_B(\ell) K\left(\tfrac{t_1 - s_1 - \ell}{H}\right) K\left(\tfrac{t_2 - s_2 - \ell}{H}\right) d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{t_1}^b d\hat{\epsilon}_{t_2}^b d\ell - Q^s \\ &= \frac{1}{T^2} \int_I \iint_{(0,T]^4} w_B(\ell) d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{s_1+\ell}^b d\hat{\epsilon}_{s_2+\ell}^b d\ell - Q^s + o_P(1) \\ &= \frac{1}{T^2} \int_I \iint_{(0,T]^2, s_1 \neq s_2} w_B(\ell) d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{s_1+\ell}^b d\hat{\epsilon}_{s_2+\ell}^b d\ell + o_P(1). \end{aligned} \quad (19)$$

where in getting the second-to-last line, the quadruple integrations over $\{(s_1, s_2, t_1, t_2) \in$

[26] This follows from (28) in the Appendix.

$(0,T]^4\}$ collapse to the double integrations over $\{(s_1, s_2, s_1+\ell, s_2+\ell) : s_1, s_2 \in (0,T]\}$.

Indeed, computing Q^s is a lot simpler than Q because there is no need to estimate conditional intensities. However, if I test the hypothesis (11) based on the statistic Q^s instead of Q, I will have to pay the price of potentially missing some alternatives - for example, those cases in which the cross correlations alternate in signs as the lag increases, in such a way that the integrated cross-correlation $\int_I \gamma(\ell)\, d\ell$ is close to zero, but the individual $\gamma(\ell)$ are not. Nevertheless, such kind of alternatives is not very common at least in our applications in default risk and high frequency finance, where the feedback from one marginal process to another is usually observed to be positively persistent, and the positive cross correlation gradually dies down as the time lag increases. In terms of computation, the statistic Q^s is much less complicated than Q since it is not necessary to estimate the sample cross covariance function $\hat{\gamma}_H(\ell)$ and the conditional intensities of the marginal processes $\hat{\lambda}_t^k$; thus two bandwidths (M and H) are saved. The benefit of this simplification is highlighted in the simulation study where the size performance of Q^s stands out from its counterpart Q.[27]

The mean and variance of Q^s are given in the following theorem. The techniques involved in the derivation are similar to those for Q.

Let us recall that in section 2, the second-order reduced form factorial product density of N^k (assumed to exist in assumption (A2)) was defined by $\varphi^{kk}(u)dtdu := E\left(dN_t^k dN_{t+u}^k\right)$ for $u \neq 0$ and $\varphi^{kk}(0)dt = E\left(dN_t^k\right)^2 = E\left(dN_t^k\right) = \lambda^k dt$. Note that there is a discontinuity point at $u = 0$ as $\lim_{u\to 0}\varphi^{kk}(u) = \left(\lambda^k\right)^2 \neq \varphi^{kk}(0)$. The reduced unconditional auto-covariance density function can then be expressed into $c^{kk}(u)dtdu := E\left(dN_t^k - \lambda^k dt)(dN_{t+u}^k - \lambda^k du\right) = [\varphi^{kk}(u) - \left(\lambda^k\right)^2]dtdu$.

Theorem 7 *Let $I \subseteq [-T,T]$ and $J_i = [-\ell_i, T-\ell_i] \cap [0,T]$ for $i = 1,2$. Under assumptions (A1-3, 4a,b and 4d) and the null hypothesis,*

$$E(Q^s) = \frac{\lambda^a \lambda^b}{T} \int_I w_B(\ell) \left(1 - \frac{|\ell|}{T}\right) d\ell.$$

With no autocorrelations:

$$Var(Q^s) = \frac{\lambda^a \lambda^b}{T^3} \int_I w_B^2(\ell) \left(1 - \frac{|\ell|}{T}\right) d\ell.$$

[27] There are two bandwidths for the simplified statistic: one for the weighting function and the other for the nonparametric estimator of the autocovariance function. We will show in simulations that for simple bivariate Poisson process and for bivariate point process showing mild autocorrelations, the empirical rejection rate (size) of the nonparametric test is stable over a wide range of bandwidths that satisfy the assumptions stipulated in the asymptotic theory of the statistic. When autocorrelation is high, the size is still close to the nominal level for some combinations of the bandwidths of the weighting function and the autocovariance estimators.

With autocorrelations:

$$\begin{aligned} Var(Q^s) &= \tfrac{1}{T^4}\iint_{I^2}\int_{J_2}\int_{J_1} w_B(\ell_1)\,w_B(\ell_2)\,c^{aa}(s_2-s_1)c^{bb}(s_2-s_1+\ell_2-\ell_1)ds_1ds_2d\ell_1d\ell_2 \\ &+\tfrac{(\lambda^b)^2}{T^4}\iint_{I^2}\int_{J_2}\int_{J_1} w_B(\ell_1)\,w_B(\ell_2)\,c^{aa}(s_2-s_1)ds_1ds_2d\ell_1d\ell_2 \\ &+\tfrac{(\lambda^a)^2}{T^4}\iint_{I^2}\int_{J_2}\int_{J_1} w_B(\ell_1)\,w_B(\ell_2)\,c^{bb}(s_2-s_1+\ell_2-\ell_1)ds_1ds_2d\ell_1d\ell_2. \end{aligned}$$

If $I=[0,T]$ and $B=o(T)$ as $T\to\infty$, then (with autocorrelations)

$$\begin{aligned} Var(Q^s) &\approx \tfrac{2}{T^3}\left[\int_0^T \mathcal{W}_2(u)du\int_{-T}^T c^{aa}(v)\,c^{bb}(v+u)\,dv + \left(\lambda^b\right)^2\omega_1\int_0^T c^{aa}(v)dv \right. \\ &\left. + (\lambda^a)^2\,\omega_1\int_0^T c^{bb}(v)dv\right], \end{aligned} \quad (20)$$

where $\omega_1=\int_0^T w(\ell)\left(1-\frac{\ell}{T}\right)d\ell$ and $\mathcal{W}_2(u)=\int_u^T w(\ell-u)\,w(\ell)\left(1-\frac{\ell}{T}\right)d\ell$.

In practice, the mean and variance can be consistently estimated with the following replacements. For $k=a,b$:

(i) replace the unconditional intensity λ^k by the estimator $\hat{\lambda}^k=N^k/T$, and

(ii) replace the unconditional auto-covariance density $c^{kk}(\ell)$ by the kernel estimator:

$$\begin{aligned} \hat{c}^{kk}_{R^k}(\ell) &= \frac{1}{T}\int_0^T\int_0^T \ddot{K}_{R^k}(t-s-\ell)\left(dN^k_s-\hat{\lambda}^k ds\right)\left(dN^k_t-\hat{\lambda}^k dt\right) \\ &= \tfrac{1}{T}\sum_{i=1}^{N^k_T}\sum_{j=1}^{N^k_T}\ddot{K}_{R^k}\left(t^k_j-t^k_i-\ell\right)-\left(1-\tfrac{|\ell|}{T}\right)\left(\hat{\lambda}^k\right)^2+o(1), \end{aligned} \quad (21)$$

where the last equality holds if $R^k/T\to 0$ as $T\to\infty$. The proof of (21) will be given in Appendix A.7, which requires that $\ddot{K}(\cdot)$ satisfy the following assumption:

Assumption (A4c) The kernel function $\ddot{K}(\cdot)$ is symmetric around zero and satisfies $\ddot{\kappa}_1\equiv\int_{-\infty}^{\infty}\ddot{K}(u)du=1$, $\ddot{\kappa}_2\equiv\int_{-\infty}^{\infty}\ddot{K}^2(u)du<\infty$, $\ddot{\kappa}_4\equiv\iiint_{(-\infty,\infty)}\ddot{K}(u)\ddot{K}(v)\ddot{K}(u+w)\ddot{K}(v+w)dudvdw<\infty$ and $\int_{-\infty}^{\infty}u^2\ddot{K}(u)du<\infty$.

5 Asymptotic Theory

5.1 Asymptotic Normality under the Null

Recall from the definition that the test statistic Q is the weighted integral of squared sample cross-covariance function between the residuals of the component processes. However, the residuals $d\hat{\epsilon}^k_t$ do not form a martingale difference process as the counting process increment dN^k_t is demeaned by its estimated conditional mean $\hat{\lambda}^k_t dt$ instead of the true conditional mean $\lambda^k_t dt$. According to the definition of ϵ^k_t, the innovations

$d\epsilon_t^k = dN_t^k - \lambda_t^k dt$ form a martingale difference process, but not the residuals $d\hat{\epsilon}_t^k = dN_t^k - \hat{\lambda}_t^k dt$.

To facilitate the proof, it is more convenient to separate the analysis of the estimation error of conditional intensity estimators $\hat{\lambda}_t^k$ from that of asymptotic distribution of the test statistic. To this end, I define the hypothetical version of Q as follows

$$\tilde{Q} = \int_I w_B(\ell)\gamma_H^2(\ell)d\ell,$$

where $\gamma_H(\ell)$ is the hypothetical cross-covariance kernel estimator between the innovations $d\epsilon_s^a$ and $d\epsilon_t^b$:

$$\begin{aligned}\gamma_H(\ell) &= \frac{1}{T}\int_0^T\int_0^T K_H(t-s-\ell)\,d\epsilon_s^a d\epsilon_t^b \\ &= \frac{1}{T}\int_0^T\int_0^T K_H(t-s-\ell)\,(dN_s^a - \lambda_s^a ds)\left(dN_t^b - \lambda_t^b dt\right).\end{aligned}$$

In the first stage of the proof, I will prove the asymptotic normality of the hypothetical test statistic $\tilde{Q}$. In the second stage (to be covered in section 5.2), I will examine the conditions under which the approximation of $\tilde{Q}$ by Q yields an asymptotically negligible error, so that Q is also asymptotically normally distributed.

Theorem 8 *Under assumptions (A1-3,4a,5,6) and the null hypothesis (11), the normalized test statistic*

$$J = \frac{\tilde{Q} - E(\tilde{Q})}{\sqrt{Var(\tilde{Q})}} \tag{22}$$

converges in distribution to a standard normal random variable as $T \to \infty$, $H \to \infty$ and $H/T \to 0$, where the mean and variance of $\tilde{Q}$ are given as follows:

$$E(\tilde{Q}) = \tfrac{1}{TH}\lambda^a\lambda^b\kappa_2\int_I w_B(\ell)\left(1-\tfrac{|\ell|}{T}\right)d\ell + o\left(\tfrac{1}{TH}\right),$$

$$\begin{aligned}&Var(\tilde{Q}) \\ =\ & \tfrac{2}{T^2H}\kappa_4\int_I w_B^2(\ell)\int_{-(T-|\ell|)}^{T-|\ell|}\left(1-\tfrac{|r|}{T}-\tfrac{|\ell|}{T}\right)\left[(\lambda^a)^2 + c^{aa}(r)\right]\left[\left(\lambda^b\right)^2 + c^{aa}(r)\right]drd\ell + o\left(\tfrac{1}{T^2H}\right) \\ =\ & \tfrac{2\left(\lambda^a\lambda^b\right)^2}{T^2H}\kappa_4\int_I w_B^2(\ell)\left(1-\tfrac{|\ell|}{T}\right)^2 d\ell \\ & + \tfrac{2}{T^2H}\kappa_4\int_I w_B^2(\ell)\int_{-(T-|\ell|)}^{T-|\ell|}\left(1-\tfrac{|r|}{T}-\tfrac{|\ell|}{T}\right)f(r)\,drd\ell + o\left(\tfrac{1}{T^2H}\right),\end{aligned}$$

where $f(x) = (\lambda^a)^2 c^{bb}(r) + \left(\lambda^b\right)^2 c^{aa}(r)\,dr + c^{aa}(r)\,c^{bb}(r)$.

If N^a and N^b do not exhibit auto-correlations, then $c^{kk}(u) \equiv 0$ for $k = a, b$ and hence the variance reduces to the first term in the last equality.

5.2 Effect of Estimation

In this section, I discuss the effect of estimating the unconditional and the $\mathcal{F}^k$-conditional intensities on the asymptotic distribution of the statistic J. I want to argue that, with the right convergence rates of the bandwidths, the asymptotic distribution of J is unaffected by both estimations.

In practice, the statistic J is infeasible because (i) $\tilde{Q}$ is a function of the conditional intensities λ_t^a and λ_t^b; and (ii) both $E(Q) = E(\tilde{Q})$ and $Var(Q) = Var(\tilde{Q})$ contain the unconditional intensities λ^a and λ^b. As discussed in section 4.4, one way to estimate the unknown conditional intensities λ_t^k (for $k = a, b$) is by means of the nonparametric kernel estimator

$$\hat{\lambda}_t^k = \int_0^T \tfrac{1}{M} \mathring{K}\left(\tfrac{t-u}{M}\right) dN_u^k,$$

On the other hand, by stationarity of $\mathbf{N}$ (assumption (A2)) the unconditional intensities λ^k (for $k = a, b$) are consistently estimated by

$$\hat{\lambda}^k = \tfrac{N_T^k}{T}.$$

Recall that Q is the same as $\tilde{Q}$ after replacing λ_t^k by $\hat{\lambda}_t^k$. Let $\widehat{E(Q)}$ and $\widehat{Var(Q)}$ be the same as $E(Q)$ and $Var(Q)$ after replacing λ^k by $\hat{\lambda}^k$ and $c^{kk}(\ell)$ by $\hat{c}_{R^k}^{kk}(\ell)$.

Theorem 9 *Suppose that $H = o(M)$ as $M \to \infty$, and that $M^5/T^4 \to 0$ and $\left(R^k\right)^5/T^4 \to 0$ as $T \to \infty$. Then, under assumptions (A4b,4c) and the assumptions in Theorems 6 and 8, the statistic $\hat{J}$ defined by*

$$\hat{J} = \frac{Q - \widehat{E(Q)}}{\sqrt{\widehat{Var(Q)}}}$$

converges in distribution to a standard normal random variable as $T \to \infty$, $H \to \infty$ and $H/T \to 0$.

As discussed in section 4.4, the conditional intensity λ_t^k of each component process N^k can also be modeled by a parametric model. Since the estimator of the parameter vector has the typical parametric convergence rate of $T^{-1/2}$ or $\left(N_T^k\right)^{-1/2}$ (which is faster than the nonparametric rate of $M^{-1/2}$), the asymptotic bandwidth condition in Theorem 9, i.e. $H = o(M)$ as $M \to \infty$ becomes redundant, and thus the result of Theorem 9 is still valid even without such condition. Similar remark applies to the auto-covariance density function $c^{kk}(\ell)$.

5.3 Asymptotic Local Power

To evaluate the local power of the Q test, I consider the following sequence of alternative hypotheses

$$\mathbf{H}_{a_T} : \gamma(\ell) = a_T \sqrt{\lambda^a \lambda^b} \rho(\ell),$$

where $a_T\rho(\ell)$ is the cross-correlation function between $d\epsilon_s^a$ and $d\epsilon_{s+\ell}^b$, and a_T is a sequence of numbers so that $a_T \to \infty$ and $a_T = o(T)$ as $T \to \infty$. The function $\rho(\ell)$, the cross-correlation function before inflated by the factor a_T, is required to be square-integrable over $\mathbb{R}$. The goal is to determine the correct rate a_T^* with which the test based on Q has asymptotic local power. For notational simplicity, I only discuss the case where N^a and N^b do not exhibit auto-correlations. The result corresponding to autocorrelated point processes can be stated similarly.

The following assumption is needed:

Assumption (A8) The joint cumulant $c_{22}(\ell_1, \ell_2, \ell_3)$ of $\{d\epsilon_s^a, d\epsilon_{s+\ell_1}^a, d\epsilon_{s+\ell_2}^b, d\epsilon_{s+\ell_3}^b\}$ is of order $o(a_T^2)$.

Theorem 10 *Suppose that assumption (A8) and the assumptions in Theorem 8 hold. Suppose further that $H = o(B)$ as $B \to \infty$. Then, under $\mathbf{H}_{a_T^*}$ with $a_T^* = H^{1/4}$, the statistic $J - \mu(K, w_B)$ (J as defined in (22)) converges in distribution to $N(0,1)$ as $H \to \infty$ and $H = o(T)$ as $T \to \infty$, where*

$$\mu(K, w_B) = \frac{\kappa_2 \int_I w_B(\ell) \left(1 - \frac{|\ell|}{T}\right) \breve{\rho}^2(\ell) d\ell}{\sqrt{2\kappa_4 \int_I w_B^2(\ell) \left(1 - \frac{|\ell|}{T}\right)^2 d\ell}}$$

and

$$\breve{\rho}^2(\ell) := \rho^2(\ell) + \int_{-T}^{T} \left(1 - \frac{|u|}{T}\right) \rho\left(\ell + \frac{u}{T}\right) \rho\left(\ell - \frac{u}{T}\right) du.$$

According to Theorem 10, a test based on Q picks up equivalent asymptotic efficiency against the sequence of Pitman's alternatives in which the cross-correlation of innovations (for each lag ℓ) grows at the rate of $a_T^* = H^{1/4}$ as the sample size T tends to infinity. It is important to note, after mapping the sampling period from $[0, T]$ to $[0, 1]$ as in (15), that the cross-covariance under $\mathbf{H}_{a_T}$ becomes $\tilde{\gamma}(\sigma) = \gamma(T\sigma) = a_T\sqrt{\lambda^a\lambda^b}\rho(T\sigma) = \tilde{a}_T\sqrt{\lambda^a\lambda^b}\tilde{\rho}(\sigma)$ by (17), where $\tilde{a}_T$ and $\tilde{\rho}(\sigma)$ are the rate and cross-correlation of innovations after rescaling. As a result, the corresponding rate that maintains the asymptotic efficiency of the test under the new scale is $\tilde{a}_T^* = H^{1/4}/T^{\nu}$, where ν is the rate of decay of the uninflated cross-correlation function ρ: $\rho(\ell) = O(\ell^{-v})$ as $\ell \to \infty$. The rate $\tilde{a}_T^*$ generally goes to zero for bivariate point processes exhibiting short and long memory cross-correlation dynamics, as long as $\nu \geq 1/4$.

6 Bandwidth Choices

According to assumption (A5), the weighting function $w(\ell)$ in the test statistic Q is required to be integrable. In practice, it is natural to choose a function that decreases with the absolute time lag $|\ell|$ to reflect the decaying economic significance of the feedback relationship over time (as discussed in section 4.3). Having this economic motivation in mind, I suppose in this section, without loss of generality, that the weighting function is a kernel function with bandwidth B, i.e. $w(\ell) \equiv w_B(\ell) = w(\ell/B)/B$. The bandwidth B is responsible for discounting the importance of the feedback strength as represented by the squared cross-covariance of innovations: the further away the time lag ℓ is from zero, the smaller is the weight $w_B(\ell)$.

6.1 Case 1: $B \ll H \ll T$

Suppose $B = o(H)$ as $H \to \infty$. This happens when B is kept fixed, or when $B \to \infty$ but $B/H \to 0$. Since $w(\ell)$ has been assumed to be a fixed function before this section, the asymptotic result in Theorem 8 remains valid. Nevertheless, I can simplify the result which is summarized in the following corollary.

Corollary 11 *Let $Q^{\mathcal{G}} \equiv \frac{TH}{\lambda^a \lambda^b} Q$. Suppose that $B = o(H)$ as $H \to \infty$. Suppose further that $I = [0, T]$. Then, with the assumptions in Theorem 8 and under the null hypothesis (11), the statistic*

$$\mathcal{M}^{\mathcal{G}} \equiv \frac{Q^{\mathcal{G}} - C^{\mathcal{G}}}{\sqrt{2D^{\mathcal{G}}}}$$

converges in distribution to a standard normal random variable as $T \to \infty$, and $H/T \to 0$ as $H \to \infty$, where

$$C^{\mathcal{G}} = \kappa_2$$

and

$$D^{\mathcal{G}} = 3\kappa_4.$$

6.2 Case 2: $H \ll B \ll T$

Suppose instead that $B \to \infty$ and $H = o(B)$ as $H \to \infty$. In this case, the smoothing behavior of the covariance estimator is dominated by that of the weighting function $w(\ell)$. As it turns out, the normalized statistic (denoted by $\mathcal{M}^{\mathcal{H}}$ in the following corollary) is equivalent to the continuous analog of Hong's (1996a) test applied to testing for cross-correlation between two time series.

Corollary 12 *Let $Q^{\mathcal{H}} \equiv \frac{TB}{\lambda^a \lambda^b} Q$. Suppose that $B \to \infty$ and that $H = o(B)$ as $H \to \infty$. Suppose further that $I = [0, T]$. Then, with the assumptions in Theorem 8 and under the null hypothesis (11), the statistic*

$$\mathcal{M}^{\mathcal{H}} \equiv \frac{Q^{\mathcal{H}} - C^{\mathcal{H}}}{\sqrt{2D^{\mathcal{H}}}}$$

converges in distribution to a standard normal random variable as $T \to \infty$, and $B/T \to 0$ as $B \to \infty$, where

$$C^{\mathcal{H}} = \int_0^T w\left(\tfrac{\ell}{B}\right)\left(1 - \tfrac{\ell}{T}\right) d\ell$$

and

$$D^{\mathcal{H}} = \int_0^T w^2\left(\tfrac{\ell}{B}\right)\left(1 - \tfrac{\ell}{T}\right)^2 d\ell.$$

6.3 Optimal Bandwidths

Choosing optimal bandwidths is an important and challenging task in nonparametric analyses. For nonparametric estimation problems, optimal bandwidths are chosen to minimize the mean squared error (MSE), and automated procedures that yield data-driven bandwidths are available and well-studied for numerous statistical models. However, optimal bandwidth selection remains a relatively unknown territory for nonparametric hypothesis testing problems. In the first in-depth analysis of how to choose the optimal bandwidth of the heteroskedasticity-autocorrelation consistent estimator for testing purpose, Sun, Phillips and Jin (2008) proposed to minimize a loss function which is a weighted average of the probabilities of type I and II error. Their theoretical comparison revealed that the bandwidth optimal for testing has a smaller asymptotic order ($O(T^{1/3})$) than the MSE-optimal bandwidth, which is typically $O(T^{1/5})$. Although the focus is on statistical inference of the simple location model, their result could serve as a guide to the present problem of nonparametric testing for Granger causality.

7 Simulations

7.1 Size and Power of Q

In the first set of size experiments, the data generating process (DGP) is set to be a bivariate Poisson process which consists of two independent marginal Poisson processes with rate 0.1. The number of simulation runs is 5000. The weighting function of Q is chosen to be a Gaussian kernel with bandwidth $B = 10$. I consider four different sample lengths ($T = 500, 1000, 1500, 2000$) with corresponding bandwidths ($M = 60, 75, 100, 120$) for the nonparametric conditional intensity estimators in such a way that the ratio M/T gradually diminishes. Figure 2 shows the plots of the empirical rejection rates against different bandwidths H of the sample cross-covariance estimator for the four different sample lengths we considered. The simulation result reveals that in finite sample the test is generally undersized at the 0.1 nominal level and oversized at the 0.05 nominal level, but the performance improves with sample length.

In the second set of experiments, the DGP is set to a more realistic one: a bivariate exponential Hawkes model (see section 1.4) with parameters

$$\mu = \begin{pmatrix} 0.0277 \\ 0.0512 \end{pmatrix}, \alpha = \begin{pmatrix} 0.0086 & 0.0017 \\ 0 \text{ or } 0.0182 & 0.0896 \end{pmatrix}, \beta = \begin{pmatrix} 0.0254 & 0.0507 \\ 0.0254 & 0.1473 \end{pmatrix}, \tag{23}$$

which were estimated by fitting the model to a high frequency TAQ dataset of PG traded in NYSE on a randomly chosen day (1997/8/8) and period (9:45am to 10:15am). For the size experiments, the parameter α_{21} was intentionally set to zero so that there is no causal relation from the first process to the second under the DGP, and we are interested in testing the existence of causality from the first process to the second only (i.e. by setting the integration range of the statistic Q to $I = [0, T]$). The number of simulation runs is 10000 with a fixed sample length 1800 (in seconds). The bandwidth of the sample cross covariance estimator is fixed at $H = 3$. A Gaussian kernel with bandwidths $B = 2$ and 20 respectively is chosen for the weighting function. For the power experiments, I set α_{21} back to the original estimate 0.0182. There is an increase, albeit mild, in the rejection rate compared to the size experiments. Figure 3 shows the plots of the rejection rates against different bandwidths M of the nonparametric conditional intensity estimators. A first observation, after comparing Figures 3(a) and 3(c), is that the empirical sizes of the test are more stable over various M when B is small. A second observation, after comparing Figures 3(b) and 3(d), is that the test seems to be more powerful when B is small. This indicates that, while a more slowly decaying weighting function gives a more consistent test against alternatives with longer causal lags, this is done at the expense of a lower power and more sensitive size to bandwidth choices.

7.2 Size and Power of Q^s

To investigate the finite sample performance of the simplified statistic Q^s, I conduct four size experiments with different parameter combinations of a bivariate exponential Hawkes model. Recall that there are only three bandwidths to choose for Q^s, namely the bandwidth B of the weighting function $w_B(\ell)$ and the bandwidths R^k of the autocorrelation function estimator $\hat{c}^{kk}_{R^k}(\ell)$ for $k = a, b$. In each of the following experiments, I generate four sets of 5000 samples of various sizes ($T = 300, 600, 900, 1200$) from a DGP and carry out a Q^s test for Granger causality from N^a to N^b on each of the samples. The DGP's of the four size experiments and one power experiment are all bivariate exponential Hawkes models with the following features:

- Size experiment 1: N^a and N^b are independent and have the same unconditional intensities with comparable and moderate self-excitatory (autoregressive) strength (Figure 4).

- Size experiment 2: N^a and N^b are independent and have the same unconditional intensities, but N^b exhibits stronger self-excitation than N^a (Figure 5).

- Size experiment 3: N^b Granger causes N^a, and both have the same unconditional intensities and self-excitatory strength (Figure 6).

- Size experiment 4: N^a and N^b are independent and have the same self-excitatory strength, but unconditional intensity of N^b doubles that of N^a (Figure 7).

- Size experiment 5: N^a and N^b are independent and have the same unconditional intensities with comparable and highly persistent self-excitatory (autoregressive) strength (Figure 8).

- Power experiment: N^a Granger causes N^b, and both have the same unconditional intensities and self-excitatory strength (Figure 9).

The nominal rejection rates are plotted against different weighting function bandwidths B (small relative to T). The bandwidths R^k of the autocovariance function estimators are set proportional to B ($R^k = cB$ where $c = 0.5$ for size experiment 5 and $c = 1$ for all other experiments).

Under those DGP's that satisfies the null hypothesis of no Granger causality (all size experiments), the empirical rejection rates of the test based on Q^s are reasonably close to the nominal rates over a certain range of B that grows with T, as shown in Figures 4-8. According to Theorem 7, I need $B = o(T)$ so that the variance can be computed by (20) in the theorem. In general, the empirical size becomes more accurate as the sample length T increases. On the other hand, the Q^s test is powerful against the alternative of a bivariate exponential Hawkes model exhibiting Granger causality from N^a to N^b, and the power increases with sample length T, as shown in Figure 9.

8 Applications

8.1 Trades and Quotes

In the market microstructure literature, there are various theories that attempt to explain the trades and quotes dynamics of stocks traded in stock exchanges. In the seminal study, Diamond and Verrecchia (1987) propose that the speed of price adjustment can be asymmetric due to short sale constraints. As a result, a lack of trades signals bad news because informed traders cannot leverage on their insights and short-sell the stock. Alternatively, Easley and O'hara (1992) argue that trade arrival is related to the existence of new information. Trade arrival affects the belief on the fundamental stock price held by dealers, who learn about the direction of new information from the observed trade sequence and adjust their bid and/or ask quotes in a Bayesian manner. It is believed that a high trade intensity is followed by more quote revisions, while a low trade intensity means a lack of new information transmitted to the market and hence leads to fewer quote revisions. As discussed in 1.3, much existing research is devoted to the testing of these market microstructure hypotheses, but the tests are generally conducted through statistical inference under strong parametric assumptions (e.g. VAR model in Hasbrouck, 1991 and Dufour and Engle, 2000; the bivariate duration model in Engle and Lunde, 2003). This problem offers an interesting opportunity to apply the nonparametric test in this paper. With minimal assumptions on the trade and quote revision dynamics, the following empirical results indicate the direction and strength of causal effect in support of the conjecture of Easley and O'hara (1992): more trade arrivals predict more quote revisions.

I obtain the data from TAQ database available in the Wharton Research Data Services. The dataset consists of all the transaction and quote revision timestamps of the stocks of Proctor and Gamble (NYSE:PG) in the 41 trading days from 1997/8/4 to 1997/9/30, the same time span as the dataset of Engle and Lunde (2003). Then, following the standard data cleaning procedures (e.g. Engle and Russell, 1998) to prepare the dataset for further analyses,

1. I employ the five-second rule when combining the transaction and quote time sequences into a bivariate point process by adding five seconds to all the recorded

quote timestamps. This is to reduce unwanted effects from the fact that transactions were usually recorded with a time delay.

2. I eliminate all transaction and quote records before 9:45am on every trading day. Stock trades in the opening period of a trading day are generated from open auctions and are thus believed to follow different dynamics.

3. Since the TAQ timestamps are accurate up to a second, this introduces a limitation to the causal inference in that there is no way to tell the causal direction among those events happening within the same second. The sampled data also constitutes a violation of assumption (A1). I treat multiple trades and quotes occurring at the same second as one event, so that an event actually indicates the occurrence of at least one event within the same second. [28]

After carrying out the data cleaning procedures, I split the data into different trading periods and conduct the nonparametric causality test for each trading day. Then, I count the number of trading days with significant causality from trade to quote (or quote to trade) dynamics. For each sampling period, let N^t and N^q be the counting processes of trade and quote revisions, respectively. The hypotheses of interest are

$$\begin{aligned} \mathbf{H}_0 &: \text{ there is no Granger causality from } N^a \text{ to } N^b\text{; vs} \\ \mathbf{H}_1 &: \text{ there is Granger causality from } N^a \text{ to } N^b. \end{aligned}$$

where $a, b \in \{t, q\}$ and $a \neq b$.

The results are summarized in Tables 2 to 4. In each case, I present the significant day count for different combinations of bandwidths (all in seconds). For each (H, B) pair, the bandwidth M of the conditional intensity estimator is determined from simulations so that the rejection rate matches the nominal size.

Some key observations are in order. First, there are more days with significant causation from trade to quote update dynamics than from quote update to trade dynamics for most bandwidth combinations. This suppports the findings of Engle and Lunde (2003). Second, for most bandwidth combinations, there are more days with significant causations (in either direction) during the middle of a trading day (11:45am – 12:45pm) than in the opening and closing trading periods (9:45am – 10:15am and 3:30pm – 4:00pm). One possible explanation is that there are more confounding factors (e.g. news arrival, trading strategies) that trigger a quote revision around the time when the market opens and closes. When the market is relatively quiet, investors

[28] For PG, 5.6% of trades, 28.1% of quote revisions and 3.6% of trades and quotes were recorded with identical timestamps (in seconds). The corresponding proportions for GM are 5.7%, 19.9% and 2.6%, respectively. Admittedly, the exceedingly number of quote revisions recorded at the same time invalidates assumption (A1), but given the low proportions for trades and trade-quote pairs with same timestamps, the distortion to the empirical results is on the conservative side. That is, if there exists a more sophisticated Granger causality test that takes into account the possibility of simultaneous quote events, the support for trade-to-quote causality would be even stronger than the support Q and Q^s tests provide, as we shall see later in Tables 2-6.

have less sources to rely on but update their belief on the fundamental stock price by observing the recent transactions. Third, the contrast between the two causation directions becomes sharper in general when the weighting function, a Gaussian kernel, decays more slowly (larger B), and it becomes the sharpest in most cases when B is 10 seconds (when the day counts with significant causation from trade to quote is the maximum). This may suggest that most causal dynamics from trades to quotes occur and finish over a time span of about $3B = 30$ seconds.

Next, I employ the simplified statistic Q^s to test the data. I am interested to see whether it implies the same causal relation from trades to quotes as found earlier, given that a test based on Q^s is only consistent against a smaller set of alternatives (as discussed in section 4.7). The result of the Q^s test on trade and quote revision sequences of PG is presented in Table 5. The result shows stronger support for the causal direction from trades to quote revisions across various trading periods of a day (compare Table 5 to Tables 2-4: the Q^s test uncovers more significant days with trade-to-quote causality than the Q test does). I also conduct the Q^s test on trades and quotes of GM, and obtain similar result that trades Granger-cause quote revisions. (See Table 6 for the test results on General Motors. Test results of other stocks considered by Engle and Lunde (2003) are similar and available upon request.) The stronger support by the Q^s test for the trade-to-quote causality suggests indirectly that the actual feedback resulting from a shock in trade dynamics to quote revision dynamics is persistent rather than alternating in signs over the time range covered by the weighting function $w(\ell)$. Given that I am testing against the alternatives with persistent feedback effect from trades to quote revisions, it is natural that the Q test is less powerful than the Q^s test. This is the price for being consistent against a wider set of alternatives[29].

8.2 Credit Contagion

Credit contagion occurs when a credit event (e.g. default, bankruptcy) of a firm leads to a cascade of credit events of other firms (see, for example, Jorion and Zhang, 2009). This phenomenon is manifested as a cluster of firm failures in a short time period. As discussed in section 1.4, a number of reduced-form models, including conditional independence and self-exciting models, are available to explain the dependence of these credit events over time, with varying level of success. Conditional independence model assumes that the probabilities of a credit events of a cross section of firms depend on some observed common factors (Das, Duffie, Kapadia and Saita, 2008; DDKS hereafter). This modeling approach easily induces cross-sectional dependence among firms, but is often inadequate to explain all the observed clustering of credit events unless a good set of common factors is discovered. One way to mitigate the model inadequacy is to introduce latent factors into the model (Duffie, Eckners, Horel and Saita, 2010; DEHS hereafter). Counterparty risk model, on the other hand, offers an appealing alternative: the occurrence of credit events of firms are directly dependent on each other (Jarrow and Yu, 2001). This approach captures directly the mutual-excitatory

[29] This includes those alternatives in which excitatory and inhibitory feedback effect from trades to quotes alternate as time lag increases.

(or serial correlation) nature of credit events that is neglected by the cross-sectional approach of conditional independence models. In a series of empirical studies, Jorion and Zhang (2007, 2009) provided the first evidence that a significant channel of credit contagion is through counterparty risk exposure. The rationale behind their arguments is that the failure of a firm can affect the financial health of other firms which have business ties to the failing firm. This empirical evidence highlights the importance of counterparty risk model as an indispensable tool for credit contagion analysis.

All the aforementioned credit risk models cannot avoid the imposition of ad-hoc parametric assumptions which are not justified by any structural models. For instance, the conditional independence models of DDKS and DEHS rely on strong log-linear assumption[30] on default probabilities, while the counterparty risk model of Jarrow and Yu adopt a convenient linear configuration. Also, the empirical work of Jorion and Zhang is based on the linear regression model. The conclusions drawn from these parametric models have to be interpreted with care as they may be sensitive to the model assumptions. Indeed, as warned by DDKS, a rejection of their model in goodness-of-fit tests can indicate either a wrong log-linear model specification or an incorrect conditional independence assumption of the default intensities, and it is impossible to distinguish between them from their test results. Hence, it is intriguing to investigate the extent of credit contagion with as few interference from model assumptions as possible. The nonparametric Granger causality tests make this model-free investigation a reality.

I use the "Bankruptcies of U.S. firms, 1980–2010" dataset to study credit contagion. The dataset is maintained by Professor Lynn LoPucki of UCLA School of Law. The dataset records, among other entries, the filing dates of Chapter 11 and the Standard Industrial Classification (SIC) codes of big bankrupting firms[31]. In this analysis, a credit event is defined as the occurrence of bankruptcy event(s). To be consistent with assumption (A1), I treat multiple bankruptcies on the same date as one bankruptcy event. Figure 10 shows the histogram of bankruptcy occurrences in 1980–2010.

I classify the bankrupting firms according to the industrial sector. More specifically, I assume that a bankruptcy belongs to manufacturing related sectors if the SIC code of the bankrupting firm is from A to E, and financial related sectors if the SIC code is from F to I. The rationale behind the classification is that the two industrial groups represent firms at the top and bottom of a typical supply chain, respectively. The manufacturing related sectors consist of agricultural, mining, construction, manufacturing, transportation, communications and utility companies, while the financial related sectors consist of wholeselling, retailing, financial, insurance, real estate and service provision companies.[32] Let N^m and N^f be the counting processes of bankruptcies from manufacturing and financial related sectors, respectively. Figure 11 plots the

[30] In the appendix of their paper, DEHS evaluates the robustness of their conclusion by considering the marginal nonlinear dependence of default probabilities on the distance-to-default. Nevertheless, the default probability is still assumed to link to other common factors in a log-linear fashion.

[31] The database includes those debtor firms with assets worth $100 million or more at the time of Chapter 11 filing (measured in 1980 dollars) and which are required to file 10-ks with the SEC.

[32] The industrial composition of bankruptcies in manufacturing related sectors are A: 0.2%; B: 6.3%; C: 4.5%; D: 58.6%; E: 30.5%. The composition in financial related sectors are F: 8.4%; G: 29.4%; H: 32.8%; I: 29.4%.

counting processes of the two types of bankruptcies. The hypotheses of interest are

$$\begin{aligned} \mathbf{H}_0 &: \text{ there is no Granger causality from } N^a \text{ to } N^b\text{; vs} \\ \mathbf{H}_1 &: \text{ there is Granger causality from } N^a \text{ to } N^b. \end{aligned}$$

where $a, b \in \{m, f\}$ and $a \neq b$.

Similar to the TAQ application, I carry out the Q test for different combinations of bandwidths (in days). The bandwidths M (for conditional intensity estimators) and B (for weighting function) are set equal to 365, 548 and 730 days (corresponding to 1, 1.5 and 2 years), while the bandwidth H (for cross-covariance estimator) ranges from 2 to 14 days.[33] The test results are displayed in Tables 7-9. For most bandwidth combinations, the Q test detects a significant credit contagion (at 5% significance level) from financial to manufacturing related sectors in periods that contain crises and recession (Asian financial crisis and 9/11 in September 1996 – July 2003; subprime mortgage crisis in September 2007 – June 2010) but not in periods of economic growth (August 2003 – August 2007). The reverse contagion becomes statistically significant too during the subprime mortgage crisis.

I also conduct the Q^s test over the period September 1996 – June 2010 that spans the financial crises and the boom in the middle. During this period, there are 350 and 247 bankruptcies in the manufacturing and financial related sectors. The normalized test statistic values (together with p-values) are presented in Table 10. The bandwidth B of the weighting function ranges from 30 to 300 days, while the bandwidths R^k of the unconditional autocorrelation kernel estimators $\hat{c}^{kk}_{R^k}(\ell)$ (for $k = m$ and f) are both fixed at 300 days. All kernels involved are chosen to be Gaussian. Over the period of interest, there is significant (at 5% significance level) credit contagion in both directions up to $B = 90$ days, but the financial-to-manufacturing contagion dominates manufacturing-to-financial contagion in the long run.

8.3 International Financial Contagion

The Granger causality test developed in this paper can be used to uncover financial contagion that spreads across international stock markets. An adverse shock felt by one financial market (as reflected by very negative stock returns) often propagates quickly to other markets in a contagious manner. There is no agreement in the concept of financial contagion in the literature[34]. For instance, Forbes and Rigobon (2002; hereafter FR) defined financial contagion as a significant increase in cross-market linkages after a shock. To measure and compare the extent of contagion over different stock market pairs, FR used a bias-corrected cross-correlation statistic for index returns. However, whether the increased cross-correlation represents a causal relationship (in Granger sense) is unclear. More recently, Aït-Sahalia, Cacho-Diaz and Laeven (2010; hereafter ACL) provided evidence of financial contagion by estimating a parametric

[33] The Q test is more sensitive to the choice of M than B, according to test results not shown in the paper (they are provided upon request). The choice of bandwidth H is guided by the restriction $H = o(M)$ from Theorem 9.

[34] See Forbes and Rigobon (2002) and the references therein for a literature review.

Hawkes jump-diffusion model to a cross-section of index returns. The contagion concept ACL adopted is in a wider sense than that of FR in that contagion can take place in both “good” and “bad” times (see footnote 2 of ACL). Based on the dynamic model of ACL, it is possible to infer the causal direction of contagion from one market to another. Nevertheless, their reduced-form Hawkes jump-diffusion model imposes a fair amount of structure on both the auto- and cross-correlation dynamics of the jumps of index returns without any guidance from structural models. The conclusion drawn from ACL regarding causal direction of contagion is model-specific and, even if the model is correct, sensitive to model estimation error. To robustify the conclusion, it is preferred to test for Granger causality of shock propagations in a nonparametric manner.

To this end, I collect daily values of the major market indices from finance.yahoo.com and compute daily log-returns from adjusted closing values. The indices in my data are picked from representative stock markets worldwide covering various time zones, including the American (Dow Jones), European (FTSE, DAX, CAC 40), Asian Pacific (Hang Seng, Straits Times, Taiwan, Nikkei), and Australian (All Ordinary) regions. The data frame, trading hours and number of observations are summarized in Table 11.

To define the days with negative shocks, I use the empirical 90%, 95% and 99% value-at-risk (VaR) for the corresponding stock indices. An event is defined as a negative shock when the daily return exceeds the VaR return. In each test, I pair up two point processes of events from two indices of different time zone, with a sampling period equal to the shorter of the two sample lengths of the two indices. The event timestamps are adjusted by the time difference between the two time zones of the two markets. Define the counting processes of shock events for indices a and b by N^a and N^b, respectively. The hypotheses of interest are

$$\begin{aligned} \mathbf{H}_0 &: \text{there is no Granger causality from } N^a \text{ to } N^b\text{; vs} \\ \mathbf{H}_1 &: \text{there is Granger causality from } N^a \text{ to } N^b. \end{aligned}$$

The results of the Q^s test applied to the pairs HSI-DJI, NIK-DJI, FTSE-DJI and AOI-DJI are shown in Tables 12-15[35]. There are a few observations. First, days with extreme negative returns exceeding 99% VaR have a much stronger contagious effect than those days with less negative returns (exceeding 95% or 90% VaR). This phenomenon is commonly found for all pairs of markets. Second, except for European stock indices, the U.S. stock market, as represented by DJI, plays a dominant role in infecting other major international stock markets. It is not hard to understand why the daily returns of European stock indices (FTSE, DAX, CAC 40) Granger-cause DJI's daily returns given the overlap of the trading hours of European stock markets and the U.S. stock market. Nonetheless, the causality from the American to European markets remains significant (for $B \leq 3$ with 95% VaR as the cutoff). Third, the test

[35] The Q^s test results for pairs involving DAX and CAC are qualitatively the same as that involving FTSE (all of them are in the European time zones), while the test results for pairs involving STI and TWI are qualitatively the same as that involving HSI (all of them are in the same Asian-Pacific time zone). I do not present these results here to reserve space, but they are available upon request.

statistic values are pretty stable over different choices of B and R^k ($k = a, b$). I used different functions of $R^k = M(B)$, such as a constant $R^k = 10$ and $R^k = 24B^{0.25}$, and found that, qualitatively, the dominating indices / markets remain the same as before (when $R^k = 10.5B^{0.3}$). Fourth, the shorter is the testing window (bandwidth B of the weighting function $w(\ell)$), the stronger is the contagious effect. For instance, with 95% value-at-risk as cutoff, DJI has significant Granger causality to HSI and NIK when $B \leq 3$ (in days) and to AOI when $B \leq 5$. This implies that contagious effect, once it starts, is most significant on the first few days, but usually dampens quickly within a week.

9 Conclusion

With growing availability of multivariate high frequency and/or irregularly spaced point process data in economics and finance, it becomes more and more of a challenge to examine the predictive relationship among the component processes of the system. One important example of such relationship is Granger causality. Most of the existing tests for Granger causality in the traditional discrete time series setting are inadequate for the irregularity of these data. Tests based on parametric continuous time models can better preserve the salient features of the data, but they often impose strong and questionable parametric assumptions (e.g. conditional independence as in doubly stochastic models, constant feedback effect as in Hawkes models) that are seldom supported by economic theories and, more seriously, distort the test results. This calls for a need to test for Granger causality (i) in a continuous time framework and (ii) without strong parametric assumptions. In this paper, I study a nonparametric approach to Granger causality testing on a continuous time bivariate point process that satisfies mild assumptions. The test enjoys asymptotic normality under the null hypothesis of no Granger causality, is consistent, and exhibits nontrivial power against departure from the null. It performs reasonably well in simulation experiments and shows its usefulness in three empirical applications: market microstructure hypothesis testing, checking the existence of credit contagion between different industrial sectors, and testing for financial contagion across international stock exchanges.

In the first application on the study of market microstructure hypotheses, the test confirms the existence of a significant causal relationship from the dynamics of trades to quote revisions in high frequency financial datasets. The next application on credit contagion reveals that U.S. corporate bankruptcies in financial related sectors Granger-cause those in manufacturing related sectors during crises and recessions. Lastly, the test is applied to study the extent to which an extreme negative shock of a major stock index transmits across international financial markets. The test confirms the presence of contagion, with U.S. and European stock indices being the major sources of contagion.

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A Appendix

A.1 List of Assumptions

(A1) The *pooled* counting process $N \equiv N^a + N^b$ is simple, i.e. $P(N(\{t\}) = 0$ or 1 for all $t) = 1$.

(A2) The bivariate counting process $\mathbf{N} = (N^a, N^b)$ is *second-order stationary* and that the *second-order reduced product densities* $\varphi^{ij}(\cdot)$ $(i, j = a, b)$ exist.

(A3) The $\mathcal{F}$-conditional intensity $\lambda^k(t|\mathcal{F}_{t^-})$ and $\mathcal{F}^k$-conditional intensity $\lambda_t^k \equiv \lambda^k(t|\mathcal{F}_{t^-}^k)$ of the counting process N_t^k exist and are predictable.

(A4a) The kernel function $K(\cdot)$ is symmetric around zero and satisfies $\kappa_1 \equiv \int_{-\infty}^{\infty} K(u)du = 1$, $\kappa_2 \equiv \int_{-\infty}^{\infty} K^2(u)du < \infty$, $\kappa_4 \equiv \iiint_{(-\infty,\infty)} K(u)K(v)K(u+w)K(v+w)dudvdw < \infty$ and $\int_{-\infty}^{\infty} u^2 K(u)du < \infty$.

(A4b) The kernel function $\mathring{K}(\cdot)$ is symmetric around zero and satisfies $\mathring{\kappa}_1 \equiv \int_{-\infty}^{\infty} \mathring{K}(u)du = 1$, $\mathring{\kappa}_2 \equiv \int_{-\infty}^{\infty} \mathring{K}^2(u)du < \infty$, $\mathring{\kappa}_4 \equiv \iiint_{(-\infty,\infty)} \mathring{K}(u)\mathring{K}(v)\mathring{K}(u+w)\mathring{K}(v+w)dudvdw < \infty$ and $\int_{-\infty}^{\infty} u^2 \mathring{K}(u)du < \infty$.

(A4c) The kernel function $\ddot{K}(\cdot)$ is symmetric around zero and satisfies $\ddot{\kappa}_1 \equiv \int_{-\infty}^{\infty} \ddot{K}(u)du = 1$, $\ddot{\kappa}_2 \equiv \int_{-\infty}^{\infty} \ddot{K}^2(u)du < \infty$, $\ddot{\kappa}_4 \equiv \iiint_{(-\infty,\infty)} \ddot{K}(u)\ddot{K}(v)\ddot{K}(u+w)\ddot{K}(v+w)dudvdw < \infty$ and $\int_{-\infty}^{\infty} u^2 \ddot{K}(u)du < \infty$.

(A4d) The kernels $K(x)$, $\mathring{K}(x)$ and $\ddot{K}(x)$ are all standard Gaussian kernels. That is: $K(x) = \mathring{K}(x) = \ddot{K}(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

(A5) The weighting function $w(\ell)$ is integrable over $(-\infty, \infty)$: i.e. $\int_{-\infty}^{\infty} w(\ell)d\ell < \infty$.

(A6) $E\left[\{N^k(B_1)N^k(B_2)N^k(B_3)N^k(B_4)\}\right] < \infty$ for $k = a, b$ and for all bounded Borel sets B_i on $\mathbb{R}$, $i = 1, 2, 3, 4$.

(A7) The rescaled counting process $\tilde{N}_u^k \equiv N_{Tu}^k/T$ (with natural filtration $\tilde{\mathcal{F}}^k$) has an $\tilde{\mathcal{F}}^k$-conditional intensity function $\tilde{\lambda}_u^k$, which is twice continuously differentiable with respect to u, and is unobservable but deterministic.

(A8) The joint cumulant $c_{22}(\ell_1, \ell_2, \ell_3)$ of $\{d\epsilon_s^a, d\epsilon_{s+\ell_1}^a, d\epsilon_{s+\ell_2}^b, d\epsilon_{s+\ell_3}^b\}$ is of order $o(a_T^2)$.

A.2 Figures

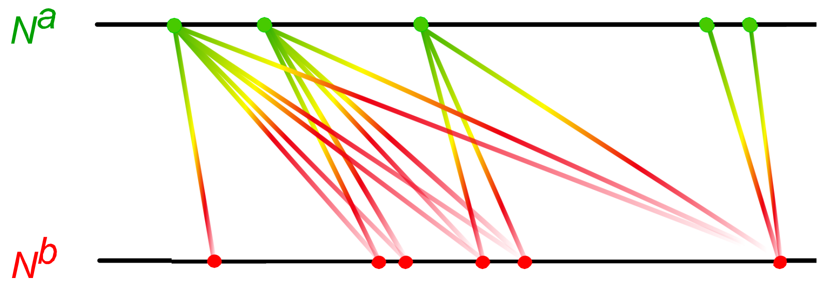
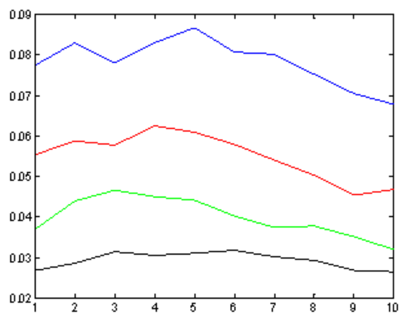
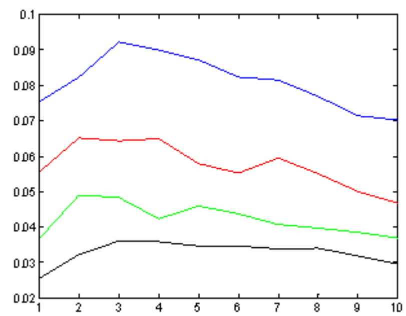


Figure 1: The statistic Q aggregates the squared contributions of residual products $d\hat{\epsilon}_s^a d\hat{\epsilon}_t^b$ for all $s < t$. The lines join all pairs of type a and type b events (shocks) at their event times (τ_i^a, τ_j^a) for all $\tau_i^a < \tau_j^a$.

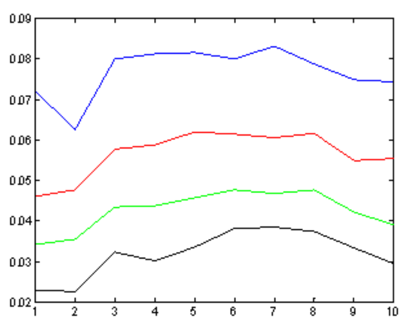
(a) $T = 500, B = 10, M = 60$



(b) $T = 1000, B = 10, M = 75$



(c) $T = 1500, B = 10, M = 100$



(d) $T = 2000, B = 10, M = 120$

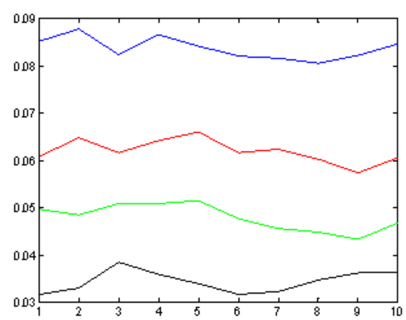


Figure 2: Size experiment of Q test, bivariate Poisson process.

Runs=5000, DGP= bivariate Poisson process (two independent Poisson processes with rate 0.1). Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

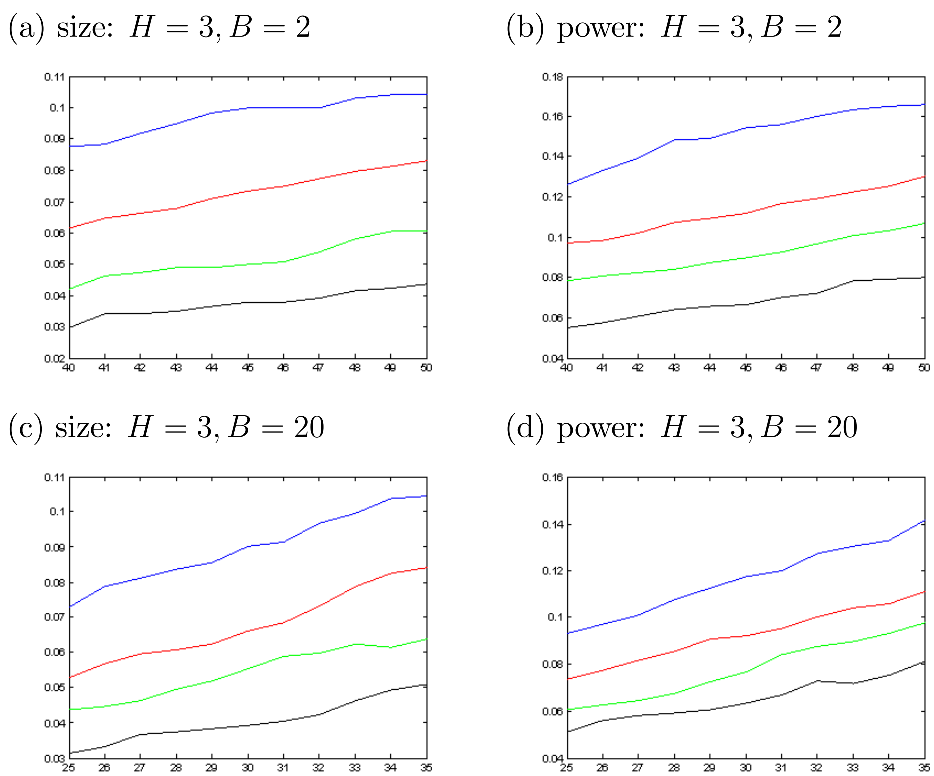
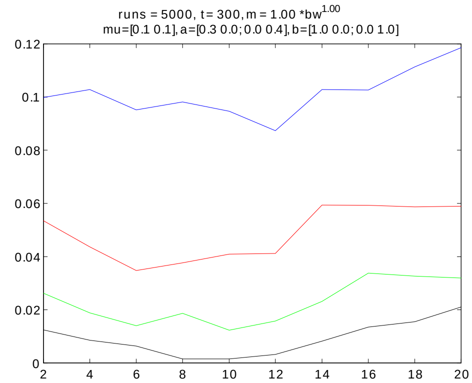
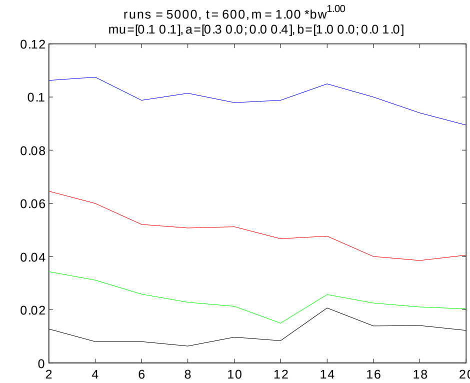


Figure 3: Size and power experiment of Q test, bivariate exponential Hawkes process. Runs=10000, T=1800, DGP= bivariate exponential Hawkes model in (23). Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

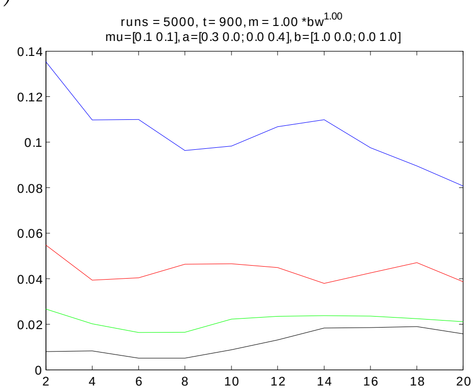
(a) $T = 300$



(b) $T = 600$



(c) $T = 900$



(d) $T = 1200$

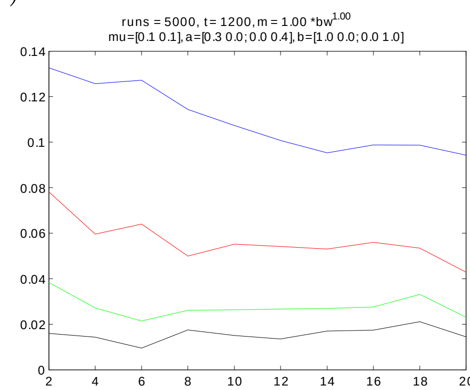
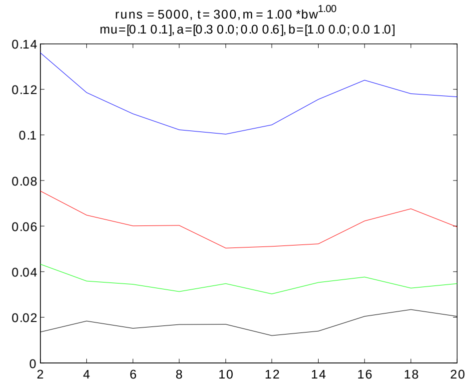


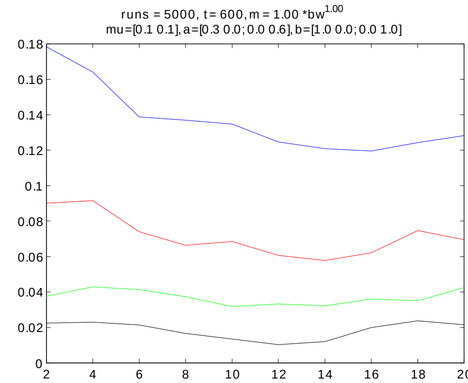
Figure 4: Size experiment 1 of Q^s test.

Runs=5000, DGP=bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

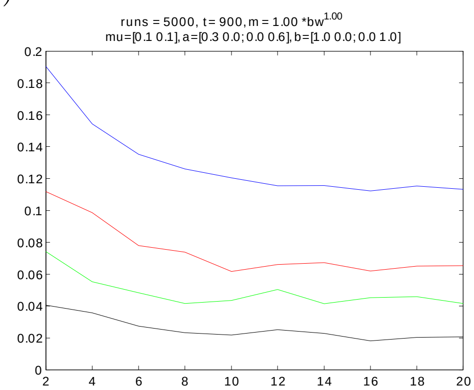
(a) $T = 300$



(b) $T = 600$



(c) $T = 900$



(d) $T = 1200$

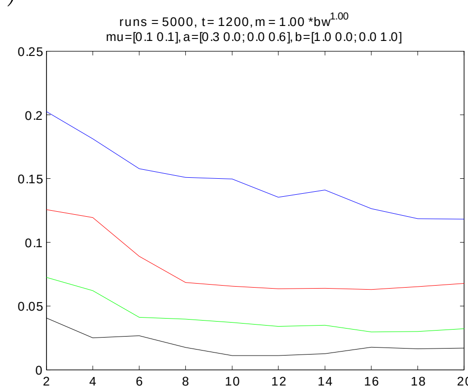
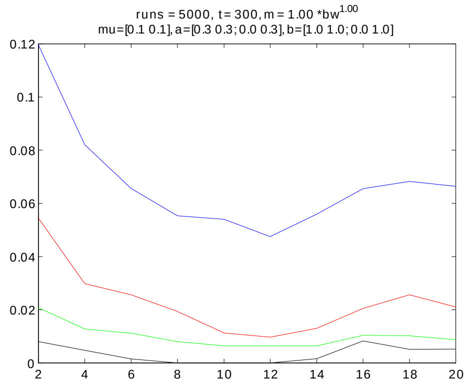


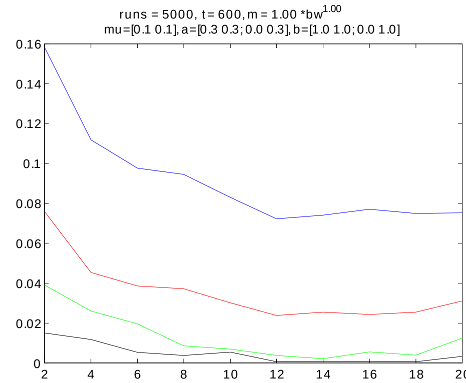
Figure 5: Size experiment 2 of Q^s test.

Runs=5000, DGP= bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

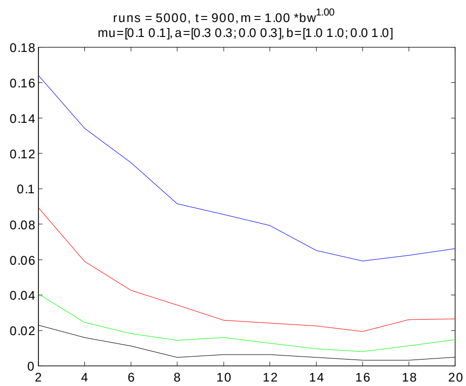
(a) $T = 300$



(b) $T = 600$



(c) $T = 900$



(d) $T = 1200$

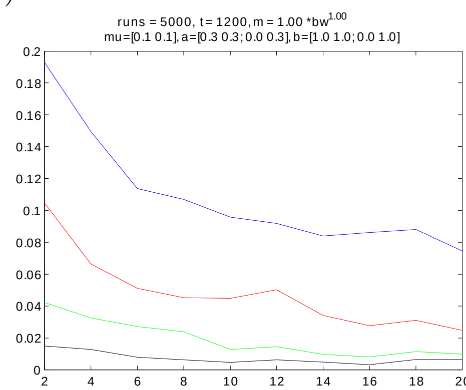
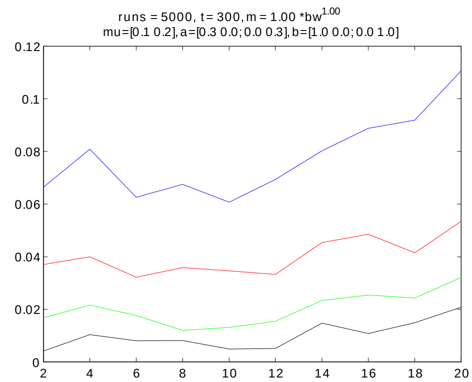


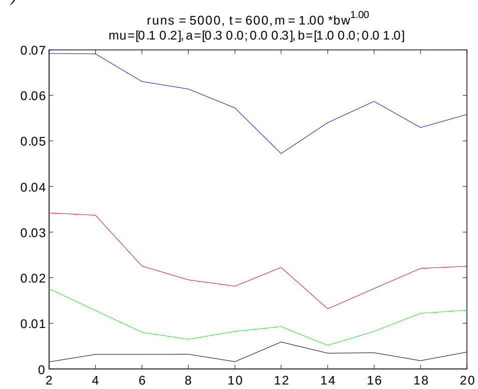
Figure 6: Size experiment 3 of Q^s test.

Runs=5000, DGP= bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.3 & 0.3 \\ 0 & 0.3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

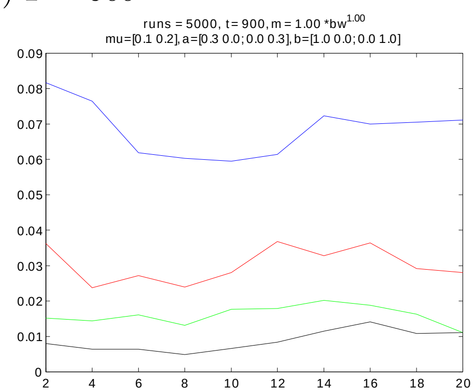
(a) $T = 300$



(b) $T = 600$



(c) $T = 900$



(d) $T = 1200$

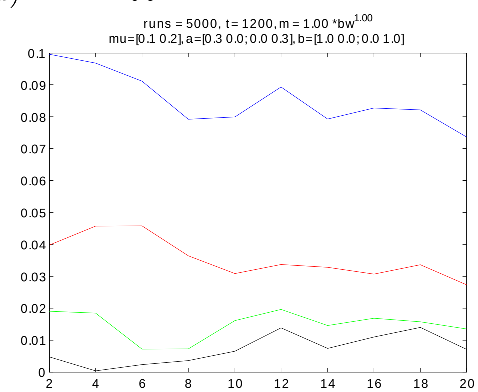


Figure 7: Size experiment 4 of Q^s test.

Runs=5000, DGP= bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

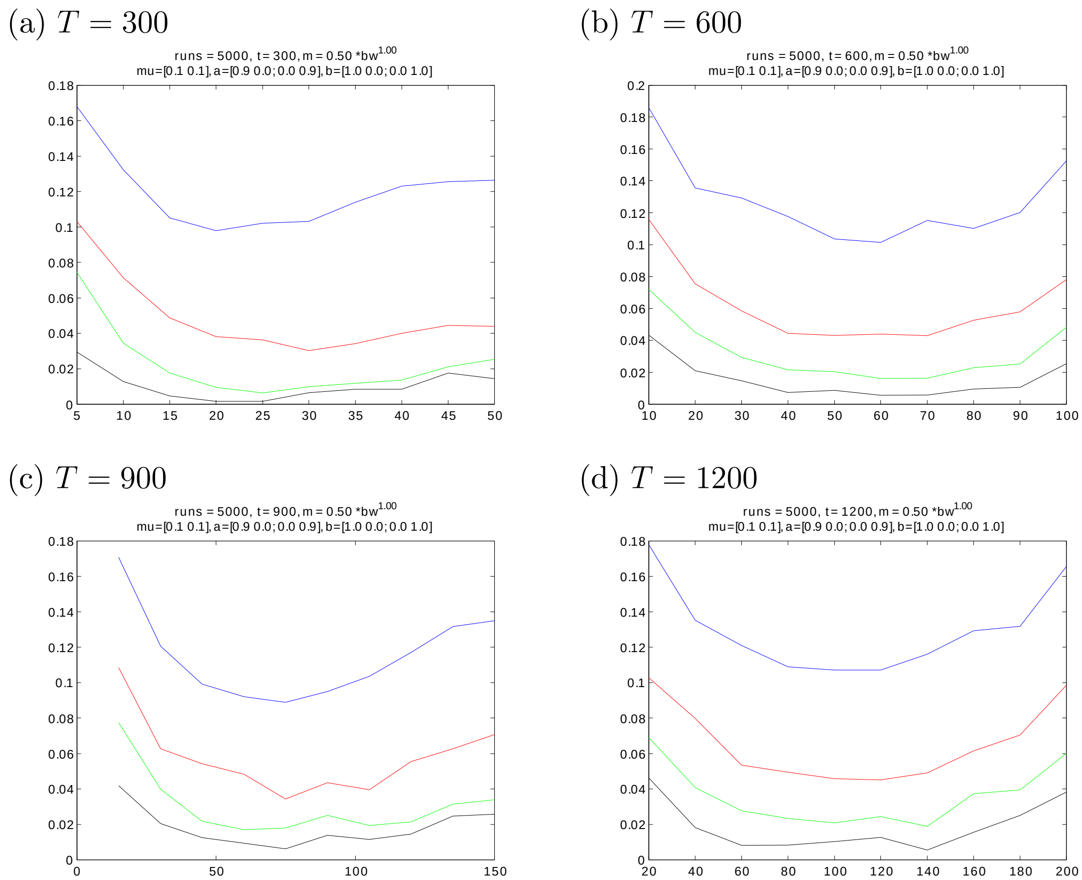


Figure 8: Size experiment 5 of Q^s test.

Runs=5000, DGP= bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

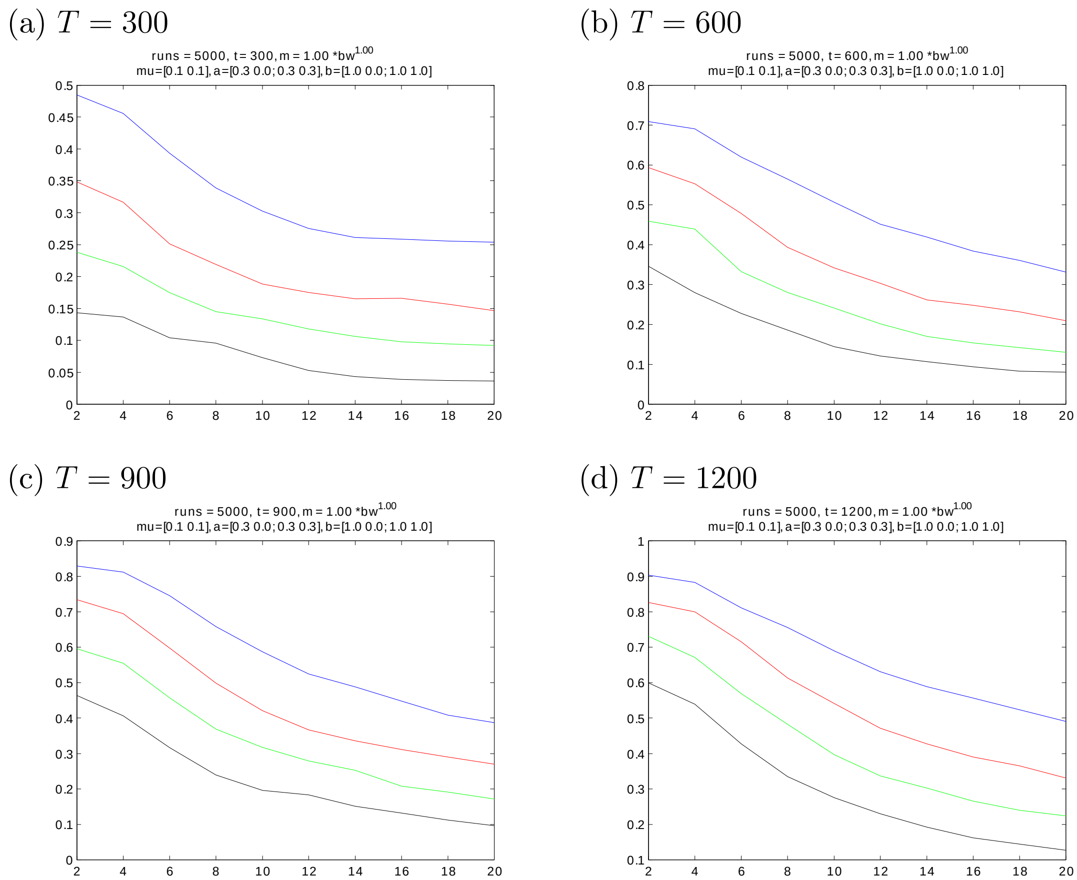


Figure 9: Power experiment of Q^s test.

Runs=5000, DGP= bivariate exponential Hawkes: $\mu = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0.3 & 0 \\ 0.3 & 0.3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Nominal size: blue=0.1; red=0.05; green=0.025; black=0.01.

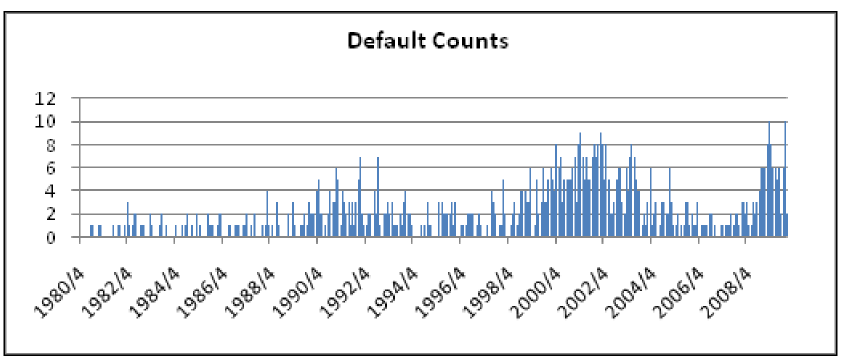


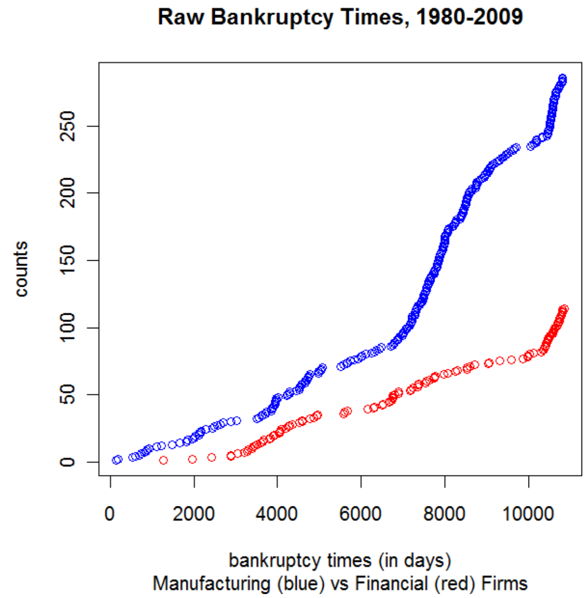
Figure 10: Histogram of bankruptcies of U.S. firms, 1980–2010.

Figure 11: Raw counts of bankruptcies in manufacturing and financial related sectors.

N_t^m (Blue): A: Agricultural; B: Mining; C: Construction; D: Manufacturing; E: Transportation, Communications, Electric, Gas; N_t^f (Red): F: Wholesale; G: Retail; H: Finance, Insurance, Real Estate; I: Services

A.3 Tables

Table 1: The asymptotic mechanisms of the two schemes.



Scheme	Observation window	Sample size	Limit	Duration
1	$[0, T]$	$n = N(T)$	$T \to \infty \Rightarrow n \to \infty$	$\tau_i - \tau_{i-1}$ fixed
2	$[0, T_0]$	$n = N(T_0)$	$n \to \infty$, T_0 fixed	$\tau_i - \tau_{i-1} \downarrow 0$

Table 2: Significant day counts (out of 41 days) of PG, 9:45am – 10:15am.

H	B	M	Trade → Quote			Quote → Trade		
sig. levels:			0.1	0.05	0.01	0.1	0.05	0.01
0.6	2	20	3	1	1	4	2	1
0.6	4	17	4	3	2	4	2	1
0.6	10	15	7	5	3	1	1	0
0.6	20	10	4	2	1	1	1	1
1	2	38	8	7	3	6	6	3
1	4	35	9	6	3	4	3	3
1	10	30	15	15	15	4	3	2
1	20	27	16	15	11	3	2	2
3	2	40	8	6	5	7	6	3
3	4	35	13	11	6	8	7	2
3	10	33	19	16	11	7	5	4
3	20	30	15	12	11	4	2	2

Mean number of trades=88.8, quotes=325.1. The bandwidth combinations give right sizes in simulations (with an estimated bivariate Hawkes model to PG data as DGP). Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 3: Significant day counts (out of 41 days) of PG, 11:45am – 12:45pm.

H	B	M	Trade → Quote			Quote → Trade		
sig. levels:			0.1	0.05	0.01	0.1	0.05	0.01
0.6	2	20	11	8	6	8	6	4
0.6	4	17	16	15	11	9	7	4
0.6	10	15	17	16	15	8	6	5
0.6	20	10	9	6	6	7	5	3
1	2	38	6	5	4	8	8	6
1	4	35	20	18	13	10	8	7
1	10	30	17	16	13	11	10	8
1	20	27	17	16	14	13	11	9
3	2	40	14	9	4	17	12	6
3	4	35	24	20	18	19	14	11
3	10	33	26	25	22	24	20	18
3	20	30	25	23	18	26	25	16

Mean number of trades=103.8, quotes=403.73. The bandwidth combinations give right sizes in simulations (with an estimated bivariate Hawkes model to PG data as DGP). Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 4: Significant day counts (out of 41 days) of PG, 3:30pm – 4:00pm.

H	B	M	Trade → Quote			Quote → Trade		
sig. levels:			0.1	0.05	0.01	0.1	0.05	0.01
0.6	2	20	1	0	0	2	1	1
0.6	4	17	7	5	3	1	1	0
0.6	10	15	8	7	7	0	0	0
0.6	20	10	6	5	3	1	1	0
1	2	38	4	3	2	4	3	1
1	4	35	5	5	4	2	2	1
1	10	30	18	18	16	6	5	2
1	20	27	13	13	13	2	2	0
3	2	40	5	5	3	6	4	3
3	4	35	10	9	7	6	6	4
3	10	33	14	13	12	7	7	5
3	20	30	10	10	9	8	6	2

Mean number of trades=93.7, quotes=361.56. The bandwidth combinations give right sizes in simulations (with an estimated bivariate Hawkes model to PG data as DGP). Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 5: Significant day counts (out of 41 days) of PG over various trading hours of a day.

PG	B	Trade → Quote			Quote → Trade		
periods	sig.:	0.1	0.05	0.01	0.1	0.05	0.01
09:45-10:15	10	13	8	0	1	0	0
	20	24	13	4	3	1	0
$\mu^t = 88.8$	30	23	14	4	2	1	0
$\mu^q = 325.1$	40	21	16	4	1	1	0
11:45-12:45	10	32	28	16	2	0	0
	20	35	34	21	3	0	0
$\mu^t = 103.8$	30	34	34	16	5	0	0
$\mu^q = 403.7$	40	33	30	14	6	0	0
15:30-16:00	10	26	12	3	0	0	0
	20	30	21	3	1	0	0
$\mu^t = 93.7$	30	26	16	6	3	1	0
$\mu^q = 361.6$	40	20	11	4	3	0	0

μ^t=mean number of trades, μ^q=mean number of quotes. The bandwidths R^k of unconditional autocorrelation estimators $\hat{c}^{kk}_{R^k}(\theta)$ (for $k = trade$ and $quote$) are set equal to B, the bandwidth of the weighting function $w_B(\theta)$.

Table 6: Significant day counts (out of 41 days) of GM over various trading hours of a day.

GM	B	Trade → Quote			Quote → Trade		
periods	sig.:	0.1	0.05	0.01	0.1	0.05	0.01
09:45-10:15	10	6	1	0	0	0	0
	20	13	8	0	1	1	0
$\mu^t = 65.4$	30	15	11	1	2	2	0
$\mu^q = 191.9$	40	17	11	4	2	2	0
11:45-12:45	10	26	16	6	9	2	1
	20	28	19	10	10	4	0
$\mu^t = 80.5$	30	26	18	8	7	2	0
$\mu^q = 217.3$	40	24	20	7	9	2	0
15:30-16:00	10	8	4	0	2	1	0
	20	12	7	1	4	1	0
$\mu^t = 65.1$	30	11	5	0	5	3	0
$\mu^q = 188.9$	40	10	5	0	6	2	1

μ^t=mean number of trades, μ^q=mean number of quotes. The bandwidths R^k of unconditional autocorrelation estimators $\hat{c}^{kk}_{R^k}(\theta)$ (for $k = trade$ and $quote$) are set equal to B, the bandwidth of the weighting function $w_B(\theta)$.

Table 7: Q tests on bankruptcy data, Sep96 – Jul03.

H	$B = M = 365$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 548$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 730$ $J^{m\to f}$	$J^{f\to m}$
2	2.32	3.56	0.09	12.66	-0.90	31.81
4	-3.85	5.87	-4.01	17.58	0.90	43.21
6	-3.14	-0.45	-2.21	10.87	5.97	49.86
8	-2.21	0.86	-0.82	15.22	9.21	49.06
10	-1.53	1.93	0.15	18.63	11.52	57.37
12	-1.06	2.62	0.86	21.12	13.32	63.88
14	-0.67	3.04	1.41	22.93	14.73	69.04

Sample sizes: $(n^m, n^m) = (209, 149)$. $m \to f$ ($f \to m$) denotes bankruptcy contagion from manufacturing related to financial related firms (and vice versa). One-sided critical values: $z_{0.05} = 1.64$; $z_{0.01} = 2.33$. Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 8: Q test on bankruptcy data, Aug03 – Aug07.

H	$B = M = 365$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 548$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 730$ $J^{m\to f}$	$J^{f\to m}$
2	-5.12	0.55	-4.56	-1.21	-2.98	-1.77
4	-3.13	1.77	-2.24	-0.24	0.01	-0.71
6	-2.70	1.17	-1.38	-0.34	1.46	-0.21
8	-1.96	0.32	-0.21	-0.68	3.17	0.04
10	-1.14	-0.07	0.96	-0.78	4.77	0.35
12	-0.46	-0.09	1.88	-0.65	5.95	0.75
14	0.04	0.10	2.53	-0.41	6.72	1.17

Sample sizes: $(n^m, n^m) = (65, 29)$. $m \to f$ ($f \to m$) denotes bankruptcy contagion from manufacturing related to financial related firms (and vice versa). One-sided critical values: $z_{0.05} = 1.64$; $z_{0.01} = 2.33$. Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 9: Q tests on bankruptcy data, Sep07 – Jun10.

H	$B = M = 365$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 548$ $J^{m\to f}$	$J^{f\to m}$	$B = M = 730$ $J^{m\to f}$	$J^{f\to m}$
2	19.37	7.58	50.42	28.11	83.53	52.44
4	10.25	14.67	46.67	39.19	89.14	73.09
6	12.80	17.67	56.57	56.38	108.40	100.61
8	15.37	20.86	65.69	65.72	125.76	117.12
10	22.14	23.29	73.90	73.13	141.37	130.41
12	23.24	25.23	81.46	79.42	155.63	141.87
14	24.43	26.87	93.37	85.00	175.19	152.16

Sample sizes: $(n^m, n^m) = (78, 71)$. $m \to f$ ($f \to m$) denotes bankruptcy contagion from manufacturing related to financial related firms (and vice versa). One-sided critical values: $z_{0.05} = 1.64$; $z_{0.01} = 2.33$. Bandwidths (in days) of (i) cross-covariance function: H; (ii) weighting function: B; (iii) conditional intensity: M. All kernels are Gaussian.

Table 10: Q^s test on bankruptcy data, September 1996 – June 2010.

B	$m \to f$	p-value	$f \to m$	p-value
30	1.79	0.037	1.78	0.038
60	1.74	0.041	1.93	0.027
90	1.66	0.049	1.99	0.024
120	1.58	0.057	1.99	0.024
150	1.51	0.066	1.96	0.025
180	1.43	0.076	1.93	0.027
210	1.35	0.088	1.89	0.029
240	1.27	0.103	1.86	0.032
270	1.18	0.119	1.82	0.034
300	1.09	0.138	1.79	0.037

$m \to f$ ($f \to m$) denotes bankruptcy contagion from manufacturing related to financial related firms (and vice versa). One-sided critical values: $z_{0.05} = 1.64$; $z_{0.01} = 2.33$. Bandwidth (in days) of the weighting function: B. Bandwidths R^k of autocovariance function estimators are set equal to 300. All kernels are Gaussian.

Table 11: Trading hours, Greenwich mean time and start dates of the sampling periods of major stock indices.

Index	Trading hours (local time)	GMT	Start date
DJI	09:30 - 16:00	-5	10/1/1928
FTSE	08:00 - 16:30	+0	4/2/1984
DAX	09:00 - 17:30	+1	11/26/1990
CAC	09:00 - 17:30	+1	3/1/1990
HSI	10:00 - 12:30, 14:30 - 16:00[36]	+8	12/31/1986
STI	09:00 - 12:30, 2:00 - 5:00	+8	12/28/1987
TWI	09:00 - 13:30	+8	7/2/1997
NIK	09:00 - 11:00, 12:30 - 15:00	+9	1/4/1984
AOI	10:00 - 16:00	+10	8/3/1984

Adjusted daily index values were collected from Yahoo! Finance. The end date of all the time series is 8/19/2011. Each time, a Granger causality test is performed on the event sequences of a pair of indices, with the sampling period equal to the shorter of the two sampling periods of the two indices.

Table 12: Q^s test applied to extreme negative shocks of DJI and HSI.

	90% VaR		95% VaR		99% VaR	
B	HSI→DJI	DJI→HSI	HSI→DJI	DJI→HSI	HSI→DJI	DJI→HSI
1	0.92	1.69	1.28	3.00	7.35	9.61
2	0.75	1.17	1.04	2.16	5.98	7.92
3	0.67	0.95	0.98	1.81	5.23	6.99
5	0.60	0.76	0.96	1.49	4.63	6.44
10	0.53	0.60	0.96	1.19	4.06	5.85
(n_1, n_2)	(608,620)		(304,310)		(61,62)	

The bandwidths of the autocovariance functions are chosen to be $R^k = 10.5B^{0.3}$.

[36] Trading hours starting from March 7, 2011: 09:30 - 12:00 and 13:30 - 16:00.

Table 13: Q^s test applied to extreme negative shocks of DJI and NIK.

	90% VaR		95% VaR		99% VaR	
B	NIK→DJI	DJI→NIK	NIK→DJI	DJI→NIK	NIK→DJI	DJI→NIK
1	0.56	1.64	1.23	2.87	5.70	10.62
2	0.46	1.13	1.00	2.05	5.21	7.84
3	0.43	0.93	0.90	1.72	5.12	7.01
5	0.41	0.74	0.82	1.42	4.93	6.52
10	0.38	0.57	0.74	1.11	4.49	6.18
(n_1, n_2)	(604,614)		(303,306)		(63,59)	

The bandwidths of the autocovariance functions are chosen to be $R^k = 10.5B^{0.3}$.

Table 14: Q^s test applied to extreme negative shocks of DJI and FTSE.

	90% VaR		95% VaR		99% VaR	
B	FTS→DJI	DJI→FTS	FTS→DJI	DJI→FTS	FTS→DJI	DJI→FTS
1	2.88	0.88	4.93	1.76	18.53	5.81
2	1.82	0.86	3.18	1.74	13.25	6.35
3	1.46	0.81	2.59	1.68	11.25	6.38
5	1.16	0.76	2.07	1.62	9.16	6.46
10	0.90	0.68	1.65	1.45	7.31	6.56
(n_1, n_2)	(621,620)		(311,310)		(63,62)	

The bandwidths of the autocovariance functions are chosen to be $R^k = 10.5B^{0.3}$.

Table 15: Q^s test applied to extreme negative shocks of DJI and AOI.

	90% VaR		95% VaR		99% VaR	
B	AOI→DJI	DJI→AOI	AOI→DJI	DJI→AOI	AOI→DJI	DJI→AOI
1	0.59	2.32	1.37	4.72	7.10	14.28
2	0.60	1.48	1.29	3.19	6.80	10.82
3	0.60	1.16	1.25	2.56	6.39	9.58
5	0.57	0.89	1.17	1.98	5.57	8.59
10	0.50	0.67	1.08	1.47	4.74	7.82
(n_1, n_2)	(679,680)		(340,341)		(68,69)	

The bandwidths of the autocovariance functions are chosen to be $R^k = 10.5B^{0.3}$.

A.4 Proof of Theorem 5

I first expand (12) into

$$\begin{aligned}\hat{\gamma}_H(\ell) &= \tfrac{1}{TH}\int_0^T\int_0^T K\left(\tfrac{t-s-\ell}{H}\right)\left(dN_s^a-\hat{\lambda}_s^a ds\right)\left(dN_t^b-\hat{\lambda}_t^b dt\right)\\ &= \tfrac{1}{TH}\int_0^T\int_0^T K\left(\tfrac{t-s-\ell}{H}\right)\left[dN_s^a dN_t^b-\hat{\lambda}_s^a ds dN_t^b - dN_s^a\hat{\lambda}_t^b dt+\hat{\lambda}_s^a\hat{\lambda}_t^b ds dt\right]\\ &= A_1+A_2+A_3+A_4. \end{aligned}\tag{24}$$

The first term is

$$A_1=\tfrac{1}{TH}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}K\left(\tfrac{t_j^b-t_i^a-\ell}{H}\right).$$

The second term, after substituting $\hat{\lambda}_t^b$ by (14), becomes

$$A_2=-\tfrac{1}{THM}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}\int_0^T K\left(\tfrac{t_j^b-s-\ell}{H}\right)\mathring{K}\left(\tfrac{s-t_i^a}{M}\right)ds.$$

Similarly, the third term is

$$A_3=-\tfrac{1}{THM}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}\int_0^T K\left(\tfrac{t-t_i^a-\ell}{H}\right)\mathring{K}\left(\tfrac{t_j^b-t}{M}\right)dt,$$

and the fourth one is

$$A_4=\tfrac{1}{THM^2}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}\int_0^T\int_0^T K\left(\tfrac{t-s-\ell}{H}\right)\mathring{K}\left(\tfrac{s-t_i^a}{M}\right)\mathring{K}\left(\tfrac{t_j^b-t}{M}\right)ds dt.$$

Note that the last three terms involve the convolution of the kernels $K(\cdot)$ and $\mathring{K}(\cdot)$ (twice for A_4).

Under assumption (A4d), I can simplify the expressions further, as it is well known that Gaussian kernels are invariant under convolution: for any $H_1, H_2>0$, the Gaussian kernel $K(\cdot)$ enjoys the property that

$$\tfrac{1}{H_1H_2}\int_{-\infty}^{\infty}K\left(\tfrac{x-z}{H_1}\right)K\left(\tfrac{z}{H_2}\right)dz=\tfrac{1}{\sqrt{H_1^2+H_2^2}}K\left(\tfrac{x}{\sqrt{H_1^2+H_2^2}}\right).$$

Using this invariance property and a change of variables, I can simplify the integrations and rewrite (24) as

$$\hat{\gamma}_H(\ell)=\tfrac{1}{T}\sum_{i=1}^{N_T^a}\sum_{j=1}^{N_T^b}\left[\tfrac{1}{H}K\left(\tfrac{t_j^b-t_i^a-\ell}{H}\right)-\tfrac{2}{\sqrt{H^2+M^2}}K\left(\tfrac{t_j^b-t_i^a-\ell}{\sqrt{H^2+M^2}}\right)+\tfrac{1}{\sqrt{H^2+2M^2}}K\left(\tfrac{t_j^b-t_i^a-\ell}{\sqrt{H^2+2M^2}}\right)\right].$$

A.5 Proof of Theorem 6

Let us prove asymptotic normality first. For notational convenience, I drop the superscript k of N_t^k, λ_t^k and their rescaled version in this proof. Let $\lambda_t^* = \int_0^T \frac{1}{M}\mathring{K}\left(\frac{t-s}{M}\right)\lambda_s ds$, then I can rewrite (18) into

$$\sqrt{M}\left(\frac{\hat{\lambda}_{Tv}-\lambda_{Tv}^*}{\sqrt{\lambda_{Tv}}}\right)+\sqrt{M}\left(\frac{\lambda_{Tv}^*-\lambda_{Tv}}{\sqrt{\lambda_{Tv}}}\right)=: X_1+X_2. \tag{25}$$

Suppose $\tilde{\mathcal{F}}^k$ denotes the natural filtration of the rescaled counting process $\tilde{N}^k$. Then, it follows from (16) that $\tilde{\lambda}_u = \lambda_{Tu}$.

With a change of variables $t = Tv$ and $s = Tu$, the first term of (25) becomes

$$\begin{aligned} X_1 &= \int_0^1 \frac{1}{\sqrt{M}}\mathring{K}\left(\frac{v-u}{M/T}\right)\left(\frac{dN_{Tu}-\lambda_{Tu}Tdu}{\sqrt{\lambda_{Tv}}}\right) \\ &= \int_0^1 \frac{1}{\sqrt{M}}\mathring{K}\left(\frac{v-u}{M/T}\right)\left(\frac{d\tilde{N}_u-\tilde{\lambda}_u du}{\sqrt{\tilde{\lambda}_v/T}}\right). \end{aligned}$$

The multiplicative model in Ramlau-Hansen (R-H, 1983) assumes that $\tilde{\lambda}_u \equiv \tilde{\lambda}_u^{(n)} = Y_u^{(n)}\alpha_u$ for each $n \equiv \tilde{N}(1) = N(T)$.[37] Let $J_u^{(n)} = 1\{Y_u^{(n)} > 0\}$ and $b_n = M/T$. Then, following the last line above, I obtain

$$\begin{aligned} X_1 &= \int_0^1 \frac{1}{\sqrt{b_n T}} J_u^{(n)}\mathring{K}\left(\frac{v-u}{b_n}\right)\left(\frac{d\tilde{N}_u^{(n)}-Y_u^{(n)}\alpha_u du}{\sqrt{Y_v^{(n)}\alpha_v/T}}\right) \\ &= \int_0^1 \frac{1}{\sqrt{b_n}} J_u^{(n)}\sqrt{Y_u^{(n)}}\mathring{K}\left(\frac{v-u}{b_n}\right)\left(\frac{d\tilde{N}_u^{(n)}/Y_u^{(n)}-\alpha_u du}{\sqrt{\alpha_v}}\right) \\ &= \sqrt{nb_n}\int_0^1 \frac{1}{b_n} J_u^{(n)}\sqrt{\frac{Y_u^{(n)}}{n}}\mathring{K}\left(\frac{v-u}{b_n}\right)\left(\frac{d\tilde{N}_u^{(n)}/Y_u^{(n)}-\alpha_u du}{\sqrt{\alpha_v}}\right). \end{aligned} \tag{26}$$

Theorem 4.2.2 of R-H states that if (i) $nJ^{(n)}/Y^{(n)} \to^P 1/\varsigma$ uniformly around v as $n \to \infty$; and (ii) α and ς are continuous at v, then

$$\sqrt{nb_n}\int_0^1 \frac{1}{b_n} J_u^{(n)}\mathring{K}\left(\frac{v-u}{b_n}\right)\left(\frac{d\tilde{N}_u^{(n)}}{Y_u^{(n)}}-\alpha_u du\right) \to^d N\left(0, \frac{\mathring{\kappa}_2\alpha_v}{\varsigma_v}\right) \tag{27}$$

as $n \to \infty$, $b_n \to 0$ and $nb_n \to \infty$. By picking $Y_u^{(n)} \equiv T$ and noting the twice continuous differentiability of $\tilde{\lambda}_u$ assumed by the theorem, assumptions (i) and (ii) are automatically satisfied. This implies that $X_1 \to^d N(0, \mathring{\kappa}_2)$ as $T \to \infty$, $M \to \infty$ and $M/T \to \infty$.

[37] The superscript (n) indicates the dependence of the relevant quantity on the sample size n.

To complete the proof, it suffices to show that the second term X_2 of (25) is asymptotically negligible relative to the first term, which was just shown to be $O_P(1)$. Indeed, by symmetry of the kernel $\mathring{K}()$ and the twice continuous differentiability of $\tilde{\lambda}_u$, I obtain

$$\begin{aligned}
\lambda^*_{Tv} - \lambda_{Tv} &= \frac{1}{T}\int_0^1 \frac{T}{M}\mathring{K}\left(\frac{v-u}{M/T}\right)\lambda_{Tu}Tdu - \lambda_{Tv} \\
&= \int_0^1 \frac{1}{m}\mathring{K}\left(\frac{v-u}{m}\right)\tilde{\lambda}_u du - \lambda_{Tv} \\
&= \int_0^1 \mathring{K}(x)\,\tilde{\lambda}_{v-mx}dx - \lambda_{Tv} \\
&= \tilde{\lambda}_v - \lambda_{Tv} + m\tilde{\lambda}'_v \int_0^1 x\mathring{K}(x)\,dx + O_P(m^2) \\
&= O_P\left(\left(\frac{M}{T}\right)^2\right).
\end{aligned}$$

If $M^5/T^4 \to 0$ (which corresponds to $nb_n^5 \to 0$), then $X_2 = \sqrt{M}\left(\lambda^*_{Tv} - \lambda_{Tv}\right)/\sqrt{\lambda_{Tv}} = O_P\left(M^{2.5}/T^2\right) = o_P(1)$, and thus is asymptotically negligible relative to X_1.

For mean-squared consistency of $\hat{\lambda}_{Tv}$, simply apply Proposition 3.2.2 of R-H.

A.6 Proof of Theorem 7

For notational simplicity, I only treat the case where $I = [0,T]$. Under the null hypothesis, the innovations from the two processes are uncorrelated, which implies that $E\left(dN^a_s dN^b_{s+\ell}\right) = E\left(dN^a_s\right)E\left(dN^b_{s+\ell}\right) = \lambda^a\lambda^b ds d\ell$, so that

$$\begin{aligned}
E\left(Q^s\right) &= \frac{1}{T^2}\int_I\int_J w_B(\ell)\,E\left(dN^a_s dN^b_{s+\ell}\right) \\
&= \frac{\lambda^a\lambda^b}{T^2}\int_I\int_J w_B(\ell)\,ds d\ell \\
&= \frac{\lambda^a\lambda^b}{T}\int_I w_B(\ell)\left(1-\tfrac{|\ell|}{T}\right)d\ell.
\end{aligned}$$

Before computing the variance, let us recall that the second-order reduced product density of N^k (which exists by assumption (A2)) was defined by $\varphi^{kk}(u)dtdu = E\left(dN^k_t dN^k_{t+u}\right)$ for $u \neq 0$, and the unconditional autocovariance density function can thus be expressed as $c^{kk}(u)dtdu = E\left(dN^k_t - \lambda^k dt)(dN^k_{t+u} - \lambda^k du\right) = \left[\varphi^{kk}(u) - \left(\lambda^k\right)^2\right]dtdu$ for $u \neq 0$. Then, under the null hypothesis, I obtain

$$\begin{aligned}
E((Q^s)^2) &= \frac{1}{T^4}\iint_{I^2}\iint_{J^2} w_B(\ell_1)\,w_B(\ell_2)\,E\left(dN^a_{s_1}dN^a_{s_2}dN^b_{s_1+\ell_1}dN^b_{s_2+\ell_2}\right) \\
&= \frac{1}{T^4}\iint_{I^2}\iint_{J^2} w_B(\ell_1)\,w_B(\ell_2)\,E\left(dN^a_{s_1}dN^a_{s_2}\right)E\left(dN^b_{s_1+\ell_1}dN^b_{s_2+\ell_2}\right).
\end{aligned}$$

I can decompose the differential as follows:

$$\begin{aligned}
& E\left(dN^a_{s_1} dN^a_{s_2} dN^b_{s_1+\ell_1} dN^b_{s_2+\ell_2}\right) \\
= \; & E\left(dN^a_{s_1} dN^a_{s_2}\right) E\left(dN^b_{s_1+\ell_1} dN^b_{s_2+\ell_2}\right) \\
= \; & \left[E\left(dN^a_{s_1} dN^a_{s_2}\right) - (\lambda^a)^2 ds_1 ds_2\right] \left[E\left(dN^b_{s_1+\ell_1} dN^b_{s_2+\ell_2}\right) - \left(\lambda^b\right)^2 d\ell_1 d\ell_2\right] \\
& + \left(\lambda^b\right)^2 \left[E\left(dN^a_{s_1} dN^a_{s_2}\right) - (\lambda^a)^2 ds_1 ds_2\right] d\ell_1 d\ell_2 \\
& + (\lambda^a)^2 \left[E\left(dN^b_{s_1+\ell_1} dN^b_{s_2+\ell_2}\right) - \left(\lambda^b\right)^2 d\ell_1 d\ell_2\right] ds_1 ds_2 \\
& + (\lambda^a)^2 \left(\lambda^b\right)^2 ds_1 ds_2 d\ell_1 d\ell_2 \\
= \; & c^{aa}(s_2 - s_1) c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
& + \left(\lambda^b\right)^2 c^{aa}(s_2 - s_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
& + (\lambda^a)^2 c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
& + (\lambda^a)^2 \left(\lambda^b\right)^2 ds_1 ds_2 d\ell_1 d\ell_2.
\end{aligned}$$

Note that the integral term associated with the last differential is $[E(Q^s)]^2$, so that

$$\begin{aligned}
Var(Q^s) \; = \; & \frac{1}{T^4} \iint_{I^2} \iint_{J^2} w_B(\ell_1) w_B(\ell_2) c^{aa}(s_2 - s_1) c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
& + \frac{1}{T^4} \iint_{I^2} \iint_{J^2} w_B(\ell_1) w_B(\ell_2) \left(\lambda^b\right)^2 c^{aa}(s_2 - s_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
& + \frac{1}{T^4} \iint_{I^2} \iint_{J^2} w_B(\ell_1) w_B(\ell_2) (\lambda^a)^2 c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2 \\
= \; & A_1 + A_2 + A_3.
\end{aligned}$$

Suppose $I = [0, T]$. I evaluate the three terms individually as follows.

(i) the first term becomes

$$A_1 = \frac{2}{T^4} \int_0^T \int_0^{\ell_2} \int_{J_2} \int_{J_1} w_B(\ell_1) w_B(\ell_2) c^{aa}(s_2 - s_1) c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2.$$

where $J_i = [0, T - \ell_i]$ for $i = 1, 2$. With a change of variables

$$(s_1, s_2, \ell_1, \ell_2) \mapsto (v = s_2 - s_1, s_2, u = \ell_2 - \ell_1, \ell_2),$$

I can rewrite A_1 into

$$\begin{aligned}
A_1 \; = \; & \tfrac{2}{T^4} \int_0^T \int_u^T \int_{-T}^{T-\ell_2} \int_0^{T-\ell_2} w_B(\ell_2 - u) w_B(\ell_2) c^{aa}(v) c^{bb}(v + u) ds_2 dv d\ell_2 du \\
= \; & \tfrac{2}{T^3} \int_0^T \int_u^T w_B(\ell_2 - u) w_B(\ell_2) \left(1 - \tfrac{\ell_2}{T}\right) \int_{-T}^{T-\ell_2} c^{aa}(v) c^{bb}(v + u) dv d\ell_2 du.
\end{aligned}$$

To simplify further, I rely on the assumption that the bandwidth of $w(\ell) \equiv w_B(\ell)$

is small relative to T, i.e. $B = o(T)$. Then, the integral $\int_{-T}^{T-\ell_2} c^{aa}(v)\, c^{bb}(v+u)\, dv$ can be well approximated by $\Gamma(u) := \int_{-T}^{T} c^{aa}(v)\, c^{bb}(v+u)\, dv$, and hence

$$A_1 \approx \tfrac{2}{T^3} \int_0^T \mathcal{W}_2(u)\Gamma(u)\, du.$$

where we defined a new weighting function by $\mathcal{W}_2(u) := \int_u^T w_B(\ell - u)\, w_B(\ell) \left(1 - \frac{\ell}{T}\right) d\ell$. Figure 12 gives a plot of $\mathcal{W}_2(u)$ when $w(\cdot)$ is a standard normal density function and T is large ($T \gg 3$).

(ii) With a change of variables $(s_1, s_2) \longmapsto (v = s_2 - s_1, s_2)$, the second term becomes

$$\begin{aligned} A_2 &= \tfrac{(\lambda^b)^2}{T^4} \iint_{I^2} \iint_{J^2} w_B(\ell_1)\, w_B(\ell_2)\, \gamma^a(s_2 - s_1) ds_1 ds_2 d\ell_1 d\ell_2 \\ &= \tfrac{(\lambda^b)^2}{T^4} \int_0^T \int_0^T w_B(\ell_1)\, w_B(\ell_2) \int_{-(T-\ell_1)}^{T-\ell_2} \int_v^{v+T-\ell_1} c^{aa}(v) ds_2 dv d\ell_1 d\ell_2 \\ &= \tfrac{(\lambda^b)^2}{T^3} \int_0^T \int_0^T w_B(\ell_1)\, w_B(\ell_2) \left(1 - \tfrac{\ell_1}{T}\right) \int_{-(T-\ell_1)}^{T-\ell_2} c^{aa}(v) dv d\ell_1 d\ell_2. \end{aligned}$$

To simplify further, I rely on the assumption that the bandwidth B of the weighting function $w_B(\cdot)$ is small relative to T, i.e. $B = o(T)$. Then, the following holds approximately:

$$\int_{-(T-\ell_1)}^{T-\ell_2} c^{aa}(v) dv \approx \int_{-T}^{T} c^{aa}(v) dv.$$

As a result, we obtain

$$A_2 \approx \tfrac{2(\lambda^b)^2}{T^3} \omega_1 \int_0^T c^{aa}(v) dv$$

where I defined the constant $\omega_1 := \int_0^T w_B(\ell) \left(1 - \frac{\ell}{T}\right) d\ell = \int_0^{T/B} w(u) \left(1 - \frac{Bu}{T}\right) du$.

(iii) With a change of variables $(s_1, s_2) \longmapsto (x = s_2 - s_1 + \ell_2 - \ell_1, s_2)$, the third term becomes

$$\begin{aligned} A_3 &= \tfrac{(\lambda^a)^2}{T^4} \iint_{I^2} \iint_{J^2} w_B(\ell_1)\, w_B(\ell_2)\, c^{bb}(s_2 - s_1 + \ell_2 - \ell_1) ds_1 ds_2 d\ell_1 d\ell_2 \\ &= \tfrac{(\lambda^a)^2}{T^4} \int_0^T \int_0^T w_B(\ell_1)\, w_B(\ell_2) \int_{-(T-\ell_2)}^{T-\ell_1} \int_{x+\ell_2-\ell_1}^{x+T-\ell_1+\ell_2-\ell_1} c^{bb}(x) ds_2 dx d\ell_1 d\ell_2 \\ &= \tfrac{(\lambda^a)^2}{T^3} \int_0^T \int_0^T w_B(\ell_1)\, w_B(\ell_2) \left(1 - \tfrac{\ell_1}{T}\right) \int_{-(T-\ell_2)}^{T-\ell_1} c^{bb}(x) dx d\ell_1 d\ell_2. \end{aligned}$$

To simplify further, I rely on the assumption that the bandwidth B of the weighting function $w_B(\cdot)$ is small relative to T, i.e. $B = o(T)$. Then, the following holds approximately:

$$\int_{-(T-\ell_2)}^{T-\ell_1} c^{bb}(v)dv \approx \int_{-T}^{T} c^{bb}(v)dv.$$

As a result, I obtain

$$A_3 \approx \tfrac{2(\lambda^a)^2}{T^3}\omega_1 \int_0^T c^{bb}(v)dv.$$

Combining the above three terms A_i for $i = 1, 2, 3$, I obtain an approximation to the variance of Q^s:

$$Var(Q^s) \approx \tfrac{2}{T^3}\left[\int_0^T \mathcal{W}_2(u)\Gamma(u)\,du + \left(\lambda^b\right)^2\omega_1 \int_0^T c^{aa}(v)dv + \left(\lambda^a\right)^2\omega_1 \int_0^T c^{bb}(v)dv\right].$$

A.7 Proof of (21)

For notational convenience, I drop the superscript k from all relevant symbols throughout this proof. Let $R/T \to 0$ as $T \to \infty$. I start by decomposing $\hat{c}_R(\ell) \equiv \hat{c}^{kk}_{R^k}(\ell)$ as follows:

$$\begin{aligned}
\hat{c}_R(\ell) &= \tfrac{1}{TR}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)\left(dN_s - \tfrac{N_T}{T}ds\right)\left(dN_t - \tfrac{N_T}{T}dt\right) \\
&= \tfrac{1}{TR}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)dN_s dN_t - \tfrac{1}{TR}\tfrac{N_T}{T}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)ds dN_t \\
&\quad - \tfrac{1}{TR}\tfrac{N_T}{T}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)dN_s^a dt + \tfrac{1}{TR}\tfrac{N_T}{T}\tfrac{N_T}{T}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)ds dt \\
&=: C_1 + C_2 + C_3 + C_4.
\end{aligned}$$

Now, the second term is

$$\begin{aligned}
C_2 &= -\tfrac{1}{TR}\tfrac{N_T}{T}\int_0^T\int_0^T \ddot{K}\left(\tfrac{t-s-\ell}{R}\right)ds dN_t^b \\
&= -\tfrac{N_T}{T^2}\sum_{j=1}^{N_T}\int_0^T \tfrac{1}{R}\ddot{K}\left(\tfrac{t_j-s-\ell}{R}\right)ds = -\tfrac{N_T}{T^2}\sum_{j=1}^{N_T}\int_{(t_j^b-T-\ell)/R}^{(t_j^b-\ell)/R}\ddot{K}(x)\,dx \\
&= -\tfrac{N_T}{T^2}\sum_{j=1}^{N_T}\left[1_{\{\ell<t_j<T+\ell\}\cap[0,T]} + o(1)\right] \\
&= -\tfrac{N_T}{T^2}\left[N_{T\wedge(T+\ell)} - N_{\ell\vee 0}\right] + o(1),
\end{aligned}$$

where the third equality made use of assumption (A4c). By stationarity of N^k, I observe that $N_{T\wedge(T+\ell)} - N_{\ell\vee 0} = \frac{T-|\ell|}{T}N_T$. Therefore, up to the leading term,

$$C_2 = -\tfrac{N_T^2}{T^2}\left(1 - \tfrac{|\ell|}{T}\right).$$

Similarly, by stationarity of N^b, the third term is, up to the leading term,

$$C_3 = -\frac{N_T^2}{T^2}\left(1 - \frac{|\ell|}{T}\right) = C_2.$$

The last term is

$$\begin{aligned} C_4 &= \frac{1}{T}\frac{N_T}{T}\frac{N_T}{T}\int_0^T \int_0^T \frac{1}{R}\ddot{K}\left(\frac{t-s-\ell}{R}\right) dsdt \\ &= \frac{N_T^2}{T^2}\frac{1}{T}\int_0^T \int_{(t-T-\ell)/R}^{(t-\ell)/R} \ddot{K}\left(x\right) dxdt \\ &= \frac{N_T^2}{T^2}\frac{1}{T}\int_0^T \left[1_{\{0<t-\ell<T\}\cap[0,T]} + o(1)\right] dt \\ &= \frac{N_T^2}{T^2}\left(1 - \frac{|\ell|}{T}\right) + o(1), \end{aligned}$$

which is $-C_2$ (neglecting the $o(1)$ terms). As a result, except for the $o(1)$ terms, I obtain

$$\hat{c}_R(\ell) = \frac{1}{TR}\sum_{i=1}^{N_T}\sum_{j=1}^{N_T} \ddot{K}\left(\frac{t_j - t_i - \ell}{R}\right) - \frac{N_T^2}{T^2}\left(1 - \frac{|\ell|}{T}\right),$$

which is (21).

A.8 Proof of Theorem 8

Let $d\hat{\epsilon}_t^k = dN_t^k - \hat{\lambda}_t^k dt$ for $k = a, b$. Then,

$$\begin{aligned} Q &= \int_I w_B(\ell)\hat{\gamma}_H^2(\ell)d\ell \\ &= \int_I w_B(\ell)\frac{1}{(TH)^2}\iiiint_{(0,T]^4} K\left(\frac{t_1-s_1-\ell}{H}\right) K\left(\frac{t_2-s_2-\ell}{H}\right) d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{t_1}^b d\hat{\epsilon}_{t_2}^b d\ell \\ &= \frac{1}{T^2}\iiiint_{(0,T]^4}\int_I w(\ell)\frac{1}{H^2}K\left(\frac{t_1-s_1-\ell}{H}\right) K\left(\frac{t_2-s_2-\ell}{H}\right) d\ell d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{t_1}^b d\hat{\epsilon}_{t_2}^b \end{aligned}$$

A.8.1 Asymptotic Mean of Q

By Fubini's theorem, the expectation of Q becomes an multiple integration with respect to $E[d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a d\hat{\epsilon}_{s_1+u}^b d\hat{\epsilon}_{s_2+u}^b]$, which, under the null hypothesis (11), can be split into $E[d\hat{\epsilon}_{s_1}^a d\hat{\epsilon}_{s_2}^a]E\left[d\hat{\epsilon}_{s_1+u}^b d\hat{\epsilon}_{s_2+u}^b\right]$. By the law of iterated expectations and the martingale property of the innovations $d\hat{\epsilon}_u^k$, it follows that $E[d\hat{\epsilon}_{u_1}^k d\hat{\epsilon}_{u_2}^k] = 0$ unless $u_1 = u_2 = u$

when it is equal to $E[\left(d\hat{\epsilon}_u^k\right)^2]$. Then, I can simplify the differential $\left(d\hat{\epsilon}_u^k\right)^2$ as follows:

$$\begin{aligned}
\left(d\hat{\epsilon}_u^k\right)^2 &= \left(dN_u^k - \hat{\lambda}_u^k du\right)^2 \\
&= \left(dN_u^k - \lambda_u^k du + \lambda_u^k du - \hat{\lambda}_u^k du\right)^2 \\
&= \left(dN_u^k - \lambda_u^k du\right)^2 + \left(\lambda_u^k du - \hat{\lambda}_u^k du\right)^2 + 2\left(dN_u^k - \lambda_u^k du\right)\left(\lambda_u^k du - \hat{\lambda}_u^k du\right) \\
&= \left(dN_u^k - \lambda_u^k du\right)^2 + o_P(du) \\
&= dN_u^k - 2dN_u^k \lambda_u^k du + \left(\lambda_u^k du\right)^2 + o_P(du) \\
&= dN_u^k + o_P(du). \qquad (28)
\end{aligned}$$

The second-to-last equality holds because of assumption (A1), which implies that $\left(dN_u^k\right)^2 = dN_u^k$ almost surely; hence the second order differential $\left(d\hat{\epsilon}_u^k\right)^2$ has a dominating first-order increment dN_u^k. It is therefore true, up to $O_P(du)$, that $E[\left(d\hat{\epsilon}_u^k\right)^2] = E[dN_u^k] = \lambda^k du$.

Now, letting $b = B/T$ and $h = H/T$, the expected value of Q is evaluated as follows:

$$\begin{aligned}
E(Q) &= \tfrac{1}{T^2H^2} \iint_{(0,T]^2} \int_I w_B(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \lambda^a \lambda^b ds dt \\
&= \tfrac{T^3}{T^2H^2} \iint_{(0,1]^2} \int_{I/T} w_B(T\sigma) K^2\left(\tfrac{v-u-\sigma}{H/T}\right) d\sigma \lambda^a \lambda^b du dv \\
&= \tfrac{1}{T^2h^2} \iint_{(0,1]^2} \int_{I/T} w_b(\sigma) K^2\left(\tfrac{v-u-\sigma}{h}\right) d\sigma \lambda^a \lambda^b du dv
\end{aligned}$$

Then, as $h \to 0$,

$$\begin{aligned}
&\iint_{(0,1]^2} \tfrac{1}{h^2} K^2\left(\tfrac{v-u-\sigma}{h}\right) du dv \\
&= \tfrac{1}{h} \int_0^1 \int_{(v-1-\sigma)/h}^{(v-\sigma)/h} K^2(x)\, dx dv \\
&= \tfrac{1}{h} \int_0^1 1_{\{0<v-\sigma<1\}\cap[0,1]} dv \int_{-\infty}^{\infty} K^2(x) dx + o\left(\tfrac{1}{h}\right) \\
&= \tfrac{1}{h}\left(1 - |\sigma|\right)\kappa_2 + o\left(\tfrac{1}{h}\right).
\end{aligned}$$

where $\kappa_2 = \int_{-\infty}^{\infty} K^2(x)dx$ (from assumption (A4a)). As a result, as $T \to \infty$, $Th = H \to \infty$ and $h = H/T \to 0$, the asymptotic mean of Q under the null hypothesis is given by

$$
\begin{aligned}
E(Q) &= \tfrac{1}{T^2h}\lambda^a\lambda^b\kappa_2\int_{I/T} w_b(\sigma)\left(1-|\sigma|\right)d\sigma + o\left(\tfrac{1}{T^2h}\right)\\
&= \tfrac{1}{T^2h}\lambda^a\lambda^b\kappa_2\int_{I}\tfrac{1}{b}w(\tfrac{\ell}{bT})\left(1-\tfrac{|\ell|}{T}\right)\tfrac{1}{T}d\ell + o\left(\tfrac{1}{T^2h}\right)\\
&= \tfrac{1}{TH}\lambda^a\lambda^b\kappa_2\int_{I} w_B(\ell)\left(1-\tfrac{|\ell|}{T}\right)d\ell + o\left(\tfrac{1}{TH}\right). \qquad (29)
\end{aligned}
$$

From (28), I also observe that $\left(d\hat{\epsilon}_u^k\right)^2 = \left(d\epsilon_u^k\right)^2 + o_P(du)$, which entails that $E(Q) = E(\tilde{Q})$.

A.8.2 Asymptotic Variance of Q Under the Null

The Case Without Autocorrelations Now, I derive the asymptotic variance of Q as $T\to\infty$, and $H/T\to 0$ as $H\to\infty$. Let $I\equiv[c_1,c_2]\subseteq[-T,T]$, where $c_1<c_2$. Consider

$$
\begin{aligned}
E\left(Q^2\right) = \; & \tfrac{1}{(TH)^4}\iint_{I^2} w(\ell_1)w(\ell_2)\int\cdots\int_{(0,T]^8} K\left(\tfrac{t_{11}-s_{11}-\ell_1}{H}\right)K\left(\tfrac{t_{21}-s_{21}-\ell_1}{H}\right)\\
& K\left(\tfrac{t_{12}-s_{12}-\ell_2}{H}\right)K\left(\tfrac{t_{22}-s_{22}-\ell_2}{H}\right)E\left[d\hat{\epsilon}^a_{s_{11}}d\hat{\epsilon}^a_{s_{12}}d\hat{\epsilon}^b_{t_{11}}d\hat{\epsilon}^b_{t_{12}}d\hat{\epsilon}^a_{s_{21}}d\hat{\epsilon}^a_{s_{22}}d\hat{\epsilon}^b_{t_{21}}d\hat{\epsilon}^b_{t_{22}}\right]d\ell_1 d\ell_2.
\end{aligned}
$$

Assume that (i) there is no cross-correlation between the two innovation processes, i.e. $\gamma(u)=0$; and (ii) there is no auto-correlation for each component process, i.e. $c^{aa}(u)=c^{bb}(u)=0$. I will relax the second assumption in the next subsection.

A key observation is that $E\left(Q^2\right)\neq 0$ only in the following cases (in all cases $s_1\neq s_2\neq t_1\neq t_2$ and $s\neq t$):

1. $\mathcal{R}_1=\{s_{11}=s_{12}=s_1,\, s_{21}=s_{22}=s_2\,,\, t_{11}=t_{12}=t_1,\, t_{21}=t_{22}=t_2\}$;
2. $\mathcal{R}_2=\{s_{11}=s_{12}=s_1,\, s_{21}=s_{22}=s_2\,,\, t_{11}=t_{21}=t_1,\, t_{12}=t_{22}=t_2\}$;
3. $\mathcal{R}_3=\{s_{11}=s_{12}=s_1,\, s_{21}=s_{22}=s_2\,,\, t_{11}=t_{22}=t_1,\, t_{12}=t_{21}=t_2\}$;
4. $\mathcal{R}_4=\{s_{11}=s_{21}=s_1,\, s_{12}=s_{22}=s_2\,,\, t_{11}=t_{12}=t_1,\, t_{21}=t_{22}=t_2\}$;
5. $\mathcal{R}_5=\{s_{11}=s_{21}=s_1,\, s_{12}=s_{22}=s_2\,,\, t_{11}=t_{21}=t_1,\, t_{12}=t_{22}=t_2\}$;
6. $\mathcal{R}_6=\{s_{11}=s_{21}=s_1,\, s_{12}=s_{22}=s_2\,,\, t_{11}=t_{22}=t_1,\, t_{12}=t_{21}=t_2\}$;
7. $\mathcal{R}_7=\{s_{11}=s_{22}=s_1,\, s_{12}=s_{21}=s_2\,,\, t_{11}=t_{12}=t_1,\, t_{21}=t_{22}=t_2\}$;
8. $\mathcal{R}_8=\{s_{11}=s_{22}=s_1,\, s_{12}=s_{21}=s_2\,,\, t_{11}=t_{21}=t_1,\, t_{12}=t_{22}=t_2\}$;
9. $\mathcal{R}_9=\{s_{11}=s_{22}=s_1,\, s_{12}=s_{21}=s_2\,,\, t_{11}=t_{22}=t_1,\, t_{12}=t_{21}=t_2\}$;
10. $\mathcal{R}_{10}=\{s_{11}=s_{12}=s_{21}=s_{22}=s$ and $t_{11}=t_{12}=t_{21}=t_{22}=t\}$.

Under the null of no cross-correlation, for cases 1 to 9, we have, up to $O(ds_1 ds_2 dt_1 dt_2)$,

$$\begin{aligned}
& E\left[d\hat{\epsilon}^a_{s_{11}} d\hat{\epsilon}^a_{s_{12}} d\hat{\epsilon}^b_{t_{11}} d\hat{\epsilon}^b_{t_{12}} d\hat{\epsilon}^a_{s_{21}} d\hat{\epsilon}^a_{s_{22}} d\hat{\epsilon}^b_{t_{21}} d\hat{\epsilon}^b_{t_{22}}\right] \\
= \; & E\left[\left(d\hat{\epsilon}^a_{s_1}\right)^2 \left(d\hat{\epsilon}^a_{s_2}\right)^2 \left(d\hat{\epsilon}^b_{t_1}\right)^2 \left(d\hat{\epsilon}^b_{t_2}\right)^2\right] \\
= \; & E\left[\left(d\hat{\epsilon}^a_{s_1}\right)^2 \left(d\hat{\epsilon}^a_{s_2}\right)^2\right] E\left[\left(d\hat{\epsilon}^b_{t_1}\right)^2 \left(d\hat{\epsilon}^b_{t_2}\right)^2\right] \\
= \; & E\left[dN^a_{s_1} dN^a_{s_2}\right] E\left[dN^b_{t_1} dN^b_{t_2}\right] \\
= \; & \left\{(\lambda^a)^2 ds_1 ds_2 + E\left[\left(dN^a_{s_1} - \lambda^a ds_1\right)\left(dN^a_{s_2} - \lambda^a ds_2\right)\right]\right\} \\
& \left\{\left(\lambda^b\right)^2 dt_1 dt_2 + E\left[\left(dN^b_{t_1} - \lambda^b dt_1\right)\left(dN^b_{t_2} - \lambda^b dt_2\right)\right]\right\} \\
= \; & \left[(\lambda^a)^2 + c^{aa}(s_2 - s_1)\right]\left[\left(\lambda^b\right)^2 + c^{bb}(t_2 - t_1)\right] ds_1 ds_2 dt_1 dt_2;
\end{aligned}$$

while for case 10, I have, up to $O(dsdt)$,

$$\begin{aligned}
& E\left[d\hat{\epsilon}^a_{s_{11}} d\hat{\epsilon}^a_{s_{12}} d\hat{\epsilon}^b_{t_{11}} d\hat{\epsilon}^b_{t_{12}} d\hat{\epsilon}^a_{s_{21}} d\hat{\epsilon}^a_{s_{22}} d\hat{\epsilon}^b_{t_{21}} d\hat{\epsilon}^b_{t_{22}}\right] \\
= \; & E\left[\left(d\hat{\epsilon}^a_s\right)^4 \left(d\hat{\epsilon}^b_t\right)^4\right] \\
= \; & E\left[dN^a_s\right] E\left[dN^b_t\right] \\
= \; & \lambda^a \lambda^b ds dt.
\end{aligned}$$

Cases 1 and 9: the innermost eight inner integrals reduce to four integrals, so that

$$
\begin{aligned}
& E\left(Q^2\right) \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{(TH)^4}\iint_{I^2} w_B(\ell_1)w_B(\ell_2)\underset{(0,T]^4}{\iiiint} K\left(\tfrac{t_1-s_1-\ell_1}{H}\right)K\left(\tfrac{t_2-s_2-\ell_1}{H}\right) \\
& K\left(\tfrac{t_1-s_1-\ell_2}{H}\right)K\left(\tfrac{t_2-s_2-\ell_2}{H}\right) ds_1ds_2dt_1dt_2d\ell_1d\ell_2 \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2T^4}{(TH)^4}\iint_{(I/T)^2}\frac{T^2}{B^2}w(\tfrac{\sigma_1}{B/T})w(\tfrac{\sigma_2}{B/T})\underset{(0,1]^2}{\iint}\int_{v_2-1}^{v_2}\int_{v_1-1}^{v_1} K\left(\frac{u-\sigma_1}{H/T}\right)K\left(\frac{v-\sigma_1}{H/T}\right) \\
& K\left(\frac{u-\sigma_2}{H/T}\right)K\left(\frac{v-\sigma_2}{H/T}\right) dudvdv_1dv_2d\sigma_1d\sigma_2 \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\int_{c_1/T}^{c_2/T} w_b(\sigma_1)w_b(\sigma_1-zh) \\
& \underset{(0,1]^2}{\iint}\int_{\frac{\sigma_1-c_2/T}{h}}^{\frac{\sigma_1-c_1/T}{h}}\int_{\frac{v_2-1-\sigma_1}{h}}^{\frac{v_2-\sigma_1}{h}}\int_{\frac{v_1-1-\sigma_1}{h}}^{\frac{v_1-\sigma_1}{h}} K(x)K(y)K(x+z)K(y+z)\,dxdydzdv_1dv_2d\sigma_1 \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\int_{c_1/T}^{c_2/T} w_b(\sigma_1)\left[w_b(\sigma_1)-zhw_b'(\ddot{\sigma}_1)\right]\underset{(0,1]^2}{\iint} 1_{\{(v_1,v_2):0\vee\sigma_1<v_i<(1+\sigma_1)\wedge 1\}}dv_1dv_2d\sigma_1 \\
& \iiint_{\mathbb{R}^3} K(x)K(y)K(x+z)K(y+z)\,dxdydz + o\left(\tfrac{1}{T^3h}\right) \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^2H}\kappa_4\int_I w_B^2(\ell_1)\left(1-\frac{|\ell_1|}{T}\right)^2 d\ell_1 + o\left(\tfrac{1}{T^2H}\right)
\end{aligned}
$$

I applied the change of variables: $(s_1,s_2,t_1,t_2)\longmapsto(u=\frac{t_1-s_1}{T}, v=\frac{t_2-s_2}{T}, v_1=\frac{t_1}{T}, v_2=\frac{t_2}{T})$ in the second equality, and $(u,v,\ell_2)\longmapsto\left(x=\frac{u-\sigma_1}{h}, y=\frac{v-\sigma_1}{h}, z=\frac{\sigma_1-\sigma_2}{h}\right)$ in the third equality. To get the fourth equality, I did a first-order Taylor expansion of $w_b(\sigma_1-zh)$ around σ_1, with $\ddot{\sigma}_1\in[\sigma_1-zh,\sigma_1]$.

Cases 2, 4, 6 and 8:

$$\begin{aligned}
& E\left(Q^2\right) \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{(TH)^4}\iint_{I^2} w_B(\ell_1)w_B(\ell_2)\int\cdots\int_{(0,T]^4} K\left(\tfrac{t_1-s_1-\ell_1}{H}\right)K\left(\tfrac{t_1-s_2-\ell_1}{H}\right) \\
& K\left(\tfrac{t_2-s_1-\ell_2}{H}\right)K\left(\tfrac{t_2-s_2-\ell_2}{H}\right)ds_1ds_2dt_1dt_2d\ell_1d\ell_2 \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\iiint_{(0,1]^3}\int_{\frac{u_1-1}{h}}^{\frac{u_1}{h}}\int_{\frac{v_2-u_1-c_2/T}{h}}^{\frac{v_2-u_1-c_1/T}{h}}\int_{\frac{v_1-u_1-c_2/T}{h}}^{\frac{v_1-u_1-c_1/T}{h}} w_b(v_1-u_1-xh)w_b(v_2-u_1-yh) \\
& K(x)\,K(x+z)\,K(y)\,K(y+z)\,dxdydzdu_1dv_1dv_2 \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\kappa_4\iiint_{(0,1]^3} w_b(v_1-u_1)w_b(v_2-u_1)1_{\{v_1-u_1\in I/T\}}1_{\{v_2-u_1\in I/T\}}du_1dv_1dv_2+o\left(\tfrac{1}{T^4h}\right) \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\kappa_4\int_0^1\int_{-s}^{1-s}\int_{-s}^{1-s} w_b(u)w_b(v)1_{\{u\in I/T\}}1_{\{v\in I/T\}}dudvds+o\left(\tfrac{1}{T^4h}\right) \\
= \;& O\left(\tfrac{1}{T^3H}\right).
\end{aligned}$$

I applied the change of variables: $(\ell_1,\ell_2,s_2)\longmapsto\left(x=\frac{v_1-u_1-\sigma_1}{H/T},y=\frac{v_2-u_1-\sigma_2}{H/T},z=\frac{u_1-u_2}{H/T}\right)$ in the second equality, and $(u_1,v_1,v_2)\longmapsto(s,u=v_1-u_1,v=v_2-u_1)$ in the fourth equality, and the fact that $\int_0^1\int_{-s}^{1-s}\int_{-s}^{1-s}w_b(u)w_b(v)dudvds=O(1)$ in the last equality.

Cases 3 and 7:

$$\begin{aligned}
& E\left(Q^2\right) \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{(TH)^4}\iint_{I^2} w_B(\ell_1)w_B(\ell_2)\int\cdots\int_{(0,T]^4} K\left(\tfrac{t_1-s_1-\ell_1}{H}\right)K\left(\tfrac{t_2-s_1-\ell_1}{H}\right) \\
& K\left(\tfrac{t_1-s_2-\ell_2}{H}\right)K\left(\tfrac{t_2-s_2-\ell_2}{H}\right)ds_1ds_2dt_1dt_2d\ell_1d\ell_2 \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\iiint_{(0,1]^3}\int_{\frac{v_1-u_2-c_2/T}{h}}^{\frac{v_1-u_2-c_1/T}{h}}\int_{\frac{v_2-1-\sigma_1}{h}}^{\frac{v_2-\sigma_1}{h}}\int_{\frac{v_1-u_1-c_2/T}{h}}^{\frac{v_1-u_1-c_1/T}{h}} w_b(v_1-u_1-xh)w_b(v_1-u_2-yh) \\
& K(x)\,K(x+z)\,K(y)\,K(y+z)\,dxdydzdu_1du_2dv_1 \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\kappa_4\iiint_{(0,1]^3} w_b(v_1-u_1)w_b(v_1-u_2)1_{\{v_1-u_1\in I/T\}}1_{\{v_1-u_2\in I/T\}}du_1du_2dv_1+o\left(\tfrac{1}{T^4h}\right) \\
= \;& \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h}\kappa_4\int_0^1\int_{t-1}^{t}\int_{t-1}^{t} w_b(u)w_b(v)1_{\{u\in I/T\}}1_{\{v\in I/T\}}dudvdt+o\left(\tfrac{1}{T^4h}\right) \\
= \;& O\left(\tfrac{1}{T^3H}\right).
\end{aligned}$$

I applied the change of variables: $(\ell_1,\ell_2,s_2)\longmapsto\left(x=\frac{v_1-u_1-\sigma_1}{H/T},y=\frac{v_1-u_2-\sigma_2}{H/T},z=\frac{v_2-v_1}{H/T}\right)$ in the second equality, and $(v_1,u_1,u_2)\longmapsto(t,u=v_1-u_1,v=v_1-u_2)$ in the fourth equality, and the fact that $\int_0^1 \int_{t-1}^t \int_{t-1}^t w_b(u)w_b(v)dudvdt = O(1)$ in the last equality.

Case 5:

$$
\begin{aligned}
& E\left(Q^2\right) \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{(TH)^4} \iint_{I^2} w_B(\ell_1)w_B(\ell_2) \int \cdots \int_{(0,T]^4} K^2\left(\tfrac{t_1-s_1-\ell_1}{H}\right) K^2\left(\tfrac{t_2-s_2-\ell_2}{H}\right) ds_1ds_2dt_1dt_2d\ell_1d\ell_2 \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h^2} \iiint_{(0,1]^3} \int_{\frac{v_2-u_1-c_2/T}{h}}^{\frac{v_2-u_1-c_1/T}{h}} \int_{\frac{v_1-u_1-c_2/T}{h}}^{\frac{v_1-u_1-c_1/T}{h}} w_b(v_1-u_1-xh)w(v_2-u_2-yh) \\
& K^2(x)K^2(y)\,dxdydu_1du_2dv_1dv_2 \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h^2}\kappa_2^2 \iiint_{(0,1]^3} w_b(v_1-u_1)w_b(v_2-u_2)1_{\{v_1-u_1\in I/T\}}1_{\{v_2-u_2\in I/T\}}du_1du_2dv_1dv_2 + o\left(\tfrac{1}{T^2h^2}\right) \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h^2}\kappa_2^2 \int_0^1\int_0^1\int_{-u_2}^{1-u_2}\int_{-u_1}^{1-u_1} w_b(u)w_b(v)1_{\{u\in I/T\}}1_{\{v\in I/T\}}dudvdu_1du_2 + o\left(\tfrac{1}{T^2h^2}\right) \\
= \; & \frac{\left(\lambda^a\lambda^b\right)^2}{T^4h^2}\kappa_2^2 \left(\int_0^1\int_{-s}^{1-s} w_b(u)1_{\{u\in I/T\}}duds\right)^2 + o\left(\tfrac{1}{T^4h^2}\right) \\
= \; & \left[E\left(Q\right)\right]^2 + o\left(\tfrac{1}{T^2H^2}\right).
\end{aligned}
$$

I applied the change of variables: $(\ell_1,\ell_2) \longmapsto \left(x = \frac{v_1-u_1-\sigma_1}{H/T}, y = \frac{v_2-u_2-\sigma_2}{H/T}\right)$ in the second equality, and $(u_1,u_2,v_1,v_2) \longmapsto (u_1,u_2,u=v_1-u_1,v=v_2-u_2)$ in the fourth equality. The last equality follows from Fubini's theorem, which gives

$$
\begin{aligned}
\int_0^1\int_{-s}^{1-s} w_b(u)1_{\{u\in I/T\}}duds &= \left(\int_{-1}^0\int_{-u}^1 + \int_0^1\int_0^{1-u}\right)\left[w_b(u)1_{\{u\in I/T\}}\right]dsdu \\
&= \int_{c_1/T}^{c_2/T} \left(1-|u|\right)w_b(u)du. \\
&= \int_{c_1}^{c_2}\left(1-\tfrac{|\ell|}{T}\right)w_B(\ell)d\ell = E(Q)
\end{aligned}
$$

Case 10:

$$
\begin{aligned}
& E\left(Q^2\right) \\
= \; & \tfrac{\lambda^a \lambda^b}{(TH)^4} \iint_{I^2} w_B(\ell_1) w_B(\ell_2) \int\!\!\int_{(0,T]^2} K^2\left(\tfrac{t-s-\ell_1}{H}\right) K^2\left(\tfrac{t-s-\ell_2}{H}\right) ds dt d\ell_1 d\ell_2 \\
= \; & \tfrac{\lambda^a \lambda^b}{T^4 h^2} \int\!\!\int_{(0,1]^2} \int_{\frac{v-u-c_2/T}{h}}^{\frac{v-u-c_1/T}{h}} \int_{\frac{v-u-c_2/T}{h}}^{\frac{v-u-c_1/T}{h}} w_b(v-u-xh) w_b(v-u-yh) K^2(x) K^2(y) dx dy du dv \\
= \; & \tfrac{\lambda^a \lambda^b}{T^4 h^2} \kappa_2^2 \int_0^1 \int_{-u}^{1-u} w_b^2(r) 1_{\{r \in I/T\}} dr du + o\left(\tfrac{1}{T^5 h^2}\right) \\
= \; & \tfrac{\lambda^a \lambda^b}{T^4 h^2} \kappa_2^2 \int_{I/T} (1-|r|) w_b^2(r) dr + o\left(\tfrac{1}{T^5 h^2}\right) \\
= \; & \tfrac{\lambda^a \lambda^b}{T^5 h^2} \kappa_2^2 \int_I \left(1 - \tfrac{|\ell|}{T}\right) w_B^2(\ell) d\ell + o\left(\tfrac{1}{T^5 h^2}\right) \\
= \; & O\left(\tfrac{1}{T^3 H^2}\right).
\end{aligned}
$$

I applied the change of variables: $(\ell_1, \ell_2) \longmapsto \left(x = \frac{v-u-\sigma_1}{H/T}, y = \frac{v-u-\sigma_2}{H/T}\right)$ in the second equality, and $(u,v) \longmapsto (u, r = v-u)$ in the third equality. The second-to-last equality follows from Fubini's theorem.

We observe that the leading terms of the asymptotic variance come from cases 1 and 9 only, thus we conclude that, as $T \to \infty$ and $H/T \to 0$ as $H \to \infty$,

$$
\begin{aligned}
Var(Q) &= E(Q^2) - [E(Q)]^2 \\
&= 2 \tfrac{\left(\lambda^a \lambda^b\right)^2}{T^2 H} \kappa_4 \int_I w_B^2(\ell_2) \left(1 - \tfrac{|\ell_2|}{T}\right)^2 d\ell_2 + o\left(\tfrac{1}{T^2 H}\right).
\end{aligned} \tag{30}
$$

The Case With Autocorrelations Suppose the two point processes N^a and N^b exhibit autocorrelations, i.e. $c^{aa}(u)$ and $c^{bb}(u)$ are not identically zero. Then, it is necessary to modify the asymptotic variance of Q. I start by noting that, up to $O(ds_1 ds_2 dt_1 dt_2)$,

$$
\begin{aligned}
& E\left[\left(d\hat{\epsilon}^a_{s_1}\right)^2 \left(d\hat{\epsilon}^a_{s_2}\right)^2\right] E\left[\left(d\hat{\epsilon}^b_{t_1}\right)^2 \left(d\hat{\epsilon}^b_{t_2}\right)^2\right] \\
= \; & E\left[dN^a_{s_1} dN^a_{s_2}\right] E\left[dN^b_{t_1} dN^b_{t_2}\right] \\
= \; & \left\{(\lambda^a)^2 ds_1 ds_2 + E\left[\left(dN^a_{s_1} - \lambda^a ds_1\right)\left(dN^a_{s_2} - \lambda^a ds_2\right)\right]\right\} \\
& \left\{\left(\lambda^b\right)^2 dt_1 dt_2 + E\left[\left(dN^b_{t_1} - \lambda^b dt_1\right)\left(dN^b_{t_2} - \lambda^b dt_2\right)\right]\right\} \\
= \; & \left[(\lambda^a)^2 + c^{aa}(s_2 - s_1)\right]\left[\left(\lambda^b\right)^2 + c^{bb}(t_2 - t_1)\right] ds_1 ds_2 dt_1 dt_2.
\end{aligned}
$$

As before, I split the computation into 10 separate cases. Since the computation techniques are analogous to the case without autocorrelations, let's focus on cases 1 and 9 which yield the dominating terms for $Var(Q)$. Under cases 1 and 9:

$$
\begin{aligned}
& E\left(Q^2\right) \\
= \; & \tfrac{1}{(TH)^4} \iint_{I^2} w_B(\ell_1) w_B(\ell_2) \iiint \int_{(0,T]^4} K\left(\tfrac{t_1-s_1-\ell_1}{H}\right) K\left(\tfrac{t_2-s_2-\ell_1}{H}\right) K\left(\tfrac{t_1-s_1-\ell_2}{H}\right) K\left(\tfrac{t_2-s_2-\ell_2}{H}\right) \\
& \left[(\lambda^a)^2 + c^{aa}(s_2-s_1)\right] \left[\left(\lambda^b\right)^2 + c^{bb}(t_2-t_1)\right] ds_1 ds_2 dt_1 dt_2 d\ell_1 d\ell_2 \\
= \; & \tfrac{T^4}{(TH)^4} \iint_{(I/T)^2} \tfrac{T^2}{B^2} w\left(\tfrac{\sigma_1}{B/T}\right) w\left(\tfrac{\sigma_2}{B/T}\right) \iint_{(0,1]^2} \int_{v_2-1}^{v_2} \int_{v_1-1}^{v_1} K\left(\tfrac{u-\sigma_1}{H/T}\right) K\left(\tfrac{v-\sigma_1}{H/T}\right) K\left(\tfrac{u-\sigma_2}{H/T}\right) K\left(\tfrac{v-\sigma_2}{H/T}\right) \\
& \left[(\lambda^a)^2 + c^{aa}(T(u-v+v_2-v_1))\right] \left[\left(\lambda^b\right)^2 + c^{bb}(T(v_2-v_1))\right] du dv dv_1 dv_2 d\sigma_1 d\sigma_2 \\
= \; & \tfrac{1}{T^4 h} \int_{c_1/T}^{c_2/T} w_b(\sigma_1) w_b(\sigma_1 - zh) \iint_{(0,1]^2} \int_{\frac{\sigma_1-c_2/T}{h}}^{\frac{\sigma_1-c_1/T}{h}} \int_{\frac{v_2-1-\sigma_1}{h}}^{\frac{v_2-\sigma_1}{h}} \int_{\frac{v_1-1-\sigma_1}{h}}^{\frac{v_1-\sigma_1}{h}} K(x) K(y) K(x+z) K(y+z) \\
& \left[(\lambda^a)^2 + c^{aa}(Th(x-y) + T(v_2-v_1))\right] \left[\left(\lambda^b\right)^2 + c^{bb}(T(v_2-v_1))\right] dx dy dz dv_1 dv_2 d\sigma_1 \\
= \; & \tfrac{1}{T^4 h} \int_{c_1/T}^{c_2/T} w_b^2(\sigma_1) \iiint_{\mathbb{R}^3} K(x) K(y) K(x+z) K(y+z) \, dx dy dz \iint_{(0,1]^2} 1_{\{(v_1,v_2): 0 \vee \sigma_1 < v_i < (1+\sigma_1) \wedge 1\}} \\
& \left[(\lambda^a)^2 + c^{aa}(T(v_2-v_1))\right] \left[\left(\lambda^b\right)^2 + c^{bb}(T(v_2-v_1))\right] dv_1 dv_2 d\sigma_1 + o\left(\tfrac{1}{T^3 h}\right) \\
= \; & \tfrac{1}{T^2 H} \kappa_4 \int_I w_B^2(\ell_1) \int_{-(T-|\ell_1|)}^{T-|\ell_1|} \left(1 - \tfrac{|r|}{T} - \tfrac{|\ell_1|}{T}\right) \left[(\lambda^a)^2 + c^{aa}(r)\right] \left[\left(\lambda^b\right)^2 + c^{aa}(r)\right] dr d\ell_1 + o\left(\tfrac{1}{T^2 H}\right).
\end{aligned}
$$

I applied the change of variables: $(s_1, s_2, t_1, t_2) \longmapsto (u = \frac{t_1-s_1}{T}, v = \frac{t_2-s_2}{T}, v_1 = \frac{t_1}{T}, v_2 = \frac{t_2}{T})$, $b = B/T$ and $h = H/T$ in the second equality, and $(u, v, \ell_2) \longmapsto \left(x = \frac{u-\sigma_1}{h}, y = \frac{v-\sigma_1}{h}, z = \frac{\sigma_1-\sigma_2}{h}\right)$ in the third equality. To get the fourth equality, I did a first-order Taylor expansion of $w_b(\sigma_1 - zh)$ around σ_1, with $\ddot{\sigma}_1 \in [\sigma_1 - zh, \sigma_1]$.

To get the last equality, I let $g(x) = \left[(\lambda^a)^2 + c^{aa}(Tx)\right]\left[\left(\lambda^b\right)^2 + c^{bb}(Tx)\right]$, and let $r = v_2 - v_1$. Suppose $\sigma_1 > 0$. Then, by Fubini's theorem, the innermost double integration (with respect to v_1 and v_2) becomes

$$
\begin{aligned}
& \int_{\sigma_1}^{1} \int_{-v_1}^{1-v_1} g(r) dr dv_1 \\
= \; & \left(\int_{-(1-\sigma_1)}^{-\sigma_1} \int_{-r}^{1} + \int_{-\sigma_1}^{0} \int_{\sigma_1}^{1} + \int_{0}^{1-\sigma_1} \int_{\sigma_1}^{1-r} \right) g(r) dv_1 dr \\
= \; & \int_{-1}^{-\sigma_1} (1+r) g(r) dr - \int_{-1}^{-(1-\sigma_1)} (1+r) g(r) dr + \int_{-\sigma_1}^{0} (1-\sigma_1) g(r) dr \\
& + \int_{0}^{1-\sigma_1} (1 - r - \sigma_1) g(r) dr.
\end{aligned}
$$

The first integral can be simplified to

$$\begin{aligned}\int_{-1}^{-\sigma_1}(1+r)\,g(r)dr &= \int_{-(1-\sigma_1)}^{0}(1+r-\sigma_1)\,g(r-\sigma_1)dr\\ &= \int_{-(1-\sigma_1)}^{0}(1+r-\sigma_1)\left[g(r)-\dot{\sigma}_1 g'(r)\right]dr\end{aligned}$$

for some $\dot{\sigma}_1 \in [0,\sigma_1]$. The second and third integrals are negligible for small σ_1 (as they are both $O(\sigma_1)$ by Taylor's expansion). Combining the first and fourth integrals, I obtain

$$\int_{\sigma_1}^{1}\int_{-v_1}^{1-v_1} g(r)drdv_1 = \int_{-(1-|\sigma_1|)}^{1-|\sigma_1|}(1-|\sigma_1|-|r|)\,g(r)dr + O(\sigma_1)$$

and hence (by switching back to $\ell_1 = T\sigma_1$)

$$\int_{-(T-|\ell_1|)}^{T-|\ell_1|}\left(1-\frac{|r|}{T}-\frac{|\ell_1|}{T}\right)g\left(\frac{r}{T}\right)dr + O\left(\frac{1}{T}\right).$$

The conclusion of the theorem follows as a result.

Similar to the mean calculation, the result in (28) implies that $\left(d\hat{\epsilon}_u^k\right)^2 = \left(d\epsilon_u^k\right)^2 + o_P(du)$, so it follows that $Var(Q) = Var(\tilde{Q})$.

A.8.3 Asymptotic normality of $\tilde{Q}$

The main tool for deriving asymptotic normality of $\tilde{Q}$ is Brown's martingale central limit theorem (see, for instance, Hall and Heyde, 1980). The proof thus boils down to three parts: (i) expressing $\tilde{Q}-E(\tilde{Q})$ as a sum of mean zero martingales, i.e. $\tilde{Q}-E(\tilde{Q}) = \sum_{i=1}^n Y_i$ where $E(Y_i|\mathcal{F}_{\tau_{i-1}}) = 0$, $n = N_T$, and $\tau_1,\ldots,\tau_n$ are the event times of the *pooled process* $N_t = N_t^a + N_t^b$; (ii) showing asymptotic negligibility, i.e. $s^{-4}\sum_{i=1}^n E(Y_i^4) \to 0$ where $s^2 = Var(\tilde{Q})$; and (iii) showing asymptotic determinism, i.e. $s^{-4}E(V_n^2 - s^2)^2 \to 0$, where $V_n^2 = \sum_{i=1}^n E(Y_i^2|\mathcal{F}_{\tau_{i-1}})$.

Martingale Decomposition Recall that the statistic $\tilde{Q}$ is defined as

$$\begin{aligned}\tilde{Q} &= \int_I w(\ell)\gamma_H^2(\ell)d\ell\\ &= \int_I w(\ell)\tfrac{1}{(TH)^2}\iiiint_{(0,T]^4} K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b d\ell\\ &= \tfrac{1}{T^2}\iiiint_{(0,T]^4}\int_I w(\ell)\tfrac{1}{H^2}K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b\end{aligned}$$

I start by decomposing $\tilde{Q}$ into four terms, corresponding to four different regions of integrations: (i) $s_1 = s_2 = s$, $t_1 = t_2 = t$; (ii) $s_1 \neq s_2$, $t_1 \neq t_2$; (iii) $s_1 \neq s_2$, $t_1 = t_2 = t$;

and (iv) $s_1 = s_2 = s$, $t_1 \neq t_2$. In all cases, integrations over regions where $s_i = t_j$ for $i, j = 1, 2$ are of measure zero because of assumption (A1): the pooled point process is simple, which implies that type a and b events cannot occur at the same time almost surely. Therefore,

$$\tilde{Q} = Q_1 + Q_2 + Q_3 + Q_4 \text{ a.s.},$$

where

$$\begin{aligned}
Q_1 &= \frac{1}{(TH)^2} \iint_{(0,T]^2} \int_I 1_{\{s \neq t\}} w(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \left(d\epsilon_s^a\right)^2 \left(d\epsilon_t^b\right)^2, \\
Q_2 &= \frac{1}{(TH)^2} \iiiint\limits_{(0,T]^4} \int_I 1_{\{s_1 \neq s_2 \neq t_1 \neq t_2\}} w(\ell) K\left(\tfrac{t_1-s_1-\ell}{H}\right) K\left(\tfrac{t_2-s_2-\ell}{H}\right) d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b, \\
Q_3 &= \frac{1}{(TH)^2} \iiint\limits_{(0,T]^3} \int_I 1_{\{s_1 \neq s_2 \neq t\}} w(\ell) K\left(\tfrac{t-s_1-\ell}{H}\right) K\left(\tfrac{t-s_2-\ell}{H}\right) d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a \left(d\epsilon_t^b\right)^2, \\
Q_4 &= \frac{1}{(TH)^2} \iiint\limits_{(0,T]^3} \int_I 1_{\{s \neq t_1 \neq t_2\}} w(\ell) K\left(\tfrac{t_1-s-\ell}{H}\right) K\left(\tfrac{t_2-s-\ell}{H}\right) d\ell \left(d\epsilon_s^a\right)^2 d\epsilon_{t_1}^b d\epsilon_{t_2}^b.
\end{aligned}$$

I will show that (i) Q_1 contributes to the mean of $\tilde{Q}$; (ii) Q_2 contributes to the variance of $\tilde{Q}$; and (iii) Q_3 and Q_4 are of smaller order than Q_2 and hence asymptotically negligible.

(i) As we saw in (29), Q_1 is of order $O_P\left(\frac{1}{TH}\right)$ which is the largest among the four terms. I decompose Q_1 to retrieve the mean:

$$\begin{aligned}
Q_1 &= \frac{1}{(TH)^2} \iint_{(0,T]^2} \int_I w(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \left(d\epsilon_s^a\right)^2 \left(d\epsilon_t^b\right)^2 \\
&= \frac{1}{(TH)^2} \iint_{(0,T]^2} \int_I w(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \left(d\epsilon_s^a\right)^2 \left[\left(d\epsilon_t^b\right)^2 - \lambda_t^b dt\right] \\
&\quad + \frac{1}{(TH)^2} \iint_{(0,T]^2} \int_I w(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \left[\left(d\epsilon_s^a\right)^2 - \lambda_s^a ds\right] \lambda_t^b dt \\
&\quad + \frac{1}{(TH)^2} \iint_{(0,T]^2} \int_I w(\ell) K^2\left(\tfrac{t-s-\ell}{H}\right) d\ell \lambda_s^a \lambda_t^b ds dt \qquad (31) \\
&\equiv Q_{11} + Q_{12} + E(\tilde{Q}).
\end{aligned}$$

The last line is obtained by (29).

Lemma 13 $Q_{11} = O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right)$ *and* $Q_{12} = O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right)$ *as* $T \to \infty$ *and* $H/T \to 0$ *as* $H \to \infty$.

Proof. Note that Q_{11}^2 contains 5 integrals. By applying a change of variables (on two variables inside the kernels), I deduce that $E(Q_{11}^2) = O\left(\frac{1}{T^3 H}\right)$ and hence the result. The proof for Q_{12} is similar. ■

(ii) I decompose Q_2 into $Q_2 = Q_{21} + Q_{22} + Q_{23} + Q_{24}$, where

$$\begin{aligned}
Q_{21} &= \frac{1}{(TH)^2}\int_{0+}^{T}\int_{0+}^{t_2^-}\int_{0+}^{t_2^-}\int_{0+}^{t_2^-}\int_I 1_{\{s_1\neq s_2\neq t_1\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b \\
Q_{22} &= \frac{1}{(TH)^2}\int_{0+}^{T}\int_{0+}^{t_1^-}\int_{0+}^{t_1^-}\int_{0+}^{t_1^-}\int_I 1_{\{s_1\neq s_2\neq t_2\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_2}^b d\epsilon_{t_1}^b \\
Q_{23} &= \frac{1}{(TH)^2}\int_{0+}^{T}\int_{0+}^{s_2^-}\int_{0+}^{s_2^-}\int_{0+}^{s_2^-}\int_I 1_{\{t_1\neq t_2\neq s_1\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon_{t_1}^b d\epsilon_{t_2}^b d\epsilon_{s_1}^a d\epsilon_{s_2}^a \\
Q_{24} &= \frac{1}{(TH)^2}\int_{0+}^{T}\int_{0+}^{s_1^-}\int_{0+}^{s_1^-}\int_{0+}^{s_1^-}\int_I 1_{\{t_1\neq t_2\neq s_2\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon_{t_1}^b d\epsilon_{t_2}^b d\epsilon_{s_2}^a d\epsilon_{s_1}^a
\end{aligned}$$

Lemma 14 $Q_2 = O_P\left(\frac{1}{TH}\right) + O_P\left(\frac{1}{TH^{1/2}}\right)$ *as* $T\to\infty$ *and* $H/T \to 0$ *as* $H\to\infty$.

Proof. Indeed, the asymptotic variance of $\tilde{Q}$ in (30) comes from Q_2. ■

(iii) It turns out that Q_3 and Q_4 are asymptotically negligible compared to Q_2.

Lemma 15 $Q_3 = O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right)$ *and* $Q_4 = O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right)$ *as* $T\to\infty$ *and* $H/T\to 0$ *as* $H\to\infty$.

Proof. Note that Q_3^2 contains 5 integrals. By applying a change of variables (on three variables inside the kernels) and combining $w(\ell_1)$ and $w(\ell_2)$ into $w^2(\ell)$ in the process, we deduce that $E(Q_3^2) = O\left(\frac{1}{T^3H}\right)$ and hence the result. The proof for Q_4 is similar. ■

As a result,

$$\tilde{Q} - E(\tilde{Q}) = Q_2 + O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right).$$

Now, I want to show that Q_2, the leading term of the demeaned statistic, can be expressed into the sum of a martingale difference sequence (m.d.s.).

Lemma 16 *Let* $n = N(T)$ *be the total event counts of the pooled process* $N = N^a + N^b$. *Then, as* $T\to\infty$ *and* $H/T\to 0$ *as* $H\to\infty$.

$$\tilde{Q} - E(\tilde{Q}) = \sum_{i=1}^{n} Y_i + O_P\left(\frac{1}{T^{3/2}H^{1/2}}\right)$$

where $Y_i = \sum_{j=1}^4 Y_{ji}$ *and* $E(Y_{ji}|\mathcal{F}_{\tau_{i-1}}^{ab}) = 0$ *for all* $i = 1,\dots,n$ *and for* $j = 1,2,3,4$ *(i.e.* $\{Y_{ji}\}_{i=1}^n$ *are m.d.s. for* $j = 1,2,3,4$*).*

Proof. The result follows by defining

$$
\begin{aligned}
Y_{1i} &= \tfrac{1}{(TH)^2}\int_{\tau_{i-1}^{+}}^{\tau_i}\int_0^{t_2^-}\int_0^{t_2^-}\int_0^{t_2^-}\int_I 1_{\{s_1\neq s_2\neq t_1\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon^a_{s_1}d\epsilon^a_{s_2}d\epsilon^b_{t_1}d\epsilon^b_{t_2},\\
Y_{2i} &= \tfrac{1}{(TH)^2}\int_{\tau_{i-1}^{+}}^{\tau_i}\int_0^{t_1^-}\int_0^{t_1^-}\int_0^{t_1^-}\int_I 1_{\{s_1\neq s_2\neq t_2\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon^a_{s_1}d\epsilon^a_{s_2}d\epsilon^b_{t_2}d\epsilon^b_{t_1},\\
Y_{3i} &= \tfrac{1}{(TH)^2}\int_{\tau_{i-1}^{+}}^{\tau_i}\int_0^{s_2^-}\int_0^{s_2^-}\int_0^{s_2^-}\int_I 1_{\{s_1\neq t_1\neq t_2\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon^a_{s_1}d\epsilon^b_{t_2}d\epsilon^b_{t_1}d\epsilon^a_{s_2},\\
Y_{4i} &= \tfrac{1}{(TH)^2}\int_{\tau_{i-1}^{+}}^{\tau_i}\int_0^{s_1^-}\int_0^{s_1^-}\int_0^{s_1^-}\int_I 1_{\{s_2\neq t_1\neq t_2\}}w(\ell)K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\epsilon^a_{s_2}d\epsilon^b_{t_2}d\epsilon^b_{t_1}d\epsilon^a_{s_1}.
\end{aligned}
$$

and noting that $E(Y_{ji}|\mathcal{F}^{ab}_{\tau_{i-1}}) = 0$ for all $i = 1, \ldots, n$. ■

Asymptotic Negligibility Next, I want to show that the summation $\sum_{i=1}^n Y_i^4$ is asymptotically negligible compared to $\left[Var(\tilde{Q})\right]^2$.

Lemma 17 *$s^{-4}\sum_{i=1}^n E(Y_i^4) \to 0$ as $T \to \infty$ and $H/T \to 0$ as $H \to \infty$, where $s^2 = Var(\tilde{Q})$.*

Proof. Consider

$$
\begin{aligned}
Y_{1i}^4 &= \tfrac{1}{T^8H^8}\iiint\int_{(\tau_{i-1},\tau_i]^4}\int\cdots\int_{(0,t_2)^{12}}\iiiint_{I^4} w(\ell_1)\ldots w(\ell_4)K\left(\tfrac{t_{111}-s_{111}-\ell_1}{H}\right)\ldots\\
&\quad K\left(\tfrac{t_{222}-s_{222}-\ell_4}{H}\right)d\ell_1\ldots d\ell_4 d\epsilon^a_{s_{111}}\ldots d\epsilon^a_{s_{222}}d\epsilon^b_{t_{111}}\ldots d\epsilon^b_{t_{222}}.
\end{aligned}
$$

A key observation is that $t_{211} = t_{212} = t_{221} = t_{222} \equiv t_2$ because there is at most one event of type b in the interval $(\tau_{i-1}, \tau_i]$ (one event if τ_i is a type b event time, zero events if τ_i is a type a event time). This reduces the four outermost integrations to just one over $t_2 \in (\tau_{i-1}, \tau_i]$. Let us focus on extracting the dominating terms. Then, to maximize the order of magnitude of $E\left(Y_{1i}^4\right)$, the next 12 integrations can be reduced to six integrations after grouping $d\epsilon^a_{ijl}$ and $d\epsilon^b_{1jl}$ into six pairs (if they were not paired, then the corresponding contribution to $E\left(Y_{1i}^4\right)$ would be zero by iterated expectations). Together with the four innermost integrations, there are 11 integrations for Y_{11i}^4, with the outermost integration running over $(\tau_{i-1}, \tau_i]$. Therefore, there are 11 integrations in $\sum_{i=1}^n E\left(Y_{1i}^4\right)$ and its outermost integration with respect to t_2 runs over $(0, T]$. As six new variables are sufficient to represent all 12 arguments in the 12 kernels, a change of variables yields a factor of $TH^6\kappa_4^2$.[38] As a result, $\sum_{i=1}^n E\left(Y_{1i}^4\right) = O\left(\frac{1}{T^7H^2}\right)$, and since $s^2 = O\left(\frac{1}{T^2H}\right)$ from (30), we have $s^{-4}\sum_{i=1}^n E(Y_{1i}^4) = O\left(\frac{1}{T^3}\right)$. The same argument applies to Y_{ji} for $j = 2, 3, 4$. By Minkowski's inequality $s^{-4}\sum_{i=1}^n E(Y_i^4) = O\left(\frac{1}{T^3}\right)$. ■

[38] 11 integrations - 6 d.f. - 4 $w() = 1$ free integration with respect to t.

Asymptotic Determinism Lastly, I want to show that the variance of $V_n^2 = \sum_{i=1}^n E(Y_i^2|\mathcal{F}_{\tau_{i-1}})$ is of a smaller order than s^4.

Lemma 18 *$s^{-4}E(V_n^2 - s^2)^2 \to 0$ as $T \to \infty$ and $H/T \to 0$ as $H \to \infty$.*

Proof. To prove that $s^{-4}E(V_n^2 - s^2)^2 \to 0$, it suffices to show that

$$E(V_n^2 - s^2)^2 = o\left(\tfrac{1}{T^4H^2}\right). \tag{32}$$

(i) Recall from lemma 16 that the i^{th} term of the martingale difference sequence in the demeaned statistic $\tilde{Q}-E(\tilde{Q})$ represents the innovation in the time interval $(\tau_{i-1}, \tau_i]$ and is given by $Y_i = Y_{1i} + Y_{2i} + Y_{3i} + Y_{4i}$, for $i = 1, 2, \ldots, n = N(T)$.

Now, note that

$$Y_i^2 = Y_{1i}^2 + Y_{2i}^2 + Y_{3i}^2 + Y_{4i}^2 + 2Y_{1i}Y_{2i} + 2Y_{3i}Y_{4i} \tag{33}$$

almost surely. The terms $Y_{1i}Y_{3i}$, $Y_{1i}Y_{4i}$, $Y_{2i}Y_{3i}$ and $Y_{2i}Y_{4i}$ are almost surely zero because of assumption (A1): the pooled process $N = N^a + N^b$ is simple, which implies that type a and b events will not occur at the same time τ_i almost surely.

(ii) Define

$$\begin{aligned} S_1 = S_2 \equiv \quad & \tfrac{1}{T^4H^4}\int_{0^+}^{T}\iiint\limits_{(0,t_2)^3}\iint\limits_{I^2} w(\ell_1)\, w(\ell_2)\, K\left(\tfrac{t_1-s_{11}-\ell_1}{H}\right) K\left(\tfrac{t_1-s_{12}-\ell_2}{H}\right) \\ & K\left(\tfrac{t_2-s_{21}-\ell_1}{H}\right) K\left(\tfrac{t_2-s_{22}-\ell_2}{H}\right) 1_{\mathcal{R}_1\cup\mathcal{R}_4\cup\mathcal{R}_7} d\ell_1 d\ell_2 \lambda_{s_1}^a \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b ds_1 ds_2 dt_1 dt_2, \end{aligned}$$

$$\begin{aligned} S_3 = S_4 \equiv \quad & \tfrac{1}{T^4H^4}\int_{0^+}^{T}\iiint\limits_{(0,s_2)^3}\iint\limits_{I^2} w(\ell_1)\, w(\ell_2)\, K\left(\tfrac{t_{11}-s_1-\ell_1}{H}\right) K\left(\tfrac{t_{12}-s_1-\ell_2}{H}\right) \\ & K\left(\tfrac{t_{21}-s_2-\ell_1}{H}\right) K\left(\tfrac{t_{22}-s_2-\ell_2}{H}\right) 1_{\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3} d\ell_1 d\ell_2 \lambda_{s_1}^a \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b ds_1 dt_1 dt_2 ds_2, \end{aligned}$$

$$\begin{aligned} S_{12} \equiv \quad & \tfrac{1}{T^4H^4}\int_{0^+}^{T}\iiint\limits_{(0,t_2)^3}\iint\limits_{I^2} w(\ell_1)\, w(\ell_2)\, K\left(\tfrac{t_1-s_{11}-\ell_1}{H}\right) K\left(\tfrac{t_1-s_{22}-\ell_2}{H}\right) \\ & K\left(\tfrac{t_2-s_{21}-\ell_1}{H}\right) K\left(\tfrac{t_2-s_{12}-\ell_2}{H}\right) 1_{\mathcal{R}_3\cup\mathcal{R}_6\cup\mathcal{R}_9} d\ell_1 d\ell_2 \lambda_{s_1}^a \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b ds_1 ds_2 dt_1 dt_2, \end{aligned}$$

and

$$\begin{aligned} S_{34} \equiv \quad & \tfrac{1}{T^4H^4}\int_{0^+}^{T}\iiint\limits_{(0,s_2)^3}\iint\limits_{I^2} w(\ell_1)\, w(\ell_2)\, K\left(\tfrac{t_{11}-s_1-\ell_1}{H}\right) K\left(\tfrac{t_{22}-s_1-\ell_2}{H}\right) \\ & K\left(\tfrac{t_{21}-s_2-\ell_1}{H}\right) K\left(\tfrac{t_{12}-s_2-\ell_2}{H}\right) 1_{\mathcal{R}_7\cup\mathcal{R}_8\cup\mathcal{R}_9} d\ell_1 d\ell_2 \lambda_{s_1}^a \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b dt_1 dt_2 ds_1 ds_2, \end{aligned}$$

where $\mathcal{R}_i$ were defined in section A.8.2. It is easy to verify from the definitions that

$$s^2 = S_1 + S_2 + S_3 + S_4 + 2S_{12} + 2S_{34} + o_p\left(\tfrac{1}{T^2 H}\right). \tag{34}$$

(iii) Define for $k = 1, 2, 3, 4$

$$V_{nk}^2 \equiv \sum_{i=1}^{n} E(Y_{ki}^2 | \mathcal{F}_{\tau_{i-1}}),$$

and for $(k, j) = (1, 2)$ and $(3, 4)$

$$V_{nkj} \equiv \sum_{i=1}^{n} E(Y_{ki} Y_{ji} | \mathcal{F}_{\tau_{i-1}}).$$

It follows from (33) that

$$V_n^2 = V_{1n}^2 + V_{2n}^2 + V_{3n}^2 + V_{4n}^2 + 2V_{12n} + 2V_{34n}. \tag{35}$$

(iv) I claim (see the proof below) that, for $k = 1, 2, 3, 4$,

$$V_{nk}^2 - S_k = o_p\left(\tfrac{1}{T^2 H}\right) \tag{36}$$

and, for $(k, j) = (1, 2)$ and $(3, 4)$,

$$V_{nkj}^2 - S_{kj} = o_p\left(\tfrac{1}{T^2 H}\right). \tag{37}$$

(v) It follows from (33)-(37) that (32) holds.

It remains to show the claims in (iv). Since the asymptotic orders of the six differences in (36) and (37) can be derived by similar techniques, let us focus on proving the first one.

To this end, I first compute $E(Y_{1i}^2 | \mathcal{F}_{\tau_{i-1}})$. Now,

$$\begin{aligned} Y_{1i}^2 \;=\; & \tfrac{1}{T^4 H^4} \int_{(\tau_{i-1}, \tau_i]^2} \int \int \cdots \int_{(0,t_2)^6} \iint_{I^2} w(\ell_1)\, w(\ell_2)\, K\left(\tfrac{t_{11} - s_{11} - \ell_1}{H}\right) K\left(\tfrac{t_{12} - s_{12} - \ell_2}{H}\right) \\ & K\left(\tfrac{t_2 - s_{21} - \ell_1}{H}\right) K\left(\tfrac{t_2 - s_{22} - \ell_2}{H}\right) d\ell_1 d\ell_2 d\epsilon_{s_{11}}^a d\epsilon_{s_{12}}^a d\epsilon_{s_{21}}^a d\epsilon_{s_{22}}^a d\epsilon_{t_{11}}^b d\epsilon_{t_{12}}^b d\epsilon_{t_{21}}^b d\epsilon_{t_{22}}^b. \end{aligned}$$

Observe that there is at most one event of type b in the interval $(\tau_{i-1}, \tau_i]$ (one event if τ_i is a type b event time, zero events if τ_i is a type a event time). This entails that

$t_{21}=t_{22}\equiv t_2$ and thus saves one integration. I can then rewrite

$$
\begin{aligned}
Y_{1i}^2 &= \tfrac{1}{T^4H^4}\int\limits_{(\tau_{i-1},\tau_i]}\int\cdots\int\limits_{(0,t_2)^6}\iint\limits_{I^2} w(\ell_1)\,w(\ell_2)\,K\left(\tfrac{t_{11}-s_{11}-\ell_1}{H}\right)K\left(\tfrac{t_{12}-s_{12}-\ell_2}{H}\right)\\
&\quad K\left(\tfrac{t_2-s_{21}-\ell_1}{H}\right)K\left(\tfrac{t_2-s_{22}-\ell_2}{H}\right)d\ell_1 d\ell_2 d\epsilon^a_{s_{11}}d\epsilon^a_{s_{12}}d\epsilon^a_{s_{21}}d\epsilon^a_{s_{22}}d\epsilon^b_{t_{11}}d\epsilon^b_{t_{12}}\left(d\epsilon^b_{t_2}\right)^2\\
&\equiv \int\limits_{(\tau_{i-1},\tau_i]} H_{11}(t_2^-)\left(d\epsilon^b_{t_2}\right)^2,
\end{aligned}
$$

where I define $H_{11}(u)$ by

$$
\begin{aligned}
H_{11}(u^-) &\equiv \int\cdots\int\limits_{(0,u)^6}\iint\limits_{I^2} w(\ell_1)\,w(\ell_2)\,K\left(\tfrac{t_{11}-s_{11}-\ell_1}{H}\right)K\left(\tfrac{t_{12}-s_{12}-\ell_2}{H}\right)K\left(\tfrac{u-s_{21}-\ell_1}{H}\right)\\
&\quad K\left(\tfrac{u-s_{22}-\ell_2}{H}\right)d\ell_1 d\ell_2 d\epsilon^a_{s_{11}}d\epsilon^a_{s_{12}}d\epsilon^a_{s_{21}}d\epsilon^a_{s_{22}}d\epsilon^b_{t_{11}}d\epsilon^b_{t_{12}}.
\end{aligned}
$$

Note that $H_{11}(u^-)$ is $\mathcal{F}$-predictable. Now, by iterated expectations, lemma 1, and the fact that $\{u\in(\tau_{i-1},\tau_i]\}\in\mathcal{F}_{u^-}$, I have

$$
\begin{aligned}
E(Y_{1i}^2|\mathcal{F}_{\tau_{i-1}}) &= E\left\{\int_{\tau_{i-1}^+}^{\tau_i} H_{11}(u^-)\left(d\hat{\epsilon}^b_u\right)^2\middle|\mathcal{F}_{\tau_{i-1}}\right\}\\
&= E\left\{\int_{\tau_{i-1}^+}^{\tau_i} H_{11}(u^-)\lambda^b_u du\middle|\mathcal{F}_{\tau_{i-1}}\right\}\\
&= E\left\{\int_{\tau_{i-1}^+}^{\tau_i} H_{11}(u^-)\frac{\lambda^b_u}{\lambda^a_u+\lambda^b_u}dN_u\middle|\mathcal{F}_{\tau_{i-1}}\right\}\\
&= H_{11}(\tau_{i-1})E\left\{\frac{\lambda^b_{\tau_i}}{\lambda^a_{\tau_i}+\lambda^b_{\tau_i}}\middle|\mathcal{F}_{\tau_{i-1}}\right\}.
\end{aligned}
$$

Note that I used the property $H_{11}(\tau_i^-)=H_{11}(\tau_{i-1})$ in the last line. Summing over i gives

$$
\begin{aligned}
V_{n1}^2 &\equiv \sum_{i=1}^n E(Y_{1i}^2|\mathcal{F}_{\tau_{i-1}})\\
&= \sum_{i=1}^n H_{11}(\tau_{i-1})E\left\{\frac{\lambda^b_{\tau_i}}{\lambda^a_{\tau_i}+\lambda^b_{\tau_i}}\middle|\mathcal{F}_{\tau_{i-1}}\right\}\\
&= \int_0^T H_{11}(u^-)E\left\{\frac{\lambda^b_u}{\lambda^a_u+\lambda^b_u}\middle|\mathcal{F}_{u^-}\right\}dN_u\\
&= \int_0^T H_{11}(u^-)\frac{\lambda^b_u}{\lambda^a_u+\lambda^b_u}dN_u.
\end{aligned}
$$

The third equality made use of the property that for $u \in (\tau_{i-1}, \tau_i]$, $N_{u^-} = N_{\tau_{i-1}}$ and hence $\mathcal{F}_{u^-} = \sigma\{(\tau_i, y_i) : 0 \leq i \leq t_{N_{u^-}}\} = \mathcal{F}_{\tau_{i-1}}$, and the last line follows from $\mathcal{F}_t$-predictability of conditional intensities λ_t^a and λ_t^b.

Let $\pi_u^b = \frac{\lambda_u^b}{\lambda_u^a + \lambda_u^b}$. Apart from the terms with $t_{11} \neq t_{12}$ and/or $s_{ij} \neq s_{kl}$ for $(i,j) \neq (k,l)$ which can be shown to be $O_p\left(\frac{1}{T^6 H^2}\right) = o_p\left(\frac{1}{T^4 H^2}\right)$. the integral $V_{n1}^2 = \int_0^T H_{11}(t_2^-)\pi_{t_2}^b dN_{t_2}$ can be decomposed by the same demeaning technique as we used for decomposing Q_1 in (31). The decomposition is represented by differentials for simplicity:

$$
\begin{aligned}
& d\ell_1 d\ell_2 \left(d\epsilon_{s_1}^a\right)^2 \left(d\epsilon_{s_2}^a\right)^2 \left(d\epsilon_{t_1}^b\right)^2 \pi_{t_2}^b dN_{t_2} \\
= \; & d\ell_1 d\ell_2 \left(d\epsilon_{s_1}^a\right)^2 \left(d\epsilon_{s_2}^a\right)^2 \left(d\epsilon_{t_1}^b\right)^2 \pi_{t_2}^b \left[dN_{t_2} - \left(\lambda_{t_2}^a + \lambda_{t_2}^b\right) dt_2\right] \\
& + d\ell_1 d\ell_2 \left(d\epsilon_{s_1}^a\right)^2 \left(d\epsilon_{s_2}^a\right)^2 \left[\left(d\epsilon_{t_1}^b\right)^2 - \lambda_{t_1}^b dt_1\right] \lambda_{t_2}^b dt_2 \\
& + d\ell_1 d\ell_2 \left(d\epsilon_{s_1}^a\right)^2 \left[\left(d\epsilon_{s_2}^a\right)^2 - \lambda_{s_2}^a ds_2\right] \lambda_{t_1}^b \lambda_{t_2}^b dt_1 dt_2 \\
& + d\ell_1 d\ell_2 \left[\left(d\epsilon_{s_1}^a\right)^2 - \lambda_{s_1}^a ds_1\right] \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b ds_2 dt_1 dt_2 \\
& + d\ell_1 d\ell_2 \lambda_{s_1}^a \lambda_{s_2}^a \lambda_{t_1}^b \lambda_{t_2}^b ds_1 ds_2 dt_1 dt_2.
\end{aligned}
$$

The first four integrals above are dominated by the first term, which can be shown to be of size $O_p\left(\frac{1}{(T^5 H^2)^{1/2}}\right) = o_p\left(\frac{1}{(T^4 H^2)^{1/2}}\right)$. The last integral is S_1 which contributes to $s^2 = Var(\tilde{Q})$ and was proven to be $O_p\left(\frac{1}{(T^4 H^2)^{1/2}}\right)$ in Theorem 8. Hence, $V_{n1}^2 - S_1 = o_p\left(\frac{1}{T^2 H}\right)$. ■

A.9 Proof of Theorem 10

First, recall that

$$
\tilde{Q} = \frac{1}{T^2} \iiiint_{(0,T]^4} \int_I w(\ell) \frac{1}{H^2} K\left(\frac{t_1 - s_1 - \ell}{H}\right) K\left(\frac{t_2 - s_2 - \ell}{H}\right) d\ell d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b.
$$

From the property of the joint cumulant of the innovations, all of which have mean zero, I can express

$$
\begin{aligned}
E[d\epsilon_{s_1}^a d\epsilon_{s_2}^a d\epsilon_{t_1}^b d\epsilon_{t_2}^b] &= E[d\epsilon_{s_1}^a d\epsilon_{s_2}^a] E[d\epsilon_{t_1}^b d\epsilon_{t_2}^b] + E[d\epsilon_{s_1}^a d\epsilon_{t_1}^b] E[d\epsilon_{s_2}^a d\epsilon_{t_2}^b] \\
&\quad + E[d\epsilon_{s_1}^a d\epsilon_{t_2}^b] E[d\epsilon_{s_2}^a d\epsilon_{t_1}^b] + c_{22}(s_2 - s_1, t_1 - s_1, t_2 - s_1) \\
&= 0 + \gamma(t_1 - s_1)\gamma(t_2 - s_2) + \gamma(t_2 - s_1)\gamma(t_1 - s_2) \\
&\quad + c_{22}(s_2 - s_1, t_1 - s_1, t_2 - s_1) \\
&= a_T^2 \lambda^a \lambda^b \left[\rho(t_1 - s_1)\rho(t_2 - s_2) + \rho(t_2 - s_1)\rho(t_1 - s_2)\right] \\
&\quad + o(a_T^2),
\end{aligned}
$$

where the last line utilizes assumption (A8).

Since $H = o(B)$, the asymptotic bias of $\tilde{Q}$ becomes

$$bias(\tilde{Q}) = \frac{a_T^2 \lambda^a \lambda^b}{TH} \kappa_2 \int_I w_B(\ell) \left(1 - \frac{|\ell|}{T}\right) \breve{\rho}^2(\ell) d\ell + o\left(\frac{a_T^2}{TH}\right).$$

The asymptotic variance of $\tilde{Q}$ under $\mathbf{H}_{a_T}$ is the same as that under $\mathbf{H}_0$ and was given in Theorem 8. If I set $a_T^* = H^{1/4}$, then the normalized statistic J converges in distribution to $N(\mu(K, w_B), 1)$.

A.10 Proof of Theorem 9

I will only prove the case with no autocorrelations, i.e. $c^{kk}(\ell) \equiv 0$ for $k = a, b$, as the error of estimating auto-covariances by their estimators can be made negligible by similar techniques as in the case for conditional intensities.

First, assuming the setup in section 4.4, the conditional intensity λ_t^k can be approximated by $\hat{\lambda}_t^k - \lambda_t^k = O_P(M^{-1/2})$ by Theorem 6.

Next, by Theorem 6, it follows that $\hat{\lambda}_t^k - \lambda_t^k = O_P(M^{-1/2})$ for $k = a, b$. By lemma **??** (see below), it is true that $T\left(Q - \tilde{Q}\right) = O_P(M^{-1/2})$. By the assumption $H = o(M)$, I thus obtain $T\left(Q - \tilde{Q}\right) = o_P(H^{-1/2})$, and hence, with $Var(TQ) = O(H^{-1/2})$,

$$T\left(Q - \tilde{Q}\right) / \sqrt{Var(TQ)} = o_P(1). \tag{38}$$

Besides, note that the approximation error of the unconditional intensity $\hat{\lambda}^k - \lambda^k$ is diminishing at the parametric rate of $O_P(T^{-1/2}) = o_P(1)$ as $T \to \infty$. Also, note that $\widehat{E(TQ)}$ is a function of unconditional intensities, so (29) implies that $\widehat{E(TQ)} - E(TQ) = o(H^{-1})$, or

$$\left[\widehat{E(TQ)} - E(TQ)\right] / \sqrt{Var(TQ)} = o(H^{-1/2}) = o(1). \tag{39}$$

Furthermore, the estimated variance $\widehat{Var(TQ)}$ is a function of unconditional intensities too, so (30) implies that $\widehat{Var(TQ)} - Var(TQ) = o(H^{-1})$, or

$$Var(TQ)/\widehat{Var(TQ)} = 1 + o(1). \tag{40}$$

Lastly, the result follows from the decomposition below with an application of

Slutsky's theorem, meanwhile making use of (38), (39) and (40):

$$\begin{aligned}\hat{J} &= \frac{TQ-\widehat{E(TQ)}}{\sqrt{\widehat{Var(TQ)}}} \\ &= \sqrt{\frac{Var(TQ)}{\widehat{Var(TQ)}}}\left[\frac{T\tilde{Q}-E(TQ)}{\sqrt{Var(TQ)}}+\frac{T\left(Q-\tilde{Q}\right)}{\sqrt{Var(TQ)}}+\frac{\widehat{E(TQ)}-E(TQ)}{\sqrt{Var(TQ)}}\right] \\ &= J+o_P(1).\end{aligned}$$

Lemma 19 $T\left(Q-\tilde{Q}\right)=O_P(M^{-1/2})$.

Proof. Recall the statistic $Q=\int_{I/T}w_b(\sigma)\hat{\tilde{\gamma}}_h^2(\sigma)d\sigma$ and its hypothetical counterpart $\tilde{Q}=\int_{I/T}w_b(\sigma)\tilde{\gamma}_h^2(\sigma)\,d\sigma$. ■

To evaluate the asymptotic order of $Q-\tilde{Q}$, I apply a change of variable $s=Tu$ and $t=Tv$ as described in section 4.6. From Theorem 6, we know that $\hat{\tilde{\lambda}}_v^k-\tilde{\lambda}_v^k=\hat{\lambda}_{Tv}^k-\lambda_{Tv}^k=O_P\left(M^{-1/2}\right)$ for $k=a,b$ (u and v fixed). It follows that $\hat{\tilde{\lambda}}_u^a\hat{\tilde{\lambda}}_v^b-\tilde{\lambda}_u^a\tilde{\lambda}_v^b=\left(\hat{\tilde{\lambda}}_u^a-\tilde{\lambda}_u^a\right)\hat{\tilde{\lambda}}_v^b+\left(\hat{\tilde{\lambda}}_v^b-\tilde{\lambda}_v^b\right)\tilde{\lambda}_u^a=O_P\left(M^{-1/2}\right)$, and hence

$$\begin{aligned}d\hat{\tilde{\varepsilon}}_u^a d\hat{\tilde{\varepsilon}}_v^b-d\tilde{\varepsilon}_u^a d\tilde{\varepsilon}_v^b &= \left(d\tilde{N}_u^a-\hat{\tilde{\lambda}}_u^a du\right)\left(d\tilde{N}_v^b-\hat{\tilde{\lambda}}_v^b dv\right)-\left(d\tilde{N}_u^a-\tilde{\lambda}_u^a du\right)\left(d\tilde{N}_v^b-\tilde{\lambda}_v^b dv\right) \\ &= -d\tilde{N}_u^a\left(\hat{\tilde{\lambda}}_v^b-\tilde{\lambda}_v^b\right)dv-d\tilde{N}_v^b\left(\hat{\tilde{\lambda}}_u^a-\tilde{\lambda}_u^a\right)du+\left(\hat{\tilde{\lambda}}_u^a\hat{\tilde{\lambda}}_v^b-\tilde{\lambda}_u^a\tilde{\lambda}_v^b\right)dudv \\ &= O_P\left(M^{-1/2}\right).\end{aligned}$$

As a result, $\hat{\tilde{\gamma}}_h(\sigma)-\tilde{\gamma}_h(\sigma)=\int_0^1\int_0^1 K_h(v-u-\sigma)\left[d\hat{\tilde{\varepsilon}}_u^a d\hat{\tilde{\varepsilon}}_v^b-d\tilde{\varepsilon}_u^a d\tilde{\varepsilon}_v^b\right]=O_P\left(M^{-1/2}\right)$. Since $\hat{\tilde{\gamma}}_h(\sigma)+\tilde{\gamma}_h(\sigma)=O_P(1)$, I deduce that

$$\begin{aligned}\hat{\tilde{\gamma}}_h^2(\sigma)-\tilde{\gamma}_h^2(\sigma) &= \left(\hat{\tilde{\gamma}}_h(\sigma)+\tilde{\gamma}_h(\sigma)\right)\left(\hat{\tilde{\gamma}}_h(\sigma)-\tilde{\gamma}_h(\sigma)\right) \\ &= O_P\left(M^{-1/2}\right),\end{aligned}$$

and thus conclude that $Q-\tilde{Q}=\int_{I/T}w_b(\sigma)\left[\hat{\tilde{\gamma}}_h^2(\sigma)-\tilde{\gamma}_h^2(\sigma)\right]d\sigma=O_P\left(T^{-1}M^{-1/2}\right)$, which implies that $T\left(Q-\tilde{Q}\right)=O_P(M^{-1/2})$.

A.11 Proof of Corollary 11

It suffices to show that the mean and variance are as given in the corollary. Denote the delta function by $\delta(\cdot)$. Since $B=o(H)$ as $H\to\infty$, the following approximation is valid:

$$\begin{aligned}&\tfrac{1}{B}w\left(\tfrac{\ell}{B}\right)\tfrac{1}{H^2}K\left(\tfrac{u-\ell}{H}\right)K\left(\tfrac{v-\ell}{H}\right) \\ =\;& \delta_\ell(0)\tfrac{1}{H^2}K\left(\tfrac{u-\ell}{H}\right)K\left(\tfrac{v-\ell}{H}\right)+o(1).\end{aligned}$$

Therefore,

$$
\begin{aligned}
Q &= \int_I w_B(\ell)\hat{\gamma}_H^2(\ell)d\ell \\
&= \tfrac{1}{T^2}\iiiint_{(0,T]^4}\int_0^T \tfrac{1}{B}w\left(\tfrac{\ell}{B}\right)\tfrac{1}{H^2}K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right)d\ell d\hat{\epsilon}^a_{s_1}d\hat{\epsilon}^a_{s_2}d\hat{\epsilon}^b_{t_1}d\hat{\epsilon}^b_{t_2} \\
&= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_{t_2-T}^{t_2}\int_{t_1-T}^{t_1}\int_0^T \tfrac{1}{B}w\left(\tfrac{\ell}{B}\right)\tfrac{1}{H^2}K\left(\tfrac{u-\ell}{H}\right)K\left(\tfrac{v-\ell}{H}\right)d\ell d\hat{\epsilon}^a_{t_1-u}d\hat{\epsilon}^a_{t_2-v}d\hat{\epsilon}^b_{t_1}d\hat{\epsilon}^b_{t_2} \\
&= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_{t_2-T}^{t_2}\int_{t_1-T}^{t_1}\left[\tfrac{1}{H^2}K\left(\tfrac{u}{H}\right)K\left(\tfrac{v}{H}\right)+o(1)\right]d\hat{\epsilon}^a_{t_1-u}d\hat{\epsilon}^a_{t_2-v}d\hat{\epsilon}^b_{t_1}d\hat{\epsilon}^b_{t_2}.
\end{aligned}
$$

Under the null hypothesis (11), I compute the mean (up to the leading term) as follows:

$$
\begin{aligned}
E(Q) &= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_{t_2-T}^{t_2}\int_{t_1-T}^{t_1}\tfrac{1}{H^2}K\left(\tfrac{u}{H}\right)K\left(\tfrac{v}{H}\right)E\left(d\hat{\epsilon}^a_{t_1-u}d\hat{\epsilon}^a_{t_2-v}\right)E\left(d\hat{\epsilon}^b_{t_1}d\hat{\epsilon}^b_{t_2}\right) \\
&= \tfrac{\lambda^a\lambda^b}{T^2}\int_0^T\int_{t-T}^{t}\tfrac{1}{H^2}K^2\left(\tfrac{u}{H}\right)dudt.
\end{aligned}
$$

By Fubini's theorem, the last line becomes

$$
\begin{aligned}
E(Q) &= \tfrac{\lambda^a\lambda^b}{T}\int_{-T}^{T}\tfrac{1}{H^2}K^2\left(\tfrac{u}{H}\right)\left(1-\tfrac{|u|}{T}\right)du \\
&= \tfrac{\lambda^a\lambda^b}{TH}\int_{-T/h}^{T/h}K^2(v)\left(1-\tfrac{|v|H}{T}\right)dv \\
&= \tfrac{\lambda^a\lambda^b}{TH}\left[\kappa_2+o(1)\right].
\end{aligned}
$$

so that $E(Q^{\mathcal{G}}) = C^{\mathcal{G}} + o(1)$. By similar techniques as I obtained (30), I compute the second moment (up to the leading term) as follows.

$$
E(Q^2) = \tfrac{1}{(TH)^4}E\left[\prod_{i,j=1,2}\int_0^T\int_{t_{ij}-T}^{t_{ij}}K\left(\tfrac{u_{ij}}{H}\right)d\hat{\epsilon}^a_{t_{ij}-u_{ij}}d\hat{\epsilon}^b_{t_{ij}}\right]
$$

The leading order terms of $E(Q^2)$ are obtained when:
(1) $t_{11}=t_{12}$, $t_{21}=t_{22}$, $t_{11}-u_{11}=t_{21}-u_{21}$, $t_{12}-u_{12}=t_{22}-u_{22}$;
(2) $t_{11}=t_{12}$, $t_{21}=t_{22}$, $t_{11}-u_{11}=t_{22}-u_{22}$, $t_{12}-u_{12}=t_{21}-u_{21}$;
(3) $t_{11}=t_{21}$, $t_{12}=t_{22}$, $t_{11}-u_{11}=t_{12}-u_{12}$, $t_{21}-u_{21}=t_{22}-u_{22}$;
(4) $t_{11}=t_{21}$, $t_{12}=t_{22}$, $t_{11}-u_{11}=t_{22}-u_{22}$, $t_{12}-u_{12}=t_{21}-u_{21}$;
(5) $t_{11}=t_{22}$, $t_{12}=t_{21}$, $t_{11}-u_{11}=t_{12}-u_{12}$, $t_{21}-u_{21}=t_{22}-u_{22}$;
(6) $t_{11}=t_{22}$, $t_{12}=t_{21}$, $t_{11}-u_{11}=t_{21}-u_{21}$, $t_{12}-u_{12}=t_{22}-u_{22}$.
Their contributions add up to

$$
\tfrac{6\left(\lambda^a\lambda^b\right)^2}{(TH)^4}\iint_{(0,T]^2}\int_{t_2-T}^{t_2}\int_{t_1-T}^{t_1}\int_A K\left(\tfrac{u_1}{H}\right)K\left(\tfrac{u_2}{H}\right)K\left(\tfrac{u_1+v}{H}\right)K\left(\tfrac{u_2+v}{H}\right)dvdu_1du_2dt_1dt_2
$$

where $A = \cap_{i=1}^{2}[t_i - T - u_i, t_i - u_i]$. After a change of variables, the last line reduces to

$$\frac{6\left(\lambda^a \lambda^b\right)^2}{T^2 H}\left[\kappa_4 + o(1)\right],$$

which dominates $\left[E(Q)\right]^2$. As a result, $Var(Q^{\mathcal{G}}) = 2D^{\mathcal{G}} + o(1)$.

A.12 Proof of Corollary 12

It suffices to show that the mean and variance are as given in the corollary. Denote the Dirac delta function at ℓ by $\delta_\ell(\cdot)$. Since $H = o(B)$ as $B \to \infty$, the following approximation is valid:

$$\tfrac{1}{B} w\left(\tfrac{\ell}{B}\right) \tfrac{1}{H^2} K\left(\tfrac{u-\ell}{H}\right) K\left(\tfrac{v-\ell}{H}\right) = \delta_\ell(u)\,\delta_\ell(v)\,\tfrac{1}{B} w\left(\tfrac{\ell}{B}\right) + o(1).$$

Therefore,

$$\begin{aligned}
Q &= \int_I w_B(\ell)\hat{\gamma}_H^2(\ell)d\ell \\
&= \tfrac{1}{T^2}\iiiint_{(0,T]^4}\int_0^T \tfrac{1}{B} w\left(\tfrac{\ell}{B}\right)\tfrac{1}{H^2}K\left(\tfrac{t_1-s_1-\ell}{H}\right)K\left(\tfrac{t_2-s_2-\ell}{H}\right) d\ell d\hat{\epsilon}^a_{s_1} d\hat{\epsilon}^a_{s_2} d\hat{\epsilon}^b_{t_1} d\hat{\epsilon}^b_{t_2} \\
&= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_{t_2-T}^{t_2}\int_{t_1-T}^{t_1}\int_0^T \tfrac{1}{B} w\left(\tfrac{\ell}{B}\right)\tfrac{1}{H^2}K\left(\tfrac{u-\ell}{H}\right)K\left(\tfrac{v-\ell}{H}\right) d\ell d\hat{\epsilon}^a_{t_1-u} d\hat{\epsilon}^a_{t_2-v} d\hat{\epsilon}^b_{t_1} d\hat{\epsilon}^b_{t_2} \\
&= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_0^T 1_{\{\ell\in\cap_{i=1}^2[t_i-T,t_i]\}}\left[\tfrac{1}{B} w\left(\tfrac{\ell}{B}\right)+o(1)\right] d\hat{\epsilon}^a_{t_1-\ell} d\hat{\epsilon}^a_{t_2-\ell} d\ell d\hat{\epsilon}^b_{t_1} d\hat{\epsilon}^b_{t_2}.
\end{aligned}$$

Under the null hypothesis (11), I compute the mean (up to the leading term) as follows:

$$\begin{aligned}
E(Q) &= \tfrac{1}{T^2}\iint_{(0,T]^2}\int_0^T 1_{\{\ell\in\cap_{i=1}^2[t_i-T,t_i]\}}\left[\tfrac{1}{B} w\left(\tfrac{\ell}{B}\right)+o(1)\right] E\left(d\hat{\epsilon}^a_{t_1-\ell} d\hat{\epsilon}^a_{t_2-\ell}\right) d\ell E\left(d\hat{\epsilon}^b_{t_1} d\hat{\epsilon}^b_{t_2}\right) \\
&= \tfrac{\lambda^a\lambda^b}{T^2}\int_0^T\int_{t-T}^t \tfrac{1}{B} w\left(\tfrac{\ell}{B}\right) d\ell dt.
\end{aligned}$$

By Fubini's theorem, the last line becomes

$$E(Q) = \tfrac{\lambda^a\lambda^b}{T}\int_0^T \tfrac{1}{B} w\left(\tfrac{\ell}{B}\right)\left(1-\tfrac{\ell}{T}\right) d\ell,$$

so that $E(Q^{\mathcal{H}}) = C^{\mathcal{H}} + o(1)$. By similar techniques as I obtained (30), I compute the variance (up to the leading term) as follows:

$$Var(Q) = 2\tfrac{\left(\lambda^a\lambda^b\right)^2}{T^2}\int_0^T \tfrac{1}{B^2} w^2\left(\tfrac{\ell}{B}\right)\left(1-\tfrac{\ell}{T}\right)^2 d\ell,$$

so that $Var(Q^{\mathcal{H}}) = 2D^{\mathcal{H}} + o(1)$.

A.13 Summary of Jarrow and Yu (2001) Model

Suppose that there are two parties (e.g. firms), a and b, whose assets are subject to the risk of default. Apart from its own idiosyncratic risk, the probability of default of each party depends on the default status of the other party. The distribution of τ^k ($k = a, b$), the time to default by party k, can be fully characterized by the *conditional intensity function*, $\lambda^k(t|\mathcal{F}_{t^-}) = \lim_{\Delta t \downarrow 0} (\Delta t)^{-1} P(\tau^k \in [t, t + \Delta t)|\mathcal{F}_{t^-})$ where $\mathcal{F} = (\mathcal{F}_t)$ is the natural filtration generated by the processes $1\{\tau^a \leq t\}$ and $1\{\tau^b \leq t\}$, i.e. $\mathcal{F}_t = \sigma\{1\{\tau^a \leq t\}, 1\{\tau^b \leq t\}\}$. Intuitively, it is the conditional probability that party k will default at time t given the history of the default status of both parties. A simple reduced form counterparty risk model is given as follows:

$$\begin{aligned} \text{for party } a \quad &: \quad \lambda^a(t|\mathcal{F}_{t^-}) = \mu^a + \alpha^{ab} 1_{\{\tau^b \leq t\}} \text{ for } t \leq \tau^a, \\ \text{for party } b \quad &: \quad \lambda^b(t|\mathcal{F}_{t^-}) = \mu^b + \alpha^{ba} 1_{\{\tau^a \leq t\}} \text{ for } t \leq \tau^b. \end{aligned}$$

This is probably the simplest bivariate default risk model with counterparty risk features represented by the parameters α^{ab} and α^{ba}. For instance, if α^{ab} is positive, then the default by party b increases the chance of default by party a, thus suggesting the existence of counterparty risk from party b to party a.

The above counterparty risk model involving two parties can be readily extended to one involving two portfolios, a and b. (e.g. two industries of firms). Each portfolio contains a large number of homogeneous parties whose individual conditional intensities of defaults take the same piecewise constant form. For $k = a, b$, let τ_i^k be the time of the i^{th} default in portfolio k, and define $N_t^k = \sum_{i=1}^{\infty} 1\{\tau_i^k \leq t\}$ which counts the number of default events in portfolio k up to time t. Now, denote the natural filtration of (N^a, N^b) by $\mathcal{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t = \sigma\{(N_s^a, N_s^b) : s \leq t\}$, and the conditional intensity of default in portfolio k at time t by $\lambda^k(t|\mathcal{F}_{t^-}) = \lim_{\Delta t \downarrow 0} (\Delta t)^{-1} P(N_{t+\Delta t}^k - N_t^k > 0|\mathcal{F}_{t^-})$. Analogous to the counterparty risk model with two parties, a counterparty risk model with two portfolios a and b is defined as follows:

$$\text{for portfolio } a: \qquad \lambda^a(t|\mathcal{F}_{t^-}) = \mu^a + \alpha^{aa} \sum_{q=1}^{\infty} 1_{\{\tau_q^a \leq t\}} + \alpha^{ab} \sum_{j=1}^{\infty} 1_{\{\tau_j^b \leq t\}}, \tag{41}$$

$$\text{for portfolio } b: \qquad \lambda^b(t|\mathcal{F}_{t^-}) = \mu^b + \alpha^{ba} \sum_{i=1}^{\infty} 1_{\{\tau_i^a \leq t\}} + \alpha^{bb} \sum_{q=1}^{\infty} 1_{\{\tau_q^b \leq t\}}. \tag{42}$$

We can rewrite (41) and (42) in terms of the counting processes N_t^k:

$$\begin{aligned} \lambda^a(t|\mathcal{F}_{t^-}) &= \mu^a + \alpha^{aa} N_t^a + \alpha^{ab} N_t^b \text{ for } t \leq \tau_i^a, \\ \lambda^b(t|\mathcal{F}_{t^-}) &= \mu^b + \alpha^{ba} N_t^a + \alpha^{bb} N_t^b \text{ for } t \leq \tau_j^b. \end{aligned}$$

With an additional exponential function (or other discount factors) to dampen the feedback effect of each earlier default event, the system of conditional intensities constitutes an *bivariate exponential* (or *generalized*) *Hawkes model* for (N^a, N^b):

$$\begin{aligned}
\lambda^a(t|\mathcal{F}_{t^-}) &= \mu^a + \alpha^{aa} \sum_{i=1}^{n^a} 1_{\{\tau_i^a \le t\}} e^{-\beta^{aa}(t-\tau_i^a)} + \alpha^{ab} \sum_{j=1}^{n^b} 1_{\{\tau_j^b \le t\}} e^{-\beta^{ab}(t-\tau_j^b)} \\
&= \mu^a + \alpha^{aa} \int_0^t e^{-\beta^{aa}(t-s)} dN_s^a + \alpha^{ab} \int_0^t e^{-\beta^{ab}(t-u)} dN_u^b,
\end{aligned}$$

$$\begin{aligned}
\lambda^b(t|\mathcal{F}_{t^-}) &= \mu^b + \alpha^{ba} \sum_{i=1}^{n^a} 1_{\{\tau_i^a \le t\}} e^{-\beta^{ba}(t-\tau_i^a)} + \alpha^{bb} \sum_{j=1}^{n^b} 1_{\{\tau_j^b \le t\}} e^{-\beta^{bb}(t-\tau_j^b)} \\
&= \mu^b + \alpha^{ba} \int_0^t e^{-\beta^{ba}(t-s)} dN_s^a + \alpha^{bb} \int_0^t e^{-\beta^{bb}(t-u)} dN_u^b.
\end{aligned}$$

To test for the existence of Granger causality based on this model, we can estimate the parameters α^{ab} and α^{ba} and test if they are significant. However, this parametric bivariate model is only one of the many possible ways that the conditional intensities of default from two portfolios can interact with one another. The nonparametric test in this paper can detect Granger causality without making a strong parametric assumption on the bivariate point process of defaults.