Learning and Selfconfirming Equilibria in Network Games*

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Abstract

Consider a set of agents who play a network game repeatedly. Agents may not know the network. They may even be unaware that they are interacting with other agents in a network. Possibly, they just understand that their payoffs depend on an unknown state that in reality is an aggregate of the actions of their neighbors. Each time, every agent chooses an action that maximizes her subjective expected payoff and then updates her beliefs according to what she observes. In particular, we assume that each agent only observes her realized payoff. A steady state of such dynamic is a **selfconfirming equilibrium** given the assumed feedback. We characterize the structure of the set of selfconfirming equilibria in network games and we relate selfconfirming and Nash equilibria. Thus, we provide conditions on the network under which the Nash equilibrium concept has a learning foundation, despite the fact that agents may have incomplete information. In particular, we show that the choice of being active or inactive in a network is crucial to determine whether agents can make correct inferences about the payoff state and hence play the best reply to the truth in a selfconfirming equilibrium. We also study learning dynamics and show how agents can get stuck in non–Nash selfconfirming equilibria. In such dynamics, the set of inactive agents can only increase in time, because once an agent finds it optimal to be inactive, she gets no feedback about the payoff state, hence does not change her beliefs and remains inactive.

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1 Motivation

Imagine an online social network, like Twitter, with many users. Let us consider a simultaneous-moves game, in which each user i decides her level of activity $a_i \geq 0$ in the social network. The payoff that agents get from their activity depends on the social interaction. In particular, when active user i receives idiosyncratic externalities, that can be positive and negative, from the other users with whom she is in contact with in the social network. The externality from user i to user j is proportional to the time that they both spend on the social network, a_i and a_j. Sticking to a quadratic specification, that allows for linear best replies, let us assume that the payoff of i from this game is[1]

$$u_i(a_i, \mathbf{a}_{-i}) = \alpha_i a_i - \frac{1}{2} a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j . \tag{1}$$

In eq. (1), I is the set of agents in the social network and a_i is the level of activity of $i \in I$, while α_i represents the individual pleasure of i from being active on the social network in isolation, which results in the *bliss point* of activity in autarchy. Parameter α_i can also be negative, and in this case i would not be active in isolation. For each $j \in I \setminus \{i\}$, there is some exogenous level of externality from j to i denoted by z_{ij}. We say that j affects i, or that j is a **peer** of i, if $z_{ij} \neq 0$.

At some point, in this paper, we will also consider an extra **global** term in the payoff function

$$u_i(a_i, \mathbf{a}_{-i}) = \alpha a_i - \frac{1}{2} a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j + \beta \sum_{k \in j \in I \setminus \{i\}} a_k . \tag{2}$$

We can interpret this extra term as an additional pleasure that i gets from being member (even if not active) of an online social network that is overall *popular*.

In this paper, the network described by the matrix $\mathbf{Z}$ of all the z_{ij}'s is *exogenous*. As a first approximation, this fits a *directed* online social network like Twitter, where users cannot decide who follows them. Under this interpretation, i receives positive or negative externalities from those who follow her, that are proportional to her activity. Payoff represents the *popularity* that i receives from being active or not in the social network. We imagine that i cannot choose the style of what she writes, since she just follows her exogenous nature. In this interpretation, a_i represents the amount of *tweets* that i writes, and this can make her more or less popular for those who follow her, according to how her style combines with the (typically unobserved) taste of each of her followers.

Since we are going to analyze learning dynamics and their steady states, we also have to specify what agents observe after their choice, because this affects how they update their beliefs. Twitter user i typically observes perfectly her own activity level a_i, but she may not observe the sign

[1] This is the class of games originally analyzed by Ballester *et al.* (2006). Bramoullé *et al.* (2014) is one of the more recent papers providing results for such linear-qadratic network games, and they discuss also how to generalize to games that have the same best–reply functions. Zenou (2016) surveys many applications.

of the externalities and the activity of others. However, she gets indirect measures of her level of popularity that come from her conversations and experiences in the real world, where her popularity from Twitter affects her social and professional real life. Players of this game may have wrong beliefs about the details of the game they are playing (e.g. the structure of the network, or the value of the parameters) and about the actions of other players. With this, they update their beliefs in response to the feedback they receive, which will be their (possibly indirectly measured) payoff. This updating process may lead to a *learning dynamic* that does not converge to a Nash equilibrium of the game.

In this paper we address the following question: assuming simple updating rules, under what circumstances do learning dynamics converge to a Nash equilibrium of the game and when, instead, do they just converge to a **selfconfirming equilibrium** where agents best reply to confirmed but possibly wrong beliefs? This question is *per se* interesting, and with our answers we provide novel theoretical tools for the analysis of network games. However, the application of the model to online social networks that we just anticipated can also help in understanding why we may easily observe apparently non–optimal best responses by economic agents in such an environment, such as agents who get stuck into "inactivity traps."

Section 2 presents our baseline model. For this setting, we characterize the set selfconfirming equilibria in Section 3, and we discuss the learning process in Section 4. In Section 5 we analyze a more general model that accounts for global externalities. We devote appendices to proofs and technical results. Appendix A analyze properties of feedback and selfconfirming equilibria in a class of games including as special cases the network games that we consider. Appendix B reports existing and results in linear algebra, that we use to find sufficient conditions for reaching interior Nash equilibria in network games. Appendix C contains the proofs of our propositions.

2 The Framework

Consider a set I of agents, with cardinality $n = |I|$ and generic element i, located in a network. Let the network be characterized by an adjacency matrix $\mathbf{Z} \in \mathbb{R}^{I\times I}$, where entry z_{ij} specifies whether agent i is linked to agent $j \neq i$ and the weight of this link, and we let $z_{ii} = 0$ by convention. In what follows we consider the possibility of asymmetric networks, so that $z_{ij} \neq z_{ji}$, and in particular the case of directed network, so that, given $i, j \in I$, we allow $z_{ij} > 0$, and $z_{ji} = 0$.[2]

We assume that there is an upper bound $\bar{w}$ and a lower bound $\underline{w}$ in the weighted externalities, that can be positive or negative, between players We let $\Theta \subseteq [\underline{w}, \bar{w}]^{I\times I}$ denote the of possible weighted networks $\mathbf{Z}$. The network game is parametrized by $\mathbf{Z} \in \Theta$ and we assume that Θ is *compact*.

[2]In some of the examples that we propose, which strictly relate to existing literature, we will consider the simplified case of $\mathbf{Z} \in \{0, 1\}^{I\times I}$ in which all active links have the same weight normalized to 1. However, this will typically not be the case.

Each agent $i \in I$ chooses an action a_i from interval $A_i = [0, \bar{a}_i]$, where the upper bound $\bar{a}_i$ is "sufficiently large".[3] For each $i \in I$, $\mathbf{A}_{-i} := \times_{j \neq i} A_j$ denotes the set of action profiles $\mathbf{a}_{-i} = (a_j)_{j \in I \setminus \{i\}}$ for players different from i. Similarly, defining $N_i := \{j \in I : z_{ij} \neq 0\}$ as the set of the neighbors of a given agent i, $\mathbf{A}_{N_i} := \times_{j \in N_i} A_j$ denotes the set of action profiles $\mathbf{a}_{N_i} := (a_j)_{j \in N_i}$ of i's neighbors.

For each $i \in I$, we posit a set (interval) $X_i = [\underline{x}_i, \bar{x}_i]$ of **payoff states for** i, with the interpretation that i's payoff his determined by his action a_i and by his payoff state x_i according to a utility function $v_i : A_i \times X_i \to \mathbb{R}$. The payoff state x_i is in turn determined by the actions of i's neighbors and is unknown to i at the time of his choice. For each agent $i \in I$ and matrix $\mathbf{Z}$, we consider a parametrized **aggregator** of the coplayers' actions $\ell_i : \mathbf{A}_{-i} \times \Theta \to X_i$ of the following form: for each $\mathbf{Z} \in \Theta$, the section of ℓ_i at $\mathbf{Z}$ is

$$\begin{array}{rccl} \ell_{i,\mathbf{Z}}: & \mathbf{A}_{-i} & \to & X_i, \\ & \mathbf{a}_{-i} & \mapsto & \sum_{j \neq i} z_{ij} a_j. \end{array}$$

This notation allows to assume that, on top of the adjacency matrix, the relationships between agents are mediated by a parameter vector $\boldsymbol{\gamma} \in \mathbb{R}_+^I$. In fact, suppose that there is a basic network $\mathbf{Z}_0$. Then we can represent the linear aggregator as

$$\mathbf{a}_{-i} \mapsto \gamma_i \left(\sum_{j \neq i} z_{0,ij} a_j \right).$$

In what follows we include in the adjacency matrix the heterogeneous parameters from vector $\boldsymbol{\gamma}$: we let $\mathbf{Z} = \boldsymbol{\Gamma} \mathbf{Z}_0$, where $\boldsymbol{\Gamma}$ is a diagonal matrix, with diagonal given by $\boldsymbol{\gamma}$, and $\mathbf{Z}_0$ is the basic network (e.g., a matrix of 0s and 1s). In words, $\mathbf{Z}$ represents a social network in which there is an additional idiosyncratic effect by which every agent i weights the effects of the others on her, and this effect is parameterized by γ_i.

Let $N_i^- := \{j \in I : z_{ij} < 0\}$ denote the set of neighbors of player i that have a negative effect on the payoff state of i. Similarly, $N_i^+ := \{j \in I : z_{ij} > 0\}$ denotes the set of neighbors of player i that have a positive effect on the payoff state of i. We also assume that

$$\forall i \in I,\ \underline{x}_i \leq \sum_{j \in N_i^-} z_{ij} \bar{a}_j,\ \bar{x}_i \geq \sum_{j \in N_i^+} z_{ij} \bar{a}_j.$$

The overall payoff function that associates each action profile $(a_i, \mathbf{a}_{-i})$ with a payoff for agent i is thus parametrized by the adjacency matrix $\mathbf{Z}$:

$$\begin{array}{rccl} u_i: & A_i \times \mathbf{A}_{-i} \times \Theta & \to & \mathbb{R}, \\ & (a_i, \mathbf{a}_{-i}, \mathbf{Z}) & \mapsto & v_i\left(a_i, \ell_i\left(\mathbf{a}_{-i}, \mathbf{Z}\right)\right). \end{array} \tag{3}$$

[3] Note that in the network literature it is common to assume $A_i = \mathbb{R}_+$. However, for the games we consider, we can always find an upper bound $\bar{a}$ on actions such that the problem is unchanged when actions are bounded above by $\bar{a}$.

Note, we assume that each agent i knows how her payoff depends on her action and her payoff state, that is, we assume that i knows function v_i, but we do *not* assume that i knows $\mathbf{Z}$. Actually, from the perspective of our analysis, agent i might even ignore that the payoff state x_i aggregates her neighbors' activities according to some weighted network structure, because *we are not modeling how i reasons strategically.*[4] If $v_{i,x_i} : A_i \to \mathbb{R}$ is strictly concave[5] for each x_i, there is a unique best reply $r_i(x_i)$ to each payoff state x_i. Although the aggregator is linear, if this "proximate" best reply function $r_i : X_i \to A_i$ is non-linear,[6] then also the best reply $r_i(\ell_i(\mathbf{a}_{-i}, \mathbf{Z}))$ is non-linear in $\mathbf{a}_{-i}$. Linearity obtains if and only if v_i is quadratic in a_i and linear in x_i. Without substantial loss of generality, among such utility functions we consider the following form:

$$\begin{array}{rccc} v_i : & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & \alpha_i a_i - \frac{1}{2} a_i^2 + a_i x_i. \end{array} \tag{4}$$

Note that v_i in eq. (4) is continuous and strictly concave in a_i. Thus, $G = \langle I, \Theta, (A_i, u_i)_{i\in I} \rangle$, with u_i defined by eqs. (3)-(4), is a parametrized nice game (see Moulin 1984 for a definition of nice game, and Appendix A for a generalization, with some results for non-linear-quadratic network games).

We assume that the game is repeatedly played by agents maximizing their instantaneous payoff. After each play agents get some feedback. Let M be an abstract set of "messages" (e.g., monetary outcomes). The information obtained by agent $i \in I$ after each round is described by a **feedback function** $f_i : A_i \times X_i \to M$. Assuming that i knows how her feedback is determined by the payoff state given her action, if she receives message m after action a_i she infers that the state x_i belongs to the "ex post information set"

$$f_{i,a_i}^{-1}(m) := \left\{ x_i' \in X_i : f_i\left(a_i, x_i'\right) = m \right\}.$$

This completes the description of the object of our analysis. The structure

$$NG = \langle I, \Theta, (A_i, X_i, v_i, \ell_i, f_i)_{i\in I} \rangle$$

is a (parameterized) **network game with feedback,** or simply **network game.** Our analysis depends on the assumptions about the payoff functions and the feedback functions.

Definition **1.** *A network game with feedback NG is* ***linear-quadratic*** *if the utility function of each player has the linear-quadratic form (4).*

[4]If the parametrized payoff functions and the parameter space Θ are common knowledge, strategic reasoning according to the epistemic assumptions of *rationality and common belief of rationality* can be captured by a simple incomplete-information version of the rationalizability concept. See, e.g., chapter 7 of Battigalli (2018) and the references therein.

[5]Or strictly quasi-concave.

[6]More precisely, not affine.

In this case, the proximate best-reply function is

$$r_i(x_i) = \begin{cases} 0, & \text{if } x_i \le -\alpha_i, \\ \alpha_i + x_i, & \text{if } \alpha < x_i < \bar{a}_i - \alpha_i, \\ \bar{a}_i, & \text{if } x_i \ge \bar{a}_i - \alpha_i. \end{cases} \tag{5}$$

Hence the best reply to the actions of others is

$$r_i(\ell_i(\mathbf{a}_{-i}, \mathbf{Z})) = \begin{cases} 0, & \text{if } \sum_{j \ne i} z_{ij} a_j \le -\alpha_i, \\ \alpha_i + \sum_{j \ne i} z_{ij} a_j, & \text{if } -\alpha_i < \sum_{j \ne i} z_{ij} a_j < \bar{a}_i - \alpha_i, \\ \bar{a}_i, & \text{if } \sum_{j \ne i} z_{ij} a_j \ge \bar{a}_i - \alpha_i. \end{cases} \tag{6}$$

DEFINITION **2.** *Feedback f_i satisfies **observability if and only if** player i **is active** (OiffA) if section f_{i,a_i} is injective for each $a_i \in (0, \bar{a}_i]$ and constant for $a_i = 0$; f_i satisfies **just observable payoffs** (JOP) relative to v_i if there is a function $\bar{v}_i : A_i \times M \to \mathbb{R}$ such that*

$$\forall (a_i, x_i) \in A_i \times X_i, v_i(a_i, x_i) = \bar{v}_i(a_i, f_i(a_i, x_i))$$

*and the section $\bar{v}_{i,a_i} : M \to \mathbb{R}$ is injective for each $a_i \in A_i$. Network game with feedback NG satisfies **observability by active players** if feedback f_i satisfies OiifA , for each player $i \in I$, and it satisfies **just observable payoffs** if f_i satisfies JOP for each player $i \in I$.*

In a game with just observable payoffs agents infer their realized payoff from the message they get, but no more than that. For example, the feedback could be a total benefit, or revenue function

$$\begin{array}{rccc} f_i: & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & \alpha_i a_i + a_i x_i, \end{array}$$

with the payoff given by the difference between benefit and activity cost $C_i(a_i)$:

$$\begin{array}{rccc} v_i: & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & f_i(a_i, x_i) - C_i(a_i). \end{array}$$

Under the reasonable assumption that agent i knows her cost function, when she chooses a_i and then gets message m, she infers that her payoff is $\bar{v}_i(a_i, m) = m - C_i(a_i)$. Thus, each section $\bar{v}_{i,a_i}$ ($a_i \in A_i$) is indeed injective. If the feedback/benefit function is $f_i(a_i, x_i) = \alpha_i a_i + a_i x_i$, then it satisifes observability if and only if i is active.

REMARK **1.** *If NG is* linear-quadratic *and satisfies* just observable payoffs*, then it satisfies observability by active players. If NG satisfies observability by active players, then*

$$f_{i,a_i}^{-1}(f_i(a_i, x_i)) = \begin{cases} X_i, & \text{if } a_i = 0, \\ \{x_i\}, & \text{if } a_i > 0 \end{cases} \tag{7}$$

for every agent $i \in I$ and action-state pair $(a_i, x_i) \in A_i \times X_i$.

Most of our analysis focuses on linear-quadratic network games with just observable payoffs. This implies that agents who are active get as a feedback a message enabling them to perfectly determine the state. Conversely, inactive agents get a message completely uninformative.

To choose an action, subjectively rational agents must have some deterministic or probabilistic conjecture about the payoff state x_i. We refer to conjectures about the state as **shallow conjectures**, as opposed to **deep conjectures,** which concern the specific network topology ($\mathbf{Z}$) and the actions of other players ($\mathbf{a}_{-i}$). In linear-quadratic network games (more generally, in nice games with feedback), it is sufficient to focus on *deterministic shallow conjectures.* Indeed, for every probabilistic conjecture $\mu_i \in \Delta(X_i)$, there exists a deterministic conjecture $\hat{x}_i \in X_i$ that justifies the same action a_i^* as the unique best reply (see the discussion in Appendix A.1).

2.1 Selfconfirming equilibrium

We analyze a notion of equilibrium which is broader than Nash equilibrium. Recall that our approach allows for the possibility of agents who are unaware of the full game around them. In a steady state, agents best respond to conjectures consistent with the feedback that they receive, which is not necessarily fully revealing. We believe that this approach fits well to a networked environment where the information that players receive is only local.[7]

Definition **3.** *A profile $(a_i^*, \hat{x}_i)_{i\in I} \in \times_{i\in I}(A_i \times X_i)$ of actions and (shallow) deterministic conjectures is a* ***selfconfirming equilibrium (SCE) at*** $\mathbf{Z}$ *if, for each $i \in I$,*

1. (subjective rationality) $a_i^* = r_i(\hat{x}_i)$,
2. (confirmed conjecture) $f_i(a_i^*, \hat{x}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)$.

We say that $\mathbf{a}^* = (a_i^*)_{i\in I}$ is a **selfconfirming action profile** at $\mathbf{Z}$ if there exists a corresponding profile of conjectures $(\hat{x}_i)_{i\in I}$ such that $(a_i^*, \hat{x}_i)_{i\in I}$ is a selfconfirming equilibrium at $\mathbf{Z}$, and we let $\mathbf{A}_{\mathbf{Z}}^{SCE}$ denote the set of such profiles. Also, for any adjacency matrix $\mathbf{Z} \in \Theta$, we denote by $\mathbf{A}_{\mathbf{Z}}^{NE}$ the set of (pure) Nash equilibria of the (nice) game determined by $\mathbf{Z}$, that is,

$$\mathbf{A}_{\mathbf{Z}}^{NE} := \left\{\mathbf{a}^* \in \times_{i\in I} A_i : \forall i \in I, a_i^* = r_i\left(\ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)\right\}.$$

Nice games satisfy all the standard assumptions for the existence of Nash equilibria.[8] Hence, we obtain the existence of selfconfirming equilibria for each $\mathbf{Z} \in \Theta$. To summarize:

[7]In a context of endogenous strategic network formationn fact, McBride (2006) apply the **conjectural equilibrium** concept, which is essentially as selfconfirming equilibrium for games with feedback (see the discussion in Battigalli et al. 2015).

[8]Since the self-map $\mathbf{a} \mapsto (r_i(\mathbf{a}_{-i}, \mathbf{Z}))_{i\in I}$ is continuous on the convex and compact set $A = \times_{i\in I}[0, \bar{a}_i]$, by Brouwer's Theorem it has a fixed point.

Remark **2.** *For every* $\mathbf{Z}$*, there is at least one Nash equilibrium, and every Nash equilibrium is a selfconfirming profile of actions:*

$$\forall\ \mathbf{Z} \in \Theta,\ \emptyset \neq \mathbf{A}_{\mathbf{Z}}^{NE} \subseteq \mathbf{A}_{\mathbf{Z}}^{SCE}.$$

3 Characterization of SCE

In this section we characterize the set $\mathbf{A}_{\mathbf{Z}}^{SCE}$ of selfconfirming equilibrium profiles of actions. We start with the simplest case in which every agent finds it subjectively optimal to be active. All our proofs are derived from the results in Appendix A and Appendix B, and are stated in Appendix C.

Proposition **1.** *Consider a network game NG satisfying* observability by active players*. Assume that, for every* $i \in I$ *and for every* $\hat{x}_i \in X_i$*,* $r_i(\hat{x}_i) > 0$*. Then, for each* $\mathbf{Z} \in \Theta$*,* $\mathbf{A}_{\mathbf{Z}}^{SCE} = \mathbf{A}_{\mathbf{Z}}^{NE}$*.*

Assume that $\alpha_i > 0$ and $\mathbf{Z} = \gamma \mathbf{Z}_0$, with $\gamma > 0$ and $\mathbf{Z}_0 \in \{0,1\}^{I \times I}$. This represents the standard case of local complementarities studied by Ballester *et al.* (2006). If $\gamma(n-1) < 1$ there is a unique Nash equilibrium which is also interior. Our proposition states that, in this case, if being inactive is not justifiable as a best reply to any shallow conjecture, then there is only one selfconfirming equilibrium action profile, which necessarily coincides with the unique Nash equilibrium.

We now consider a more general case in which agents may be inactive. Let I_0 denote the **set of players for whom being inactive is justifiable**. Note that, by Lemma A in Appendix A.1,

$$I_0 = \{i \in I : \min r_i(X_i) = 0\}.$$

Also, for each $\mathbf{Z} \in \Theta$ and nonempty subset of players $J \subseteq I$, let $\mathbf{A}_{J,\mathbf{Z}}^{NE}$ denote the set of Nash equilibria of the auxiliary game with player set J obtained by letting $a_i = 0$ for each $i \in I \backslash J$, that is,

$$\mathbf{A}_{J,\mathbf{Z}}^{NE} = \left\{ \mathbf{a}_J^* \in \times_{j \in J} A_j : \forall j \in J, a_j^* = r_j\left(\ell_j\left(\mathbf{a}_{J\backslash\{j\}}^*, \mathbf{0}_{I\backslash J}, \mathbf{Z}\right)\right)\right\},$$

where $\mathbf{0}_{I\backslash J} \in \mathbb{R}^{I\backslash J}$ is the profile that assigns 0 to each $i \in I\backslash J$. If $J = \emptyset$, let $\mathbf{A}_{J,\mathbf{Z}}^{NE} = \emptyset$ by convention. We relate the set of self confirming equilibria to the sets of Nash equilibria of the auxiliary games.

Proposition **2.** *Suppose that network game with feedback NG is* linear-quadratic *and satisfies* just observable payoffs*. Then, for each* $\mathbf{Z} \in \Theta$*, the set of selfconfirming action profiles is*

$$\mathbf{A}_{\mathbf{Z}}^{SCE} = \bigcup_{I\backslash J \subseteq I_0} \mathbf{A}_{J,\mathbf{Z}}^{NE} \times \left\{\mathbf{0}_{I\backslash J}\right\},$$

that is, in each SCE profile $\mathbf{a}^*$*, a subset* $I\backslash J$ *of players for whom being inactive is justifiable choose* 0*, and every other player chooses the best reply to the actions of his coplayers. Therefore, in each SCE profile* $\mathbf{a}^*$ *and for each player* $i \in I$*,*

$$\begin{aligned} a_i^* &= 0 \Rightarrow \underline{x}_i \leq -\alpha_i, \\ a_i^* &> 0 \Rightarrow \left(\alpha_i + \sum_{j \in I} z_{ij} a_j^* > 0\ \wedge a_i^* = \min\left\{\bar{a}_i, \alpha_i + \sum_{j \in I} z_{ij} a_j^*\right\}\right). \end{aligned} \quad (8)$$

In every SCE we can partition the set of agents in two subsets. Agents in $J \subseteq I$ choosing a strictly positive action, and agents in $I \setminus J$ choosing a null action. Start considering these last ones. If they play $a_i^* = 0$, they get null payoff independently of others' actions. But, since every conjecture $\hat{x}^i \in (-\infty, -\alpha_i]$ is compatible with this payoff, their conjecture is confirmed. On the contrary, agents in J choosing a strictly positive action $a_i^* > 0$ receive a message that enables them to recover perfectly $\hat{x}^i$. Notice that every agent makes a choice that is a best reply to her own conjecture. However, differently from the case in Proposition 1, here there are some agents (the ones in the set $I \setminus J$) who do not necessarily have correct conjectures. Given the specific structure of linear quadratic games, for every i, the presence of inactive agents in her neighborhood, is payoff irrelevant. For this reason the set of self confirming equilibria can be characterized from the set of Nash equilibria of the auxiliary game.

Suppose, for example, that $I_0 = I$. This means that, for every subset $J \subseteq I$ we can find a self confirming equilibria in which agents in J are inactive. Furthermore, suppose that there is a unique interior Nash equilibrium for the auxiliary game corresponding to every subset of active players. Then $|\mathbf{A}_{\mathbf{Z}}^{SCE}| = 2^{|I|}$, that is, there are exactly 2^n SCE action profiles. Appendix A.3 discusses the equilibrium characterization for the generalized case of non linear-quadratic network games.

3.1 Assumptions about the network

Now we focus on the network $\mathbf{Z}$. In what follows, remember that we can always represent $\mathbf{Z}$ as $\mathbf{Z} = \mathbf{\Gamma}\mathbf{Z}_0$, where $\mathbf{\Gamma}$ is a diagonal matrix, and $\mathbf{Z}_0$ is the basic underlying representation of the network. Thus, matrix $\mathbf{Z}$ represents a basic network combined with an additional idiosyncratic effect by which every agent i weights the effects of the others on her. This effect is modeled by the parameter γ_i.[9]

We list below some additional properties of matrix $\mathbf{Z}$ that are not maintained assumptions. Rather, in some of the following results (for which we refer also to Appendix B) we will use some of these assumptions, in other results we will use other assumptions. Finally, in some of the main results, we will show that alternative assumptions will provide alternative sufficient conditions.

Assumption **1.** *Matrix* $\mathbf{Z}$ *of size* n *has* bounded values, *i.e.* $|z_{ij}| < \frac{1}{n}$ *for all* i *and* j.

Assumption **2.** *Matrix* $\mathbf{Z}$ *has the* same sign property *i.e., for every* i, j, $sign(z_{ij}) = sign(z_{ji})$, *where the sign function can have values* -1, 0 *or* 1.[10]

Assumption **3.** *Matrix* $\mathbf{Z}$ *is* negative, *i.e.* $z_{ij} < 0$ *for all* i *and* j,

[9]Then the payoff of $i \in I$ at a given profile $\mathbf{a}$ of the original game is

$$u_i(\mathbf{a}) = \alpha a_i - \frac{1}{2}a_i^2 + a_i \gamma_i \sum_{j \in I} z_{0,ij} a_j = \alpha a_i - \frac{1}{2}a_i^2 + a_i \sum_{j \in I} z_{ij} a_j \quad .$$

[10]The sign condition is the one used in Bervoets *et al.* (2016) to prove convergence to Nash equilibria in network games, under a particular form of learning.

We recall here that the spectral radius $\rho(\mathbf{Z})$ of $\mathbf{Z}$ is the largest absolute value of its eigenvalues.

Assumption **4.** *Matrix* $\mathbf{Z}$ *is* limited*, i.e.* $\rho(\mathbf{Z}) < 1$.

Assumption **5.** *Matrix* $\mathbf{Z}$ *is* symmetrizable*, i.e. it can be written as* $\mathbf{Z} = \mathbf{\Gamma Z}_0$*, with* $\mathbf{\Gamma}$ *diagonal and* $\mathbf{Z}_0$ *symmetric. Moreover,* $\mathbf{\Gamma}$ *has all positive entries in the diagonal.*

Note that if $\mathbf{Z}$ is symmetrizable then all its eigenvalues are real. Moreover, since $\mathbf{\Gamma}$ has all positive entries, Assumption 5 implies the sign condition from Assumption 2.
Our final assumption is discussed in Bramoullé *et al.* (2014) and combines Assumptions 4 and 5 above.

Assumption **6.** $\mathbf{Z} = \mathbf{\Gamma Z}_0$ *is* symmetrizable-limited*, i.e.* $\mathbf{Z}$ *is symmetrizable and, for every* i, j*,* $z_{ij} = z_{0,ij}\sqrt{\gamma_i \gamma_j}$*, is limited.*

Our previous results from Section 3, about the characterization of selfconfirming equilibria, state that we can choose any subset of agents and have them inactive in a SCE. However we cannot ensure that the other agents are active, because their best response in the reduced game could be null. The next result goes in the direction of specifying under which sufficient conditions this does not happen. Given the matrix $\mathbf{Z}$, and given $J \subseteq I$, we call $\mathbf{Z}_J$ the submatrix who has only rows and columns corresponding to the elements of J.

Proposition **3.** *Consider a set* $J \subseteq I$*. Let us assume that* $\mathbf{Z}_J$ *satisfies at least one one of the three conditions below:*

1. *[(i)]*
2. *it has bounded values (Assumption 1),*
3. *it is negative and limited (Assumptions 3* and *4),*
4. *or it is symmetrizable–limited (Assumption 6).*

Then, we have the two following results:

1. $\mathbf{A}^{NE}_{J,\mathbf{Z}} = \left\{\mathbf{a}^{NE}_J\right\}$*, such that* $\mathbf{a}^{NE}_J > 0$*;*
2. *There exists* $\mathbf{a}^* \in \mathbf{A}^{SCE}_{\mathbf{Z}}$ *such that* $\mathbf{a}^* = \left\{\mathbf{a}^{NE}_J\right\} \times \left\{\mathbf{0}_{I \setminus J}\right\}$.

4 Learning process

The definition of selfconfirming equilibrium is a static definition. If agents happen to have those conjectures, and play accordingly, then they have no reason to move away from it. However we may wonder how agents get to play SCE action profiles, and if these profiles are stable.

We first notice that existing results show that SCE have solid learning motivations. Consider a sequence of actions' profiles, at time t given by $(\mathbf{a}_t)_{t=0}^{\infty}$. An existing theorem states the following result: *If a trajectory* $(\mathbf{a}_t)_{t=0}^{\infty}$ *is consistent with adaptive learning*[11] *and* $a_t \to a^*$, *then* a^* *is a selfconfirming equilibrium action profile.*

Of course, the limit of the trajectory may or may not be a Nash equilibrium. Let us now consider a best response dynamics. This generates trajectories that—by construction—are consistent with adaptive learning. With this, we prove convergence (under reasonable assumptions), hence convergence to an SCE.

To ease the analysis we consider best reply dynamics for shallow conjectures. In details, given conjecture $\hat{x}_{i,t}$, $a^*_{i,t} = r(\hat{x}_{i,t})$ is the best reply to this conjecture. Notice that we may or may not be in a selfconfirming equilibrium. At the end of the period, given the feedback received, agents update conjectures. If conjectures are confirmed then an agent keeps past conjecture, otherwise she updates using as new conjecture the conjectures that would have been correct in the past period. In details,

$$\hat{x}_{i,t+1} = \begin{cases} \hat{x}_{i,t} & \text{if } a^*_{i,t} = 0 \\ \ell_i\left(\mathbf{a}^*_{-i,t}, \mathbf{Z}\right) & \text{if } a^*_{i,t} > 0 \end{cases} . \tag{9}$$

This rules states that if $a^*_{i,t} = 0$, past conjectures are kept. This since $a^*_{i,t} = 0$ can be played only if $\hat{x}_{i,t} \in (-\infty, -\alpha)$. Then, any of these conjectures is confirmed by the feedback, as previously argued. On the contrary, if $a^*_{i,t} > 0$, feedback is such that agents can perfectly infer the level of $\ell_i\left(\mathbf{a}^*_{-i,t}, \mathbf{Z}\right)$, and so they update conjectures accordingly. This is one possible adaptive learning dynamics. Previous theorem implies that if the dynamics described above converges, then it must converge to a selfconfirming equilibrium.

The implicit interpretation of SCE is based on the idea of an underlying learning process: players repeatedly take a decision, receive feedback and update actions accordingly. Under this view a SCE is a stable point of the dynamics where players keep on taking a decision that is a consistent best response to the feedback that they receive. In this section we consider the notion of stability, in the simplest possible case of resistance to small perturbations, as in Bramoullé and Kranton (2007). However, we will not consider perturbations to the strategy profile, but perturbations on the profile of beliefs.

Definition 4 (Learning process). *Players start at time* 0 *with a vector of beliefs (shallow deterministic conjectures)* $\hat{\mathbf{x}}_0 = (\hat{x}_{i,0})$. *At each time step players take actions according to rationality: for each player* i *we have* $a^*_{i,t} = \max\{\alpha_i + \hat{x}_{i,t}, 0\}$.
At the end of each time step players update beliefs such that, if $a^*_{i,t} = 0$, *then* $\hat{x}_{i,t+1} = \hat{x}_{i,t}$*; if instead* $a^*_{i,t} > 0$, *then* $\hat{x}_{i,t+1} = \frac{u_i(\mathbf{a}^*_t)}{a^*_{i,t}} - \alpha + \frac{1}{2}a^*_{i,t}$.

[11] A trajectory $(\mathbf{a}_t)_{t=0}^{\infty}$ is consistent with adaptive learning if for every $\hat{t}$, there exists some T such that, for every $t > \hat{t} + T$ and $i \in I$, $a_{i,t}$ is a best reply to some *deep* conjecture μ_i that assigns probability 1 to the set action profiles $\mathbf{a}_{-i}$ consistent with the feedback received from $\hat{t}$ through $t-1$. See Chapter 6 of Battigalli (2018).

Note that we are in principle able to express the dynamics just in terms of beliefs. If we consider the case of linear best replies, from equations (8) and (9), but the system is not linear because

$$\hat{x}_{i,t+1} = \begin{cases} \hat{x}_{i,t} & \text{if} \quad \hat{x}_{i,t} \leq \alpha \\ \sum_{j \in I} z_{ij} a^*_{j,t} & \text{if} \quad \hat{x}_{i,t} > \alpha \end{cases},$$

and for any other player j, we have that $a^*_{j,t} = \max\{\alpha_j + \hat{x}_{j,t}, 0\}$.

Clearly a SCE of the game, as defined in the beginning of Section 3, will always be a steady state point of this learning dynamics. Consider $\hat{\mathbf{x}}$ to be also the vector of beliefs once the steady state is reached.

Example 1. Consider the case of 4 players, with the network matrix $\mathbf{Z} \in \{0, 0.2\}^{I \times I}$ shown in the left panel of Figure 1, and $\alpha = 0.1$. This is a case of complements, with only positive externalities. Starting from the beliefs depicted at the beginning of the bottom–left plot in Figure 1, the learning dynamics (expressed in the top–left plot also for actions) converges to the unique Nash equilibrium of the network game (whose strategies are the dotted lines).

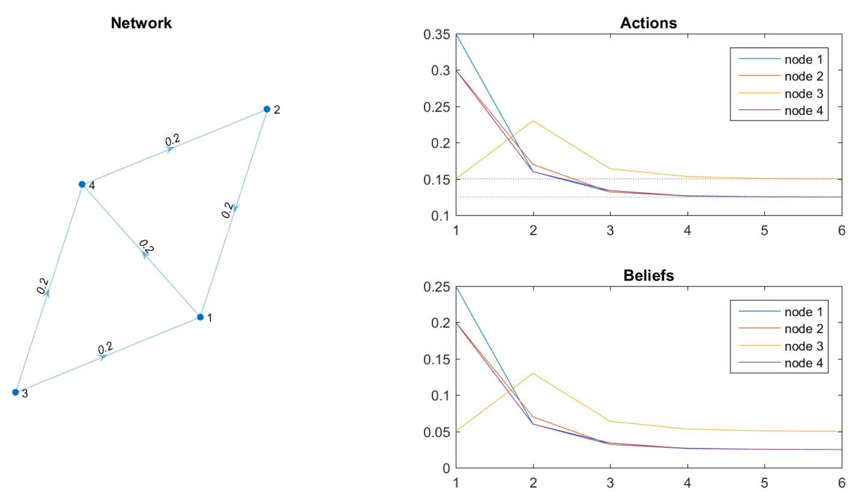

Figure 1: A case of strategic complements, with 4 players, where the learning dynamics converges to the Nash equilibrium.

We can then define stable steady states, with respect to the beliefs of the players.

Definition **5.** *A selfconfirming strategy profile* $\mathbf{a}^* \in \mathbf{A}^{SCE}_{\mathbf{Z}}$ *is* locally stable *if it is consistent with a vector of beliefs* $\hat{\mathbf{x}}$*, such that there is an* $\epsilon > 0$*, such that for every* $\hat{\mathbf{x}}'$*, with* $\|\hat{\mathbf{x}}' - \hat{\mathbf{x}}\| < \epsilon$*, the learning dynamics starting from* $\hat{\mathbf{x}}'$ *converges back to* $\hat{\mathbf{x}}$*.*

4.1 Results

Each SCE will be characterized by some active agents. So, given a strategy profile $\mathbf{a}$, let us call $I_{\mathbf{a}} \subseteq I$ the set of active players

$$I_{\mathbf{a}} = \{i \in I : r(\hat{x}_i) > 0\}.$$

In this way, given a strategy profile $\mathbf{a}$, $\mathbf{Z}_{I_{\mathbf{a}}}$ is the submatrix whose rows and columns are all and only those players that are active in $\mathbf{a}$. This allows us to characterize locally stable selfconfirming equilibria.

PROPOSITION **4.** *Consider $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$. $\mathbf{a}^*$ is locally stable if:*

- *Assumption 4 holds for matrix $\mathbf{Z}_{I_{\mathbf{a}^*}}$;*
- *for every $i \in I \backslash I_{\mathbf{a}^*}$ we have that $\alpha + \hat{x}_i < 0$.*

Moreover, we provide alternative sufficient conditions, based on linear algebra, to characterize all those subsets of I that characterize a locally stable selfconfirming strategy profile

PROPOSITION **5.** *Consider a selfconfirming strategy profile $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$. If $\mathbf{Z}_{I_{\mathbf{a}^*}}$ satisfies at least one one of the three conditions below:*

1. *it has bounded values (Assumption 1),*
2. *negative (Assumptions 3 and 4),*
3. *or it is symmetrizable–limited (Assumption 6),*

then for every $J \subseteq I_{\mathbf{a}^*}$, there exists another locally stable selfconfirming equilibrium $\mathbf{a}^{**} \in \mathbf{A}_{\mathbf{Z}}^{SCE}$ such that

1. *$\mathbf{A}_{J,\mathbf{Z}}^{NE} = \left\{\mathbf{a}_J^{NE}\right\}$, such that $\mathbf{a}_J^{NE} > \mathbf{0}_J$;*
2. *$\mathbf{a}^{**} = \left\{\mathbf{a}_J^{NE}\right\} \times \left\{\mathbf{0}_{I\backslash J}\right\}$.*

So, we already know that we can have selfconfirming equilibria that are not Nash equilibria, because some agents are inactive even if it is not a best response for them. The following example shows that we can reach them also starting with initial beliefs inducing all positive actions at the beginning of the learning dynamics. Things actually depend on the initial beliefs, and starting from different beliefs, we can get to the unique interior Nash, or to a SCE which is not Nash

EXAMPLE 2. Consider the case of 4 players, with the network matrix $\mathbf{Z} \in \{-0.2, 0, 0.2\}^{I\times I}$ shown in the left panel of Figure 2, and, for every i, $\alpha_i = 0.1$. This is a case of general externalities, that can be positive or negative. The central plots and the right plots show the learning dynamics of actions and beliefs that start from different starting beliefs. In one case (the central one) we converge to the unique Nash equilibrium of this game (the dotted lines in the top center and right plots), in the other (the right one) the learning dynamics puts, after 2 rounds, one player out from the active agents, and the remaining 3 converge to a selfconfirming equilibrium which is not Nash.

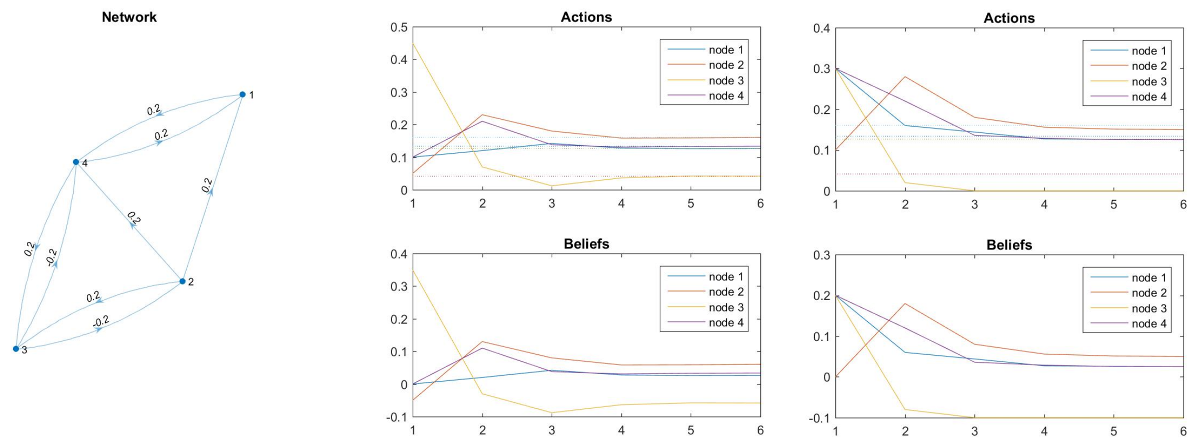

Figure 2: General strategic externalities. Starting from different beliefs on the same network (left panel), the learning process may converge to the unique Nash equilibrium (center panel) or to a SCE which is not a Nash equilibrium (right panel)

5 Local and Global externalities

We consider an extension to the case of equation (4), in which we add a global externality term with no strategic effects. For each $i \in I$, we posit an interval $Y_i = [\underline{y}_i, \bar{y}_i]$, $\beta \in \mathbb{R}$, and we consider the following aggregator[12]

$$\begin{array}{rccl} g_{i,\beta,\mathbf{Z}}: & \mathbf{A}_{-i} & \to & Y_i, \\ & \mathbf{a}_{-i} & \mapsto & \beta \sum_{j \neq i} a_j. \end{array}$$

We assume that every agent $i \in I$ knows Y_i. Then, we define $y_i = g_i(\mathbf{a}_{-i}, \beta)$. The new parametrized payoff function is

$$\begin{array}{rccl} v_i: & A_i \times X_i \times Y_i & \to & \mathbb{R} \\ & (a_i, x_i, y_i) & \mapsto & \alpha_i a_i - \frac{1}{2} a_i^2 + a_i x_i + y_i \end{array}, \tag{10}$$

[12]This aggregator g sums up the actions of all the agents in the network except agent i. We could have considered agent i as well, but we opted for this specification not to change the first order condition with respect to the case with just local externalities.

where both x_i and y_i are unknown. Deterministic conjectures for each $i \in I$ are now defined as the pair $(\hat{x}_i, \hat{y}_i) \in (X_i, Y_i)$. We recall that we keep the assumption that $f_i = v_i$. We provide now the definition of selfconfirming equilibrium for the case with global externalities.

Definition **6.** *A profile $(a_i^*, \hat{x}_i, \hat{y}_i)_{i \in I} \in \times_{i \in I} (A_i \times X_i \times Y_i)$ of actions and (shallow) deterministic conjectures is a* ***selfconfirming equilibrium at*** $\mathbf{Z}$ *and β of a linear quadratic network game with feedback and global externalities if, for each $i \in I$,*

1. (subjective rationality) $a_i^* = r_i(\hat{x}_i)$,
2. (confirmed conjecture) $f_i(a_i^*, \hat{x}_i, \hat{y}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right), g_i\left(\mathbf{a}_{-i}^*, \beta\right)\right)$.

Notice that the rationality condition is unchanged with respect to the case of just local externalities since first order conditions are not affected by the global externality term. To compare this game with the linear quadratic network game with just local externalities, we consider the case in which agents observe own payoff, so that for each $i \in I$, $f_i = u_i$. Then, we can characterize any selfconfirming equilibrium as follows:

Proposition **6.** *Fix $\mathbf{Z} \in \Theta$. Every selfconfirming equilibrium profile $\left(a_i^*, \hat{x}^i, \hat{y}^i\right)_{i \in I} \in \times_{i \in I}\left(A_i \times \mathbb{R}^2\right)$ is such that, for every $i \in I$*

1. *if $a_i^* = 0$, then $\hat{x}_i \in (-\infty, -\alpha]$, $\hat{y}_i = y_i$*
2. *if $a_i^* > 0$, then $a_i^* = \alpha + \hat{x}_i$, $\hat{y}_i = y_i + a_i^*(x_i - \hat{x}_i)$*

We discuss how the presence of the global externality term in the payoff function, changes radically the characterization of selfconfirming equilibria. Recall that players observe own realized payoff. Indeed when global externalities are considered, **observability by active players does not hold** anymore. Inactive players have correct conjectures about the global externality, but may have correct or incorrect conjectures about the local externality part, as far as they justify their being inactive. Active players, on the other hand, are not able to determine precisely the magnitude of the local vs global effects. Given any strictly positive action a_i^*, for every agent i, $(\hat{y}_i - y_i) = a_i^*(x_i - \hat{x}_i)$. Then in equilibrium if agent i overestimates (underestimates) the local externality, she must compensate this error by underestimating (overestimating) the global externality. Then, with respect to the case of just local externalities, we have that i) active agents choose a best response to a (generically) wrong conjecture about x; ii) it immediately follows that it is not possible to characterize any selfconfirming equilibrium in terms of a Nash equilibrium of the auxiliary games when just active players are considered.

5.1 Learning with Global Externalities

We now consider the learning process that originates from an adaptive updating of conjectures, as we did for the case of just local externalities. To ease the analysis we focus entirely on the case of strictly positive justifiable actions. We will obtain this with the simple assumption that $\alpha > 0$ and that all the elements in $\mathbf{Z}$ are not negative. This case, however, is a bit more complex since, at each time, there are infinitely many collections of feasible pairs $(\hat{x}_{i,t}, \hat{y}_{i,t})_{i \in I}$ compatible with adaptive learning. For every $i \in I$, and each time t, let $m^*_{i,t} = f\left(a^*_{i,t}, x_{i,t}, y_{i,t}\right)$ be the message agents receive. Then, fixing $\hat{x}_{i,t}$, $\hat{y}_{i,t}$ is uniquely determined. In details, at each time period, agent i chooses a pair $(\hat{x}_{i,t}, \hat{y}_{i,t})$, that is confirmed given the message received at the previous period. We have that $\hat{y}_{i,t} = m^*_{i,t-1} - \alpha a^*_{i,t-1} + \frac{1}{2}\left(a^*_{i,t-1}\right)^2 - a^*_{i,t-1}\hat{x}_{i,t}$. Given message $m^*_{i,t-1}$, and considering that agents perfectly recall their past actions, given $\hat{x}_{i,t}$, $\hat{y}_{i,t}$ is uniquely determined. We can just focus on the dynamics of $\hat{x}_{i,t}$. The dynamics of $\hat{x}_{i,t}$ is given by the following

$$\hat{x}_{i,t+1} = \frac{m^*_{i,t} - \hat{y}_{i,t+1}}{a^*_{i,t}} - \alpha + \frac{1}{2} a^*_{i,t} \tag{11}$$

To avoid bifurcations at each time period, we need to use simplificative assumptions. Define $c_{i,t} = \frac{\hat{x}_{i,t}}{\hat{y}_{i,t}}$. Then

Assumption **7.** *For each $i \in I$ and for each $t \in \mathbb{N}$, $c_{i,t} = v_{i,t+1} = c_i$.*

We call c_i the *perceived centrality* of player i.

From equation (11), and substituting observed payoff in the message, we get that the learning dynamics is

$$\hat{x}_{i,t+1} = x_{i,t} + \frac{y_{i,t}}{a^*_{i,t}} - \frac{\hat{y}_{i,t+1}}{a^*_{i,t}} \tag{12}$$

Substituing $c_{i,t} = \frac{\hat{x}_{i,t}}{\hat{y}_{i,t}}$ we get

$$\hat{x}_{i,t+1} = \frac{c_{i,t}}{1 + v_{i,t} a^*_{i,t}} \left(a^*_{i,t} x_{i,t} + y_{i,t}\right) \tag{13}$$

We define the *true centrality* of player i as

$$c'_{i,t} = \frac{x_{i,t}}{y_{i,t}} \quad .$$

The value of $c'_{i,t}$ is always between 0 and $\frac{\sum_{k \neq i} z_{i,j}}{\beta}$. Then, we can assume that also the perceived centrality $c_i \in \left(0, \frac{\sum_{k \neq i} z_{i,j}}{\beta}\right]$. The dynamics, then, can be written as

$$\hat{x}_{i,t+1} = c_i y_{i,t} \frac{a^*_{i,t} c'_{i,t} + 1}{a^*_{i,t} c_i + 1} \quad ,$$

which implies that the conjecture is correct only when $c_i = c'_{i,t}$.

We look at best responses $a^*_{i,t+1} = \alpha + \hat{x}_{i,t+1}$, and study existence and characterization of the system of equations given by this learning process. Recall that $y_{i,t} = \beta \sum_{j \neq i} a_{j,t}$. To find a fixed point we look at the system of n equations

$$H_i(\mathbf{a}^*, \mathbf{c}, \mathbf{Z}) = \alpha + c_i \left(\beta \sum_{j \neq i} a^*_j \right) \frac{a^*_i c'_i + 1}{a^*_i c_i + 1} - a^*_i = 0 \quad . \tag{14}$$

For comparison, we study the system of equations that provide the Nash Equilibrium of this network game, namely:

$$F_i(\mathbf{a}^*, \mathbf{Z}) = \alpha + \sum_{j \in I} z_{ij} a^*_j - a^*_i = 0 \quad . \tag{15}$$

Proposition **7.** *If the system defined by (15) admits a solution, then for each vector* $\mathbf{v}$ *of perceived centralities, also the system defined by (14) admits a solution. If we call* $\mathcal{A} \subset [\alpha, \infty)^n$ *the space of the solutions of the system, then the system implies a homeomorphism between all vectors* $\mathbf{c}$ *and* $\mathcal{A}$. *This homeomorphism respects the partial ordering between vectors in the two spaces.*

Previous result is a static results, but the homeomorphism is implied by the particular learning dynamics that we are imposing, which is based on constant belief centralities. Here below is instead a real dynamical result, that provides sufficient conditions for convergence of our learning dynamics. We impose that externalities are not too big, and that the global externality is not too big. Namely, we say that, for each player i, $0 < c_i \beta(n-1) < \sum_{k \neq i} z_{i,j} < 2$.[13]

Proposition **8.** *If, for each player* i, *we have that* $0 < c_i \beta(n-1) < \sum_{j \neq i} z_{i,j} < 2$, *then the dynamics defined by the learning process (14) always converge to its unique solution, which is stable.*

[13]It should be noted that we are not requiring that $|\sum_{j \neq i} z_{i,j}| < 1$, which would imply Assumption 4.

Appendix A A generalization. Selfconfirming equilibria in parametrized nice games with aggregators

In this section we develop a more general analysis of selfconfirming equilibria in a class of games that contains the linear-quadratic network games with feedback. To ease reading, we make this section self-contained repeating some definitions from the main text.

A **parametrized nice game with aggregators and feedback** is a structure

$$G = \left\langle I, \Theta, (A_i, \ell_i, v_i, f_i)_{i\in I} \right\rangle$$

where

- I is the finite **players set**, with cardinality $n = |I|$ and generic element i.
- $\Theta \subseteq \mathbb{R}^m$ is a *compact* **parameter space**.
- $A_i = [0, \bar{a}_i] \subseteq \mathbb{R}_+$, a *closed interval*, is the **action space** of player i with generic element $a_i \in A_i$.
- $X_i = [\underline{x}_i, \bar{x}_i] \subseteq \mathbb{R}$, a *closed interval*, is the a **space** of **payoff states** for i.
- $\ell_i : \mathbf{A}_{-i} \times \Theta \to X_i$ (where $\mathbf{A}_{-i} = \times_{j\in I\setminus\{i\}} A_j$) is a *continuous* parametrized **aggregator** of the actions of i's coplayers such that its *range* $\ell_i (\mathbf{A}_{-i} \times \Theta)$ is *connected.*[14]
- $v_i : A_i \times X_i \to \mathbb{R}$ is the **payoff (utility) function** of player i, which is *strictly quasi-concave* in a_i and *continuous,*[15] and from which we derive the **parameterized payoff function**

$$\begin{array}{rccl} u_i : & A_i \times \mathbf{A}_{-i} \times \Theta & \to & \mathbb{R}, \\ & (a_i, \mathbf{a}_{-i}, \theta) & \mapsto & v_i (a_i, \ell_i (\mathbf{a}_{-i}, \theta)). \end{array}$$

  Thus, $x_i = \ell_i (\mathbf{a}_{-i}, \theta)$ is the payoff relevant state that i has to guess in order to choose a subjectively optimal action. With this, for each $\theta \in \Theta$, $\left\langle I, (A_i, u_{i,\theta})_{i\in I} \right\rangle$ is a nice game (Moulin, 1979), and $\left\langle I, \Theta, (A_i, u_i)_{i\in I} \right\rangle$ is a parametrized nice game. We let

$$\begin{array}{rccl} r_i : & X_i & \to & A_i \\ & x_i & \mapsto & \arg\max_{a_i\in A_i} v_i (a_i, x_i) \end{array}$$

  denote the **best reply function** of player i.

[14] Since the range of each section $\ell_{i,\theta}$ must be a closed interval, we require that the union of the closed intervals $\ell_{i,\theta} (\mathbf{A_{-i}})$ $(\theta \in \Theta)$ is also an interval, which must be closed because Θ is compact and ℓ_i continuous.

[15] That is, v_i is jointly continuous in (a_i, x_i) and, for each $x_i \in [\underline{x}_i, \bar{x}_i]$, the section $v_{i,x_i} : [0, \bar{a}_i] \to \mathbb{R}$ has a unique maximizer a_i^* (that typically depends on x_i), it is strictly increasing on $[0, a_i^*]$, and it is strictly decreasing on $[a_i^*, \bar{a}_i]$. Of course, the monotonicity requirement holds vacuously when the relevant subinterval is a singleton.

- Let $M \subseteq \mathbb{R}$ be a set of "messages," $f_i : A_i \times X_i \to M$ is a **feedback function** that describes what i observes (a "message," e.g., a monetary outcome) after taking any action a_i given any payoff state x_i.[16]

On top of the formal assumptions stated above, we maintain the following *informal assumption* about players' knowledge of the game:

- Each player i *knows* v_i *and* f_i.

Unless we explicitly say otherwise, we instead do not assume that i knows θ, or function ℓ_i, or even that i understands that his payoff is affected by the actions of other players. However, since i knows the feedback function $f_i : A_i \times X_i \to M$ and the action he takes, what i infers about the payoff state x_i after he has taken action a_i and observed message m is that

$$x_i \in f_{i,a_i}^{-1}(m) := \left\{x_i' : f_i\left(a_i, x_i'\right) = m\right\}.$$

Appendix A.1 Conjectures

Definition **A.** *A* ***shallow conjecture*** *for i is a probability measure $\mu_i \in \Delta(X_i)$. A (deep)* ***conjecture*** *for i is a probability measure $\bar{\mu}_i \in \Delta(\mathbf{A}_{-i} \times \Theta)$. An action a_i^* is* ***justifiable*** *if there exists a shallow conjecture μ_i such that*

$$a_i^* \in \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i);$$

in this case we say that μ_i ***justifies*** *a_i^*. Similarly, we say that (deep) conjecture $\bar{\mu}_i \in \Delta(\mathbf{A}_{-i} \times \Theta)$* ***justifies*** *$a_i^*$ if the shallow conjecture induced by $\bar{\mu}_i$ ($\mu_i = \bar{\mu}_i \circ \ell_i^{-1} \in \Delta(X_i)$) justifies a_i^*.*

Remark **3.** *If $a_i \mapsto v_i(a_i, x_i)$ is* strictly concave *for each x_i, then also $a_i \mapsto \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i)$ is strictly concave and the map*

$$\mu_i \mapsto \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i)$$

is a continuous function.[17]

The following lemma summarizes well known results about nice games (see, e.g., Battigalli 2018) and some straightforward consequences for the more structured class of nice games with aggregators considered here:

Lemma **A.** *The best reply function $r_i : X_i \to A_i$ is continuous, hence its range $r_i(X_i)$ is a closed interval, just like X_i. Furthermore, for each given $a_i^* \in A_i$, the following are equivalent:*

[16] Here the assumption that M is a set of real numbers is without loss of generality, because the same holds for the set of payoff states X_i.

[17] When $\Delta(X_i)$ is endowed with the topology of weak convergence of measures.

- a_i^* *is justifiable,*

- $a_i^* \in r_i(X_i)$ *(that is, a_i^* is justified by a deterministic shallow conjecture),*

- *there is no a_i such that $v_i(a_i^*, x_i) < v_i(a_i, x_i)$ for all $x_i \in X_i$ (that is, a_i^* is not dominated by any other pure action).*

COROLLARY **B.** *Suppose that the aggregator ℓ_i is* onto*. Then, an action of player i is justifiable if an only if it is justified by a deep conjecture.*

Proof. The "if" part is trivial. For the "only if" part, fix a justifiable action a_i^* arbitrarily. By Lemma A, there is some $x_i \in X_i$ such that $a_i^* = r_i(x_i)$. Since the aggregator ℓ_i is onto, there is some $(\mathbf{a}_{-i}, \theta) \in \ell_i^{-1}(x_i)$ such that

$$a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}, \theta).$$

Hence a_i^* is justified the deep conjecture $\delta_{(\mathbf{a}_{-i}, \theta)}$, that is, the Dirac measure supported by $(\mathbf{a}_{-i}, \theta)$. ■

With this, from now on we restrict our attention to (shallow, or deep) *deterministic conjectures.*

Appendix A.2 Feedback properties

DEFINITION **B.** *Feedback f_i satisfies* ***observable payoffs*** *(OP) relative to v_i if there is a function $\bar{v}_i : A_i \times M \to \mathbb{R}$ such that*

$$v_i(a_i, x_i) = \bar{v}_i(a_i, f_i(a_i, x_i))$$

for all $(a_i, x_i) \in A_i \times X_i$; if the section $\bar{v}_{i,a_i}$ is injective for each $a_i \in A_i$, then we say that f_i satisfies ***just observable payoffs*** *(JOP) relative to v_i. Game G satisfies (just) observable payoffs if, for each player $i \in I$, feedback f_i satisfies (J)OP relative to v_i.*

If f_i satisfies JOP, we may assume without loss of generality that $f_i = v_i$, because, for each action a_i, the partitions of X_i induced by the preimages of v_{i,a_i} and f_{i,a_i} coincide:

REMARK **4.** *Feedback f_i satisfies JOP relative to v_i if and only if*

$$\forall a_i \in A_i, \ \left\{v_{i,a_i}^{-1}(u)\right\}_{u \in v_{i,a_i}(X_i)} = \left\{f_{i,a_i}^{-1}(m)\right\}_{m \in f_{i,a_i}(X_i)}. \tag{a}$$

Proof. (Only if) Fix $a_i \in A_i$. Since f_i satisfies JOP relative to v_i, $v_{i,a_i}(X_i) = (\bar{v}_{i,a_i} \circ f_{i,a_i})(X_i)$ (by OP), for each $u \in v_{i,a_i}(X_i)$ there is a unique message $m_{a_i,u} = \bar{v}_{i,a_i}^{-1}(u)$ (by injectivity of $\bar{v}_{i,a_i}$), and

$$\begin{aligned} v_{i,a_i}^{-1}(u) &= \{x_i \in X_i : v_i(a_i, x_i) = u\} \\ &= \{x_i \in X_i : \bar{v}_i(a_i, f_i(a_i, x_i)) = u\} \\ &= \{x_i \in X_i : f_i(a_i, x_i) = m_{a_i,u}\} = f_{i,a_i}^{-1}(m_{a_i,u}), \end{aligned}$$

which implies eq. (a).

(If) Suppose that eq. (a) holds. For every $a_i \in A_i$ and $m \in f_{i,a_i}(X_i)$ select some $\xi_i(a_i, m) \in f_{i,a_i}^{-1}(m)$. Let

$$D := \bigcup_{a_i \in A_i} \{a_i\} \times f_{i,a_i}(X_i)$$

With this,

$$\xi_i : D \to X_i$$

is a well defined function. Domain D is the set of action-message pairs for which the definition of $\bar{v}_i$ matters. Define $\bar{v}_i$ as follows:

$$\bar{v}_i(a_i, m) = \begin{cases} v_i(a_i, \xi_i(a_i, m)) & \text{if } (a_i, m) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, eq. (a) implies that

$$\forall (a_i, x_i) \in A_i \times X_i,\ \bar{v}_i(a_i, f_i(a_i, x_i)) = v_i(a_i, x_i).$$

Hence, OP holds. Furthermore, for all $a_i \in A_i$, $m', m'' \in f_{a_i}(X_i)$,

$$\begin{aligned} m' &\neq m'' \Rightarrow \xi_i(a_i, m') \neq \xi_i(a_i, m') \\ &\Rightarrow v_i(a_i, \xi_i(a_i, m')) \neq v_i(a_i, \xi_i(a_i, m'')) \\ &\Rightarrow \bar{v}_i(a_i, m') \neq \bar{v}_i(a_i, m') \end{aligned}$$

where the second implication follows from eq. (a) ($\xi_i(a_i, m')$ and $\xi_i(a_i, m')$ belong to different cells of the coincident partitions, hence yield different utilities), and the third holds by construction. Therefore, $\bar{v}_{i,a_i}$ is injective for every a_i, which means the JOP holds. ■

Definition **C.** *Feedback f_i satisfies **observability if and only if i is active** (OiffA) if section f_{i,a_i} is injective for each $a_i > 0$ and constant for $a_i = 0$. Game G satisfies **observability by active players** if OiffA holds for each i.*

Remark **5.** *If NG is* linear-quadratic *and satisfies* just observable payoffs*, then it satisfies observability by active players.*

Proof. By Remark 4 JOP implies that, for each $a_i \in A_i$,

$$\left\{v_{i,a_i}^{-1}(u)\right\}_{u \in v_{i,a_i}(X_i)} = \left\{f_{i,a_i}^{-1}(m)\right\}_{m \in f_{i,a_i}(X_i)}.$$

The linear-quadratic form of v_i implies that, for every $x_i \in X_i$,

$$v_{i,0}^{-1}(v_{i,0}(x_i)) = X_i$$

$$\forall a_i > 0, v_{i,0}^{-1}\left(v_{i,0}\left(x_i\right)\right) = \left\{x_i\right\}.$$

These equalities imply that $f_{i,0}$ is constant and f_{i,a_i} is injective for $a_i > 0$, that is, NG satisfies observability by active players. ■

DEFINITION **D.** *Feedback f_i satisfies* ***own-action independence*** *(OAI) of feedback about the state if, for all* justifiable *actions a_i^*, a_i^o and all payoff states $\hat{x}_i, x_i \in X_i$,*

$$f_i\left(a_i^*, \hat{x}_i\right) = f_i\left(a_i^*, x_i\right) \Rightarrow f_i\left(a_i^o, \hat{x}_i\right) = f_i\left(a_i^o, x_i\right).$$

Game G satisfies own-action independence of feedback about the state if, for each player $i \in I$, feedback f_i satisfies OAI.

In other words, OAI says that if player i cannot distinguish between two payoff states $\hat{x}_i$ and x_i when he chooses some given justifiable action a_i^*, then he cannot distinguish between these two states when he chooses any other justifiable action a_i^o. This is equivalent to requiring that the partitions of X_i of the form $\left\{f_{i,a_i}^{-1}\left(m\right)\right\}_{m \in f_{i,a_i}(X_i)}$ coincide across justifiable actions, i.e., across actions $a_i \in r_i\left(X_i\right)$ (see Lemma A).

The following lemma—which holds for any game, not just nice games—states that, under payoff observability and own-action independence, an action is justified by a confirmed conjecture if and only if it is a best reply to the actual payoff state:

LEMMA **C.** *If f_i satisfies* payoff observability *relative to v_i and* own-action independence *of feedback about the state, then for all $\left(a_i^*, x_i\right) \in A_i \times X_i$ the following are equivalent:*

1. *there is some $\hat{x}_i \in X_i$ such that $a_i^* \in \arg\max_{a_i \in A_i} v_i\left(a_i, \hat{x}_i\right)$ and $f_i\left(a_i^*, \hat{x}_i\right) = f_i\left(a_i^*, x_i\right)$,*

2. *$a_i^* \in \arg\max_{a_i \in A_i} v_i\left(a_i, x_i\right)$.*

Proof.(Cf. Battigalli *et al.* 2015, Battigalli 2018) It is obvious that (2) implies (1) independently of the properties of f_i. To prove that (1) implies (2), suppose that f_i satisfies OP-OAI and let $\hat{x}_i$ be such that (1) holds. Let a_i^o be a best reply to the actual state x_i. We must show that also a_i^* is a best reply to x_i. Note that both a_i^* and a_i^o are justifiable; hence, by OAI, $f_i\left(a_i^*, \hat{x}_i\right) = f_i\left(a_i^*, x_i\right)$ implies $f_i\left(a_i^o, \hat{x}_i\right) = f_i\left(a_i^o, x_i\right)$. Using OP, condition (1), and OAI as shown in the following chain of equalities and inequalities, we obtain

$$v_i\left(a_i^*, x_i\right) \overset{\text{(OP)}}{=} \bar{v}_i\left(a_i^*, f_i\left(a_i^*, x_i\right)\right) \overset{(1)}{=} \bar{v}_i\left(a_i^*, f_i\left(a_i^*, \hat{x}_i\right)\right) \overset{\text{(OP)}}{=} v_i\left(a_i^*, \hat{x}_i\right) \overset{(1)}{\geq}$$
$$v_i\left(a_i^o, \hat{x}_i\right) \overset{\text{(OP)}}{=} \bar{v}_i\left(a_i^o, f_i\left(a_i^o, \hat{x}_i\right)\right) \overset{\text{(1,OAI)}}{=} \bar{v}_i\left(a_i^o, f_i\left(a_i^o, x_i\right)\right) \overset{\text{(OP)}}{=} v_i\left(a_i^o, x_i\right).$$

Since a^o is a best reply to x_i and $v_i\left(a_i^*, x_i\right) \geq v_i\left(a_i^o, x_i\right)$, it must be the case that also a_i^* is a best reply to x_i. ■

Corollary **D.** *Suppose that G satisfies* payoff observability *and* own-action independence *of feedback about the state, then the sets of selfconfirming action profiles and Nash equilibrium action profiles coincide for each θ:*

$$\forall\theta \in \Theta,\ \mathbf{A}_{\theta}^{SCE} = \mathbf{A}_{\theta}^{NE}.$$

Proof By Remark 2, we only have to show that $\mathbf{A}_{\theta}^{SCE} \subseteq \mathbf{A}_{\theta}^{NE}$. Fix any $\mathbf{a}^* = (a_i^*)_{i\in I} \in \mathbf{A}_{\theta}^{SCE}$ and any player i. By definition of SCE, there is some $\hat{x}_i \in X_i$ such that $a_i^* \in r_i(x_i^*)$ and $f_i(a_i^*, \hat{x}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \theta\right)\right)$. By Lemma C $a_i^* \in r_i\left(\ell_i\left(\mathbf{a}_{-i}^*, \theta\right)\right)$. This holds for each i, hence $\mathbf{a}^* \in \mathbf{A}_{\theta}^{NE}$. ■

Corollary D provides sufficient conditions for the equivalence between SCE and NE. Next, we give sufficient conditions that allow a characterization of $\mathbf{A}_{\theta}^{SCE}$ by means of Nash equilibria of auxiliary games.

Appendix A.3 Equilibrium Characterization

If $a_i \in [0, \bar{a}_i]$ is interpreted as an activity level (e.g., effort) by player i, then it makes sense to say that i is **active** if $a_i > 0$ and **inactive** otherwise. Let I_0 denote the **set of players for whom being inactive is justifiable**. Note that, by Lemma A,

$$I_0 = \{i \in I : \min r_i(X_i) = 0\}.$$

Also, for each $\theta \in \Theta$ and nonempty subset of players $J \subseteq I$, let $\mathbf{A}_{J,\theta}^{NE}$ denote the set of Nash equilibria of the auxiliary game with player set J obtained by letting $a_i = 0$ for each $i \in I\backslash J$, that is,

$$\mathbf{A}_{J,\theta}^{NE} = \left\{\mathbf{a}_J^* \in \times_{j\in J} A_j : \forall j \in J, a_j^* = r_j\left(\ell_j\left(\mathbf{a}_{J\backslash\{j\}}^*, \mathbf{0}_{I\backslash J}, \theta\right)\right)\right\},$$

where $\mathbf{0}_{I\backslash J} \in \mathbb{R}^{I\backslash J}$ is the profile that assigns 0 to each $i \in I\backslash J$. If $J = \emptyset$, let $\mathbf{A}_{J,\theta}^{NE} = \emptyset$ by convention.

Lemma **E.** *Suppose that the parametrized nice game with aggregators and feedback G satisfies* observability by active players. *Then, for each θ, the set of selfconfirming action profiles is*

$$\mathbf{A}_{\theta}^{SCE} = \bigcup_{I\backslash J\subseteq I_0} \mathbf{A}_{J,\theta}^{NE} \times \left\{\mathbf{0}_{I\backslash J}\right\}.$$

Proof Let J be the set of players i such that $a_i^* > 0$. Fix $\theta \in \Theta$ arbitrarily. Let $\mathbf{a}^* \in \mathbf{A}_{\theta}^{SCE}$ and fix any $i \in I$. If $a_i^* = 0$, then 0 is justifiable for i, that is $i \in I_0$. If $a_i^* > 0$, OiifA implies that f_{i,a_i^*} is injective, that is, action a_i^* reveals the payoff state, hence the (shallow) conjecture justifying a_i^* is correct: $a_i^* = r_i\left(\ell_i\left(\mathbf{a}_{-i}^*, \theta\right)\right)$. Thus, $\mathbf{a}^* = \left(\mathbf{a}_J^*, \mathbf{a}_{I\backslash J}^*\right)$ so that $a_i^* = 0$ for each $i \in I\backslash J \subseteq I_0$, and $a_j^* = r_j\left(\ell_j\left(\mathbf{a}_J^*, \mathbf{0}_{I\backslash J}, \theta\right)\right) > 0$ for each $j \in J$. Hence,

$$\mathbf{a}^* = \left(\mathbf{a}_J^*, \mathbf{a}_{I\backslash J}^*\right) \in \mathbf{A}_{J,\theta}^{NE} \times \left\{\mathbf{0}_{I\backslash J}\right\} \text{ with } I\backslash J \subseteq I_0.$$

Let $I\backslash J \subseteq I_0$ and $\left(\mathbf{a}_J^*, \mathbf{a}_{I\backslash J}^*\right) \in \mathbf{A}_\theta^{NE} \times \left\{\mathbf{0}_{I\backslash J}\right\}$. Since G satisfies OiffA, for each $i \in I\backslash J$, any conjecture justifying $a_i^* = 0$ (any $\hat{x}_i \in r_i^{-1}(0)$) is trivially confirmed. For each $j \in J$, $a_j^* > 0$ is by assumption the best reply to the correct, hence confirmed, conjecture $x_j^* = \ell_i\left(\mathbf{a}_J^*, \mathbf{0}_{I\backslash J}, \theta\right)$. Hence, $\left(\mathbf{a}_J^*, \mathbf{a}_{I\backslash J}^*\right) = \left(\mathbf{a}_J^*, \mathbf{0}_{I\backslash J}\right) \in \mathbf{A}_\theta^{SCE}$. ■

Appendix B Interior Nash equilibria

Propositions 1 and 2 in Section 3 show that in our framework there exists an equivalence between any selfconfirming equilibrium and the Nash equilibrium of a reduced game in which only active agents are considered. Moreover, we can set any subset of agents to be inactive. We now provide some results about existence of these selfconfirming equilibria, that will be useful in proving Proposition 3 in Section 3. We first present sufficient conditions that are present in the literature for the existence of interior Nash equilibria, then we provide some original results.

In this appendix we formulate the problem as a linear algebra problem. We consider a square matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$ such that $z_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. We call $\mathbf{I}$ the identity matrix, $\lambda_{max}(\mathbf{Z})$ the maximal eigenvalue of $\mathbf{Z}$, $\rho(\mathbf{Z})$ the spectral radius of $\mathbf{Z}$ (i.e. the largest absolute value of its eigenvalues), $\mathbf{1}$ is the vector of all 1's, $\mathbf{0}$ is the vector of all 0's, and $\gg$ is the strict partial ordering between vectors (meaning that all the elements in the first vector are pairwise strictly greater than the elements in the second vector).

Proposition **F.** *If for all i, $z_{ii} = 0$, for all $i \neq j$, $z_{ij} \leq 0$, and if $\rho(\mathbf{Z}) < 1$, then $(\mathbf{I} - \mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$.*[18]

There are also results when the sign of the externalities are mixed. We recall that the matrix $\mathbf{Z}$ is symmetrizable if there exists a diagonal matrix $\mathbf{\Gamma}$ and a symmetric matrix $\mathbf{Z}_0$ such that $\mathbf{Z} = \mathbf{\Gamma}\mathbf{Z}_0$. Note that if $\mathbf{Z}$ is symmetrizable then all its eigenvalues are real. If for all i, $z_{ii} = 0$, and $\mathbf{Z}$ is symmetrizable, we define the symmetric matrix $\tilde{\mathbf{Z}}$ to be such that $\tilde{z}_{ij} = z_{ij}\sqrt{\gamma_i\gamma_j}$.

Proposition **G.** *If for all i, $z_{ii} = 0$, $\mathbf{Z}$ is symmetrizable, and if $|\lambda_{max}(\tilde{\mathbf{Z}})| < 1$, then $(\mathbf{I} - \mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$.*[19]

We provide here below an alternative condition, which does also guarantee all positive solutions.

Proposition **H.** *Consider a square matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$ such that:*

- $z_{ii} = 0$ *for all* $i \in \{1, \ldots, n\}$;
- $|z_{ij}| < \frac{1}{n}$ *for all* $i, j \in \{1, \ldots, n\}$.

[18] This is Theorem 1 in Ballester *et al.* (2006). The same result is in Appendix A in Stańczak *et al.* (2006).

[19] See Section VI of Bramoullé *et al.* (2014), generalizing Proposition 2 therein. Note that in their payoff specification externalities have a *minus* sign, while in (4) we have a *plus* sign: this is why we have a condition on the maximal eigenvalue and not on the minimal eigenvalue.

Then $(\mathbf{I}-\mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$.

Proof: Call $\mathbf{B} = (\mathbf{I}-\mathbf{Z})$. First of all, by *Gershgorin circle theorem*,[20] $\mathbf{B}$ has all eigenvalues strictly between 0 and 2, so $det(\mathbf{B}) \neq 0$.
Consider the n vectors $\mathbf{b}^1, \ldots, \mathbf{b}^n$ given by the n rows of $\mathbf{B}$, and take the hyperplane in $\mathbb{R}^n$ passing by those n points:

$$H = \{\mathbf{h} \in \mathbb{R}^n : \exists \boldsymbol{\alpha} \in \mathbb{R}^n \text{ with } \boldsymbol{\alpha}' \cdot \mathbf{1} = 1 \text{ and } \mathbf{h} = \mathbf{B}'\boldsymbol{\alpha}\} \quad .$$

Now, consider the following vector

$$\mathbf{v} = \mathbf{B}^{-1}\mathbf{1} \quad .$$

Component v_i of $\mathbf{v}$ is exactly the sum of the elements in i^{th} row of $\mathbf{B}^{-1}$. However, $\mathbf{v}$ is also a vector perpendicular to H. That is because for any $\mathbf{h} \in H$ we have

$$\begin{aligned} \mathbf{h} \cdot \mathbf{v} &= \left(\mathbf{B}'\boldsymbol{\alpha}\right)' \cdot \mathbf{B}^{-1}\mathbf{1} \\ &= \boldsymbol{\alpha}'\mathbf{1} \\ &= \sum_{i=1}^{n} \alpha_i = 1 \quad , \end{aligned}$$

which is a constant.

Now, we want to show that H does not pass thorugh the convex region of vectors with all non-postitive elements: $H \cap (-\infty, 0]^n = \emptyset$.
In fact, it is impossible to find $\boldsymbol{\alpha} \in \mathbb{R}^n$, such that $\boldsymbol{\alpha}' \cdot \mathbf{1} = 1$ and $\mathbf{B}'\boldsymbol{\alpha} \ll \mathbf{0}$.
If it was the case, by absurdum, we could take $k = \arg\max_{i \in \{1,\ldots,n\}}\{\alpha_i\}$ ($\alpha^k > 0$ because $\sum_{i=1}^n \alpha_i = 1$), and write

$$\alpha \mathbf{b}^k = \alpha_k + \sum_{j \neq k} \alpha_j b_{jk} > \alpha_k - \sum_{j \neq k} |\alpha_j||z_{jk}| > \alpha_k \left(1 - \sum_{j \neq k} |z_{jk}|\right) > 0 \quad ,$$

which would be a contradiction.

Finally, we show that if an hyperplane H satisfies $H \cap (-\infty, 0]^n = \emptyset$, then its perpendicular vector from the origin has all positive elements, and this would close the proof .
We do so by induction on n .

1. $n = 2$: This is easy to show graphically;

2. **Induction hypothesis**: Suppose it is true for $n = m - 1$;

[20] https://en.wikipedia.org/wiki/Gershgorin_circle_theorem

3. **Induction step**: In $\mathbb{R}^m$, a vector $\mathbf{v}$ from the origin is perpendicular to an hyperplane H not passing through the origin can be obtained in the following way. For each dimension $i \in \{1, \ldots, m\}$ take $V^{\neg i} = \{\mathbf{v} \in \mathbb{R}^m : v_i = 0\}$. Call $H^{\neg i}$ the intersection of H with $V^{\neg i}$, and take a vector $\mathbf{v}_{\neg i} \in V^{\neg i}$ from the origin that is perpendicular to $H^{\neg i}$. By the induction hypothesis $\mathbf{v}_{\neg i}$ has all positive elements. We can obtain the vector $\mathbf{v}$ from the origin that is perpendicular to H by rescaling each $\mathbf{v}_{\neg i}$, such that $\mathbf{v}_{\neg i}$ is the projection of $\mathbf{v}$ on $H^{\neg i}$. By construction, $\mathbf{v}$ will have all positive elements.

Notice that, if $\mathbf{Z}$ satisfies the conditions of Proposition H, then it must also hold that $|\lambda_{max}(\mathbf{Z})| < 1$, because of *Gershgorin circle theorem*. However, the condition that $|\lambda_{max}(\mathbf{Z})| < 1$ is in general not sufficient to guarantee that $(\mathbf{I} - \mathbf{Z})^{-1} \mathbf{1} \gg \mathbf{0}$.

Appendix C Proofs

Proof of Proposition 1

Proof. Since every agent is active, **state observability by active players** implies *own action independence of the feedback about the state*. Then, the result derives from Corollary D in Appendix A. ∎

Proof of Proposition 2

Proof. By Remark 5, NG satisfies observability by active players. Hence, Lemma E in Appendix A and the best reply equation (6) yield the result. ∎

Proof of Proposition 3

Proof. Condition (i), (ii) and (iii) correspond, respectively, to the conditions in Propositions H, F and G from Appendix B. ∎

Proof of Proposition 4

Proof. If for every $i \in I \backslash I_{\mathbf{a}^*}$ we have that $\alpha + \hat{x}_i < 0$, then changing their $\hat{x}_i$ such that the inequality is still strict, will not make them become active.
So, let us focus on the subset $I_{\mathbf{a}^*}$ of active agents. If we perturb locally the beliefs, we will perturb locally also their actions. Assumption 4 garantees that the discrete dynamical system defined for actions by (8) and (9) is stable. So, the variation to beliefs can always be small enough such that:
all their actions remain strictly positive;
we are in a neighborhood of $\mathbf{a}^*$ in the actions' space, such that the discrete dynamical system defined for actions by (8) and (9) converges back to $\mathbf{a}^*$. ∎

Proof of Proposition 5

Proof. When we remove elements from $J_{\mathbf{a}}$ and set them to 0, it is as if we delete corresponding rows and columns in the $\mathbf{Z}_{J_{\mathbf{a}}}$ matrix. By the Cauchy interlace theorem applied to symmetrizable matrices (see Kouachi 2016) we know that the eigenvalues of the new matrix are between the minimal and the maximal eigenvalues of the old matrix. ■

Proof of Proposition 6

A selfconfirming equilibrium is such that, for all $i \in I$, rationality implies

$$a_i^* = \max\{0, \alpha_i + \hat{x}_i\} \quad .$$

Each agent then thinks that

$$m^* = \alpha_i a_i^* - \frac{1}{2}(a_i^*)^2 + a_i^* \hat{x}_i + \hat{y}_i \quad ,$$

so that

$$\hat{y}_i = m^* - \alpha_i a_i^* + \frac{1}{2}(a_i^*)^2 - a_i^* \hat{x}_i \quad .$$

Substituting the expression of the true payoff function

$$m^* = \alpha_i a_i^* - \frac{1}{2}(a_i^*)^2 + a_i^* x_i + y_i$$

into it, we get the dependence between $\hat{y}_i$ and $\hat{x}_i$:

$$\hat{y}_i = y_i + a_i^* (x_i - \hat{x}_i) \quad .$$

The first and second items in the proposition are derived, respectively, if $a_i^* = 0$ or $a_i^* > 0$.

Proof of Proposition 7

Proof. First, we derive some properties. Each equation in the system given by (14) can be written also as a parabola $b_1 a_i^2 + b_2 a_i + b_3 = 0$, in the following way

$$\begin{aligned} H_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) \quad = \quad & \underbrace{c_i}_{\equiv b_1} a_i^2 + \underbrace{\left(1 - \alpha c_i - c_i \left(\sum_{j \in I} z_{ij} a_{j,t}\right)\right)}_{\equiv b_2} a_i \\ & - \underbrace{\left(1 + c_i \left(\beta \sum_{j \neq i} a_{j,t}\right)\right)}_{\equiv b_3} = 0 \quad . \end{aligned} \qquad \text{(b)}$$

So, the solution a_i^* to $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) = 0$ lies in the right–arm of an upward parabola, where $\frac{d\ell_i}{da_i}\Big|_{a_i = a_i^*} > 0$. With respect to c_i, each $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is a linear equation.
Note also that each a_i is bounded in the interval

$$\alpha < a_i < \alpha + \left(\sum_{j \in N_i} z_{ij} a_j\right) + \beta \frac{\sum_{k \neq i} a_k}{a_i} \quad .$$

Considering that a_i^* is increasing in b_3 and decreasing in b_2, it is easy to see that each a_i^* increases in each a_j, with $j \neq i$.

Second, we show that there is a homeomorphism. There is a continuous function defined from each $\mathbf{c} \in [0,1]^n$ to an element $\mathbf{a} \in \mathcal{A}$, that is because

- either $c_i = 0$ and then $a_i^* = \alpha$;
- or $c_i > 0$ and then each a_i^* is continuously increasing in each x_j with $j \neq i$.

$$\lim_{c_i \to 0} a_i^* = \alpha \quad .$$

a_i^* is bounded above by

$$\alpha + \left(\sum_{j \in N_i} z_{ij} a_j\right) + \beta \frac{\sum_{j \neq i} a_j}{a_i^*} \quad .$$

Since the system defined by (15) admits a solution, also this system has a finite solution.

This function is one–to–one and invertible, because for each $\mathbf{a} \in \mathcal{A}$, we obtain a unique vector $\mathbf{c} \in [0,1]^n$, and since we obtain it from a linear system of equations, also the inverse function from $\mathcal{A}$ to $[0,1]^n$ is continuous.

To analyze the relation between $\mathbf{a}^*$ and $\mathbf{c}$, we can apply the implicit function theorem to $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$.
We can compute

$$\frac{dF_i}{dc_i} = \frac{\beta \sum_{j \neq i} a_{j,t}}{(a_i c_i + 1)^2}$$

Now, since

$$\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) = -(a_i c_i + 1) F_i(\mathbf{x}, \mathbf{c}) \quad ,$$

we have that $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$, with respect to a_i, has the same zeros as $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$, and that, for each a_i, $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is negative if and only if $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is positive. As they are both continuous functions, this means that since $\frac{d\ell_i}{da_i}\Big|_{a_i = a_i^*} > 0$, we have $\frac{dF_i}{da_i}\Big|_{a_i = a_i^*} < 0$. So, we obtain that

$$\frac{da_i}{dv_i}\Bigg|_{a_i = a_i^*} = -\left.\frac{\partial F_i / \partial c_i}{\partial F_i / \partial a_i}\right|_{a_i = a_i^*} > 0 \quad . \tag{c}$$

This shows that a_i^* is increasing with v_i, and the other way round. ∎

Proof of Proposition 8

Proof. We consider the system (14)

$$F_i(\mathbf{a},\mathbf{v},\mathbf{Z}) = \alpha + c_i \left(\beta \sum_{j\neq i} a_{j,t}\right) \frac{a_i c_i' + 1}{a_i c_i + 1} - a_i = 0 \quad ,$$

with $c'_{i,t} = \frac{\sum_{j\in I} z_{ij} a_{j,t}}{\beta \sum_{j\neq i} a_{j,t}}$. We can compute its Jacobian, with respect to $\mathbf{a}$, and check that each row of the Jacobian sum to less than 1, so that the process is always a contraction. The Jacobian J is such that:

$$\begin{cases} J_{ij} &= \frac{v_i}{a_i c_i + 1}\left(\beta + a_i z_{ij}\right) \\ J_{ii} &= c_i \left(\beta \sum_{j\neq i} a_j\right)\left(\frac{c_i'}{a_i c_i + 1} - c_i \frac{a_i c_i' + 1}{(a_i c_i + 1)^2}\right) - 1 \end{cases}$$

The sum of each row of the Jacobian is

$$\sum_j J_{ij} = \frac{c_i}{a_i c_i + 1}\left(\beta\left(\sum_{j\neq i} a_j\right)\left(c_i' - c_i \frac{a_i c_i' + 1}{a_i c_i + 1}\right) + a_i \left(\sum_{j\neq i} z_{i,j}\right) + \beta(n-1)\right) - 1 \tag{d}$$

Let us analyze expression (d) with respect to a_i, for any $a_i \geq 0$.
As $a_i \to \infty$, we have that expression (d) is equal to

$$\sum_{j\neq i} z_{i,j} - 1 \quad , \tag{e}$$

whose absolute value is less than one by assumption.
If $a_i \to 0$, espression (d) becomes

$$c_i \beta \left(\left(\sum_{j\neq i} a_j\right)\left(c_i' - c_i\right) + (n-1)\right) - 1 \quad . \tag{f}$$

An interior maximum or minimum of the numerical expression (d), with respect to a_i, must satisfy first order condition

$$\begin{aligned} -\left(\frac{c_i}{a_i c_i + 1}\right)^2 \left(\beta\left(\sum_{j\neq i} a_j\right)\left(c_i' - c_i \frac{a_i c_i' + 1}{a_i c_i + 1}\right) + a_i \left(\sum_{j\neq i} z_{i,j}\right) + \beta(n-1)\right) & \\ + \frac{c_i}{a_i c_i + 1}\left(\beta\left(\sum_{j\neq i} a_j\right)\left(\frac{c_i}{a_i c_i + 1}\right)\left(c_i' - c_i \frac{a_i c_i' + 1}{a_i c_i + 1}\right) + \left(\sum_{j\neq i} z_{i,j}\right)\right) &= 0 \end{aligned}$$

Last expression can be simplified and results in

$$v_i \beta (n-1) = \sum_{j\neq i} z_{i,j} \quad ,$$

which is endependent on a_i. So, the only candidates for being minima or maxima for espression (d) are its value in the extrema, namely (e) and (f).

Also, the sign of the first derivative of (d) with respect to a_i is equal to the sign of $\sum_{j \neq i} z_{i,j} - c_i \beta(n-1)$. So, if $c_i \beta(n-1) < \sum_{j \neq i} z_{i,j}$ we have that (d) is strictly increasing in a_i, and then (e) is strictly greater than (f).

The value of (e) is between -1 and 1, by assumption, because $0 < \sum_{j \neq i} z_{i,j} < 2$.

The quantity in (f) is minimized by $v_i \to 0$; and $c'_i \to 0$. In this case (f) goes to -1 from the right, and for any $c_i > 0$ it will be greater than -1. This complete the proof, because we have shown that any row of the Jacobian J sums to a number between -1 an 1. ■

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