Looking for the stars: Estimating the natural rate of interest

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Preliminary version; this version: October 30, 2018

Abstract

Natural rate of interest or r-star and the natural rate of output growth are important policy benchmarks widely used by central banks to determine the stance of an economy. It is well recognized that r-star, linearly related to the natural rate of output growth within the New Keynesian framework, is subject to low-frequency fluctuations. To track its evolution over time, we propose an unobserved components model with similar cycles based on the work of Holston et al. (2017). Our model takes an estimate of the time-varying natural rate of output growth as input via a first-stage model based on a first-difference version of Okun's law with time-varying parameters. In the second-stage, the full model is estimated using Kalman filter. We also show that the similar cycles imply a Taylor rule and a hybrid New Keynesian Phillips curve. For US, EA and UK, our estimates suggest that the decline of natural rate of output growth started from the 1960s, while r-star for US and EA started to fall from 1985. R-star of UK started low during 1960s, but rose and stayed relatively high in the 80s until a big drop took place during the GFC.

Keywords: *Natural rate of interest; Potential growth rate; Trend growth; Secular stagnation; Phillips curve; Monetary policy rules; Unobserved components models; Similar cycles; Kalman filter*

JEL Classification: C32; E43; E52; O40

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1 Introduction

There has been a lively debate over the past decade on whether the natural rate of interest, or the so-called r-star, in developed economies is around zero. The century-old notion of r-star originates from Wicksell (1898) who defines it as the rate of bank loans that neither stimulates nor curbs commodity prices. R-star has re-emerged as a monetary policy benchmark and started gaining popularity since the beginning of 21-st century and nowadays is widely used in central banks. Following literature, we define the natural rate of interest as the rate that is compatible with an economy growing at its potential giving stable maximum employment and output in an environment of stable inflation. Accordingly, the corresponding output growth in such an environment is the natural rate of output growth, or potential output growth rate[1].

In this paper, we propose a model that builds on the definition given above for estimating the unobserved natural rate of interest and output growth. As a result, we also obtain two coincidental indicators. One is the real interest rate gap which is the deviation of real interest rate from its natural rate and measures the monetary stance; that is, an accommodative monetary policy leads the real rate to exceed the r-star. The output gap is defined similarly.

Papers by Laubach and Williams (2003) and Holston et al. (2017) set an important framework for estimating the r-star. Importantly, they argue that central banks cannot reply on r-star or the real interest rate gap exclusively when making monetary policy; because there are gradual shifts in both the natural rate of output growth and natural rate of interest. A simple equilibrium relationship suggests that the gradual shifts of r-star is driven by that of potential output growth rate and by the time-variation of factors including time preference and risk aversion of economic agents. Their contributions can be seen as relating the r-star to economic growth. While Laubach and Williams (2003) initially studies only US, Holston et al. (2017) considers similar models for three more economies. Within this strand of literature, Garnier and Wilhelmsen (2005) extend analysis to EA and document some modifications to the model of Laubach and Williams (2003). Fries et al. (2016) consider a mixed frequency extension for four European countries, and find a significant drop in r-star during the GFC. McCririck et al. (2017) embrace a Bayesian approach for estimating the model of Holston et al. (2017) using Australian data, and link the drop in r-star to the widening gaps between policy rate and market interest rates. All these researches find that the uncertainty around the estimate of r-star and the potential output growth rate is quite large, emphasizing that cautions should be made when one makes the claim about recent

[1]We use natural rate of output growth, potential output growth rate and trend growth rate interchangeably in this paper.

value of r-star. This is also noted by Matthew and Justin (2017) and Lewis and Vazquez-Grande (2017). Another strand of literature studying the natural rate is based on dynamic stochastic general equilibrium (DSGE) models with nominal rigidities. Lombardi and Sgherri (2007) find it critical to account for time-variation of the underlying productivity and inflation trend to ensure consistency between the estimated r-star and the evolution of economy. Johannsen and Mertens (2016) model the effect of zero lower bound on nominal interest rates and find the drop in natural rate is less profound if one takes stochastic volatility into account. Del Negro et al. (2017) build a comprehensive DSGE model with financial frictions which reconciles findings from a simple VAR model with local mean, and use convenience bond yield to explain the changes in inflation expectation and the drop in natural rate. Finally, Gerali and Neri (2017) provides evidence on the difference between drivers of the natural rate in US and EA, where the former is mainly related to technology and investment shocks while the latter to risk premium shocks.

Our paper follows the framework of Holston et al. (2017) and makes several modifications by noting some important pitfalls if one fits their model directly to economies other than the US. Firstly, we pin down the potential output growth rate by making the link between observed growth rate of output and unemployment explicit via a first-difference version of Okun's law with time-varying parameters. Secondly, we model the gap variables, i.e. deviations from natural rate, through a similar cycles model. Harvey (2011) uses similar cycles to model the interaction between output gap and inflation gap and finds the implied Phillips curve provides better fit than a regression specification for US data. Our model extends to tri-variate stochastic similar cycles. Furthermore, our model also takes into account the reaction function of central banks via a Taylor rule, which is ignored in Holston et al. (2017). Our econometric model takes two-stage estimation and is able to reconcile some unintuitive findings from previous literature and perform robustly across different economies. Empirically, we find that for US and EA, r-star starts to drop from 1985 while the natural rate of output growth starts to fall from the initial sample period. For UK, r-star starts low in the 1960s and rises up until the onset of the GFC.

The paper is organized as follows. Section 2 discusses the model of Holston et al. (2017) and introduces our model with detailed two-stage estimation procedure. Section 3 gives estimation results from both stages, followed some robustness checks in terms of different model specification. We conclude and comment on future research in Section 4.

2 Empirical methodology

The natural rate of interest is a long-run equilibrium or flexible price steady-state concept in the DSGE literature. For example, the recent research by Barsky et al. (2014) and Cúrdia et al. (2015) define the natural rates of macroeconomic variables as the long-run rate when an economy is growing at its potential. Under this framework, the natural rate of interest is such that the growth rate of output and unemployment rate are at their natural rates in an environment with stable inflation. However, it is clear that although equilibrium is termed long-run, natural rates may be subject to low-frequency fluctuations due to advancement of technology and changing time preference of the representative agent that are difficult to detect (Stock and Watson, 1998). Holston et al. (2017) thus develop a New Keynesian modeling framework (the HLW model hereafter) based on a Phillips curve and an intertemporal IS curve to describe the stochastic driving forces behind output gap and real interest rate gap but allows for low-frequency gradual shifts in the potential growth rate of output and the natural rate of interest. Empirically, the HLW does not seem to adequately capture output gap and real interest rate gap compared with estimates produced by various institutions. Before we introduce our model and estimation procedure, the next section discusses the HLW model and some of its limitations.

2.1 The HLW model and discussions

On a balanced growth path, the intertemperal utility maximization by representative agent with CES preference in standard monetary DSGE models implies a constant inflation steady-state that links the equilibrium real interest rate r^* with the equilibrium per capita consumption growth g_c via

$$r^* = \frac{1}{\alpha} g_c + z, \tag{1}$$

where α is the risk aversion or the intertemporal elasticity in consumption, and z is the rate of time preference that is inversely related to discount rate. Off equilibrium, the natural rate of interest is time-varying in response to shifts in the right-hand side variables of equation (1). Based on this link, the HLW model assumes a law of motion for the natural rate given by

$$r_t^* = g_t^* + z_t, \tag{2}$$

where g_t^* is the potential growth rate of output and z_t is a stochastic process capturing fluctuations in other determinants of r_t^*[2]. The HLW model imposes a unity risk aversion, i.e. $\alpha = 1$.

The econometric treatment of the rest of the HLW model is given within a New Keynesian framework (e.g. Woodford, 2001 and Galí, 2015) with a Phillips curve identifying the output gap, i.e. the deviation of output from its potential. The latter is also affected by the real interest rate gap $\psi_{r,t} = r_t - r_t^*$ where r_t is the real interest rate. In particular, Holston et al. (2017) estimate the following equations:

$$\begin{aligned} &\text{IS curve:} \qquad && \psi_{y,t+1} = a_1\psi_{y,t} + a_2\psi_{y,t-1} + \frac{a_y}{2}(\psi_{r,t} + \psi_{r,t-1}) + \epsilon_{\psi_y,t} \\ &\text{Phillips curve:} && \pi_{t+1} = b_\pi \pi_t + (1-b_\pi)\bar{\pi}_{t-1} + b_y\psi_{y,t} + \epsilon_{\pi,t}, \end{aligned} \tag{3}$$

where the output gap $\psi_{y,t} = y_t - y_t^*$; y_t and y_t^* are 100 times the logarithm of real GDP the potential rate of output, respectively. π_t denotes the annulized core CPI inflation, with $\bar{\pi}_t = \frac{1}{3}\sum_{i=0}^{2}\pi_{t-i}$. The error terms $\epsilon_{\psi_y,t}$ and $\epsilon_{\pi,t}$ capture transitory shocks to output and inflation. Low-frequency shifts in r_t^* is modeled via (2) with z_t being a random walk. g_t^* is a random walk that drives an integrated random walk of order 2 for the potential output growth rate y_t^*, namely

$$\begin{aligned} y_{t+1}^* &= y_t^* + g_t^* + \epsilon_{y^*,t}, \\ g_{t+1}^* &= g_t^* + \epsilon_{g^*,t}. \end{aligned} \tag{4}$$

The measurement equation in the HLW model decompose the output and real rate as

$$\begin{aligned} y_t &= y_t^* + \psi_{y,t}, \\ r_t &= r_t^* + \psi_{r,t}. \end{aligned} \tag{5}$$

The star variables are the nonstationary local mean of the above system and they are of primary interest because they serve as policy benchmarks. The stationary gap variables are important as well because they are coincidental indicators the reflect the stance of the economy. Equation (2) makes it clear that the potential or trend growth rate of output g_t^* is a major source of low-frequency time-variation of r_t^*, an argument supported by (1) and used extensively at institutions (see, e.g. Laubach and Williams, 2003, Garnier and Wilhelmsen, 2005, Fries et al., 2016, Lewis and Vazquez-Grande, 2017 and McCririck et al., 2017).

[2]For example, theoretical researches by Woodford (2011) and Hamilton et al. (2016) show that changes in the time-preference of consumers may lead to fluctuations in the equilibrium real rate; Laubach (2009) shows the impact of budget deficits and debt on the interest rate; Bernanke et al. (2005) provides evidence of the impact of global supply of savings on domestic long-term interest rate; Chatterjee (2016) documents the global factor affecting nominal rate due to coordinated monetary policy across developed economies.

The HLW model uses the median unbiased estimator of Stock and Watson (1998) to pin down the variation of the potential growth rate g_t^* and that of the r-star specific component z_t[3]. One pitfall of the HLW model estimation is that when deciding the variation of g_t^*, HLW apply the median unbiased estimator based on a "handicapped" version of the model which discards the IS curve. As a result, the variation of g_t^* is incorrectly calibrated, as thus is that of z_t and r_t^*[4]. It turns out these two signal-to-noise ratios are highly consequential on the final estimates of the model (see Garnier and Wilhelmsen, 2005 and Matthew and Justin, 2017 for a discussion). From the comparison between our estimates and theirs in section 3, it is likely that the variation of two stars in the HLW model are biased downward. Secondly, the HLW model fits US data quite sensibly, yet its fit to EA and UK data is problematic. The EA output gap shows a prolonged stagnation between 1980 and 2000, which is in stark contrast to institutional estimates. The UK real interest rate gap attains very negative values between 1975 and 1980, down to lower than -10% followed by a 30-year long positive real rate regime, which calls for cautious take on this result. Some possible explanations for these puzzling results can be attributed to the nearly fixed starting values used by the HLW model for unobserved components; highly consequential use of 4-quarter moving average measure of ex-ante inflation expectations; and the lack of information content in the data to pin down the output gap. We introduce our model in the next section with remedies to all these concerns.

2.2 Implementation of two-stage estimation

Our model builds on the HLW model that links r-star to the potential growth rate and emphasizes that one source of gradual changes in r-star comes from the changes in potential growth rate of an economy. In other words, the two unobservables: potential output y_t^* and real interest rate r_t are cointegrated due to the common random walk g_t^*. One pathology of the HLW model, especially when fitted to EA and UK, comes from the weak identification of g_t^* and z_t[5]. To better estimate these two components, we isolate them and propose a two-stage procedure by firstly estimating g_t^* following the model of Li and Mendieta-Muñoz (2018), and secondly estimating an

[3]As Stock and Watson (1998) noted in linear state space models, due to small sample size, maximum likelihood estimation tends to underestimate the innovation variance, and thus variation for a nonstationary unobserved process. This phenomenon is well documented in macroeconometric studies. If estimated freely, g_t^* and z_t in the HLW model are found to be a deterministic trend and a constant, respectively.

[4]In particular, they propose a three-step estimation method where they firstly use the model without the IS curve to determine the signal-to-noise ratio $\lambda_{g^*} = \sigma_{g^*}/\sigma_{y^*}$ and secondly determine $\lambda_z = \sigma_z/\sigma_{\psi_y}$. In the third step, they plug in λ_{g^*} and λ_z into the model and estimate other parameters and unobserved processes.

[5]They are identified in the model, but information about their variation may be weak in the data due to, for example, measurement errors.

unobserved components model with similar cycles and a model-consistent measure of inflation expectation.

2.2.1 The trend growth rate

As in (5), the output y_t can be written as the sum of a time-varying local mean y_t^* and a stationary cycle $\psi_{y,t}$. It follows

$$y_t = y_t^* + stationary.$$

The potential output y_t^* is an integrated random walk of order 2 as in (4). This means by taking difference, we have

$$\Delta y_t = g_t^* + stationary, \tag{6}$$

where g_t^* is an integrated random walk of order 1. This is a reduced form model which enables us to directly estimate g_t^*. Let $Y_t = \exp y_t$ denote the actual output in level. We have the identity

$$Y_t = \frac{Y_t}{H_t}\frac{H_t}{N_t}\frac{N_t}{L_t}L_t = P_t Q_t E_t L_t,$$

where H_t, N_t, L_t represent hours worked, total employment, and labor force, respectively. Therefore, $Y_t/H_t = P_t$, $H_t/N_t = Q_t$ and $N_t/L_t = E_t$ indicate labor productivity, hours worked per worker, and the employment rate, respectively. Taking the first difference of logarithm in the above equation, we have

$$\Delta y_t = p_t + q_t + e_t + l_t.$$

This means that the growth rate of output Δy_t is the sum of growth rates of labor productivity p_t, hours worked per worker q_t, the employment rate e_t and the labor force l_t.

When an economy is at its potential, the growth rate should maintain a constant employment rate E_t, or $e_t = 0$[6]. Also, that goods market clear indicates the growth rate of supply of goods equals that of demand, i.e. $\Delta y_{S,t} = \Delta y_{D,t}$, and any disequilibrium in the goods market is thus captured by the growth rate of employment, i.e. $e_t = \Delta y_{S,t} - \Delta y_{D,t}$. So a necessary condition for an economy at its potential is $\Delta y_t = p_t + q_t + l_t$. Let $u_t = 1 - E_t$ denote the unemployment

[6] Alternatively, the potential gorwth rate should be such that the employment rate E_t is at its potential. This is to say $1 - E_t$ should be the natural rate of unemployment, or NAIRU. In literature, NAIRU is a low-frequency nonstationary stochastic process, meaning that e_t is an innovation term of limited variation which can be assumed to be accommodated by the transitory shock $\epsilon_{\Delta y,t}$ in (7).

rate. Using the fact $\Delta u_t = -E_{t-1}(E_t/E_{t-1} - 1) = -E_{t-1}e_t$, we can easily show that

$$\Delta y_{D,t} = \Delta y_{S,t} - \frac{1}{E_{t-1}}\Delta u_t.$$

This gives rise to a time-varying parameter model (TVPM) if one assumes the discrepancy between the left- and right-hand side of the above equation is due to transitory shocks $\epsilon_{\Delta y,t}$ hitting the system. From (6) and the above equation, we can thus write

$$\Delta y_t = g_t^* + O_t \Delta u_t + \epsilon_{\Delta y,t}. \tag{7}$$

Equation (7) is a first-difference version of Okun's law with time-varying Okun coefficient O_t which measures the inverse relationship between changes in the unemployment rate and growth rate of output. The variation in g_t^* is captured by $p_t + q_t + l_t$, which measures the long-run growth rate of labor productivity $p_t + q_t$ and of the labor force l_t, that are free from aggregate demand fluctuations[7].

Since g_t^* is an integrated random walk of order 1, it can be modeled by a smooth local level model with a stochastic drift (Harvey, 1990). Assuming O_t is also a gradual shift that follows a random walk, the full model thus reads

$$\begin{aligned}
\Delta y_t &= g_t^* + O_t \Delta u_t + \epsilon_{\Delta y,t}, & \epsilon_{\Delta y,t} &\sim N(0, \sigma^2_{\Delta y}),\\
g_{t+1}^* &= g_t^* + \mu_t, & &\\
\mu_{t+1} &= \mu_t + \epsilon_{\mu,t}, & \epsilon_{\mu,t} &\sim N(0, \sigma^2_{\mu}),\\
O_{t+1} &= O_t + \epsilon_{O,t}, & \epsilon_{O,t} &\sim N(0, \sigma^2_{O}).
\end{aligned} \tag{8}$$

Due to nonstationarity, we use diffuse initialization for g_1^*, μ_1 and O_1 (Koopman, 1997). One can specify an agnostic moving average dynamics for $\epsilon_{\Delta y,t}$ with stochastic volatility as in Li and Mendieta-Muñoz (2018) to mitigate possible error autocorrelation and heterskedasticity.

Importantly, model (8) may suffer from endogeneity problem due to possible correlation between Δu_t and $\epsilon_{\Delta_y,t}$. Kim (2006) shows that in such a case, maximum likelihood estimation of the TVPM via Kalman filter leads to invalid inference. To tackle this, Kim (2006) proposes a Heckman-type two-step bias correction procedure. Suppose that we have a m-dimensional

[7] Li and Mendieta-Muñoz (2018) argue that, on a balanced growth path with maximum employment, g_t^* serves as a "threshold growth rate" that equals the sum of labor force and productivity growth (Klump et al., 2008). If $\Delta u_t = 0$, g_t^* represents a natural or long-run output growth rate since it is the minimum level of output required to reduce u_t given labor force and productivity growth.

vector of instruments z_t for all t; and that there is a standard TVPM that we can use to project Δu_t onto the space spanned by z_t, i.e.

$$\Delta u_t = z_t'\gamma_t + \epsilon_{\Delta u,t}, \qquad \epsilon_{\Delta u,t} \sim N(0, \sigma^2_{\Delta u}), \tag{9}$$

where γ_t is $m \times 1$ vector of time-varying coefficients where each component $\gamma_{i,t}$ follows a random walk with innovation variance σ_{γ_i}, $i = 1, ..., m$. Kalman filter allows for decomposition of Δu_t into a predicted value $E(\Delta u_t | \mathcal{F}_{t-1})$ and an orthogonal prediction error $\hat{\epsilon}_{\Delta u,t} = \sigma_{\Delta u}\hat{\epsilon}^*_{\Delta u,t}$ where $\mathcal{F}_{t-1}$ is the information set available at $t-1$ and $\hat{\epsilon}^*_{\Delta u,t}$ is standard normal.

If we assume that $E(\hat{\epsilon}^*_{\Delta u,t}\epsilon_{\Delta y,t}) = \rho\sigma_{\Delta y}$, we can write

$$\epsilon_{\Delta y,t} = \rho\sigma_{\Delta y}\hat{\epsilon}^*_{\Delta u,t} + \epsilon^*_{\Delta y,t}, \qquad \epsilon^*_{\Delta y,t} \sim N(0, (1-\rho^2)\sigma^2_{\Delta y}).$$

Substituting the above equation into the measurement equation of the TVPM (8), we have

$$\Delta y_t = g^*_t + O_t\Delta u_t + \rho\sigma_{\Delta y}\hat{\epsilon}^*_{\Delta u,t} + \epsilon^*_{\Delta y,t}. \tag{10}$$

The standardized prediction errors $\hat{\epsilon}^*_{\Delta u,t}$ in (10) augment the measurement equation in (8) as bias correction terms similar to Heckman (1976)'s two-step procedure for sample selection. A t-test for the maximum likelihood estimate of ρ can be used to check the necessity of this procedure. Lastly, to estimate the model using the two-step bias correction procedure, we need to deal with the "limited variation" of g^*_t or the so-called "pile-up" problem documented by Stock and Watson (1998). We firstly estimate the TVPM treating g^*_t as a constant and apply the exponential Wald statistic for structural breaks to determine the signal-to-noise ratio (SNR) $\lambda = \frac{\sigma_\mu}{\sigma_{\Delta y}}$; and secondly re-estimate the model by imposing $\sigma^2_\mu \equiv \lambda^2\sigma^2_{\Delta y}$. In this first stage, we estimate the parameter vector $\theta_1 = (\sigma_{\Delta u}, \sigma_{\gamma_1}, ..., \sigma_{\gamma_m}, \rho, \sigma_{\Delta y}, \sigma_O, \lambda)'$.

2.2.2 Unobserved components model with similar cycles

The trend growth rate g^*_t is estimated using a reduced form model with diffuse initialization. We find this appealing compared to the estimation of the HLW model which initializes the state vector from HP filter. In this section, we introduce our model and and its estimation.

Harvey (2011) finds that the US Phillips curve can be well modeled by a bivariate unobserved components model with similar cycles. We follow his approach by introducing a trivariate similar cycles model for both the Phillips curve and IS curve in the spirit of (3). In particular, denoting

$\psi_t = (\psi_{y,t}, \psi_{r,t}, \psi_{\pi,t})'$ and an auxiliary cycle vector $\tilde{\psi}_t$, we specify the following model for the gap variables,

$$\begin{bmatrix} \psi_{t+1} \\ \tilde{\psi}_{t+1} \end{bmatrix} = \left\{ \varphi \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \otimes I_3 \right\} \begin{bmatrix} \psi_t \\ \tilde{\psi}_t \end{bmatrix} + \begin{bmatrix} \epsilon_{\psi,t} \\ \epsilon_{\tilde{\psi},t} \end{bmatrix}, \quad \epsilon_{\psi,t}, \epsilon_{\tilde{\psi},t} \sim N(0, \Sigma_\psi), \tag{11}$$

where $\psi_{y,t}$ is the output gap; $\psi_{r,t}$ is the real interest rate gap; $\psi_{\pi,t}$ is the inflation gap; $\varphi \in (-1, 1)$ is a damping factor ensuring stationarity of stochastic cycles; $\omega \in (0, 2\pi)$ is the angular frequency of the cycles such that $\tau = 2\pi/\omega$ is the cycle period; and the stochastic forces driving the cycles are such that $E(\epsilon_{\psi,t}\epsilon'_{\tilde{\psi},t}) = 0$ with covariance matrix

$$\Sigma_\psi = \begin{bmatrix} \sigma^2_{\psi_y} & \rho_{yr}\sigma_{\psi_y}\sigma_{\psi_r} & \rho_{\pi y}\sigma_{\psi_\pi}\sigma_{\psi_y} \\ & \sigma^2_{\psi_r} & \rho_{r\pi}\sigma_{\psi_r}\sigma_{\psi_\pi} \\ & & \sigma^2_{\psi_\pi} \end{bmatrix}. \tag{12}$$

$\rho_{\pi y}$ and ρ_{yr} reflect the Phillips curve and IS curve, respectively. $\rho_{r\pi}$ captures the Taylor principle that models the reaction function of central banks. It is easy to see that the Taylor rule implied by the similar cycles is given by

$$i_t = \pi^e_t + r^*_t + \beta^\psi_{TA}\psi_{\pi,t} + \gamma^\psi_{TA}\psi_{y,t} + \epsilon_{i,t}.$$

The regression equation can be rearranged such that the left-hand side variable is the real interest gap $\psi_{r,t} = r_t - \pi^e_t - r^*_t$. β^ψ_{TA} and γ^ψ_{TA} are the Taylor coefficients; $\epsilon_{i,t}$ is a monetary shock. This specification is in line with the original definition of Taylor rule in Taylor (1993) and Taylor (1999) where Taylor suggests $\beta^\psi_{TA} \approx 0.5$. One can calculate $\text{Cov}(\psi_{r,t}, \psi_{y,t})$ and $\text{Cov}(\psi_{r,t}, \psi_{\pi,t})$ and solve for β^ψ_{TA} and γ^ψ_{TA}. For example, β^ψ_{TA} is given by

$$\beta^\psi_{TA} = \frac{\text{Var}(\psi_{y,t})\text{Cov}(\psi_{r,t}, \psi_{\pi,t}) - \text{Cov}(\psi_{r,t}, \psi_{y,t})\text{Cov}(\psi_{\pi,t}, \psi_{y,t})}{\text{Var}(\psi_{\pi,t})\text{Var}(\psi_{y,t}) - \text{Cov}(\psi_{\pi,t}, \psi_{y,t})^2} = \frac{\sigma_{\psi_r}(\rho_{r\pi} - \rho_{yr}\rho_{\pi y})}{\sigma_{\psi_\pi}(1 - \rho^2_{\pi y})}, \tag{13}$$

which captures the reaction of real interest rate to the inflation cycle. Notice that the unconditional moments of the cycles reduce to those of the cycle innovations because the multipliers in front of unconditional moments cancel out in the numerator and denominator. Since i_t reacts one-to-one to π^e_t in our model, the magnitude of β^ψ_{TA} indicates the activeness of monetary policy rule (Lubik and Schorfheide, 2004). The implied Phillips curve and IS curve coefficients β^ψ_{PC} and β^ψ_{IS} can be computed similarly, allowing comparisons between similar cycles specification and regression specifications such as equation (3).

The similar cycles model is a parsimonious specification for the gap variables which imposes identical autocorrelation function for $\psi_{y,t}$, $\psi_{r,t}$ and $\psi_{\pi,t}$. Indeed, from a New Kenyesian perspective, these gap variables should all resonate with real activities, i.e. the business cycle. From Figure 1, which shows the EA output gap, interest rate gap and inflation cap obtained from HP filter, we see that all cycles share the same frequency with similar amplitudes.

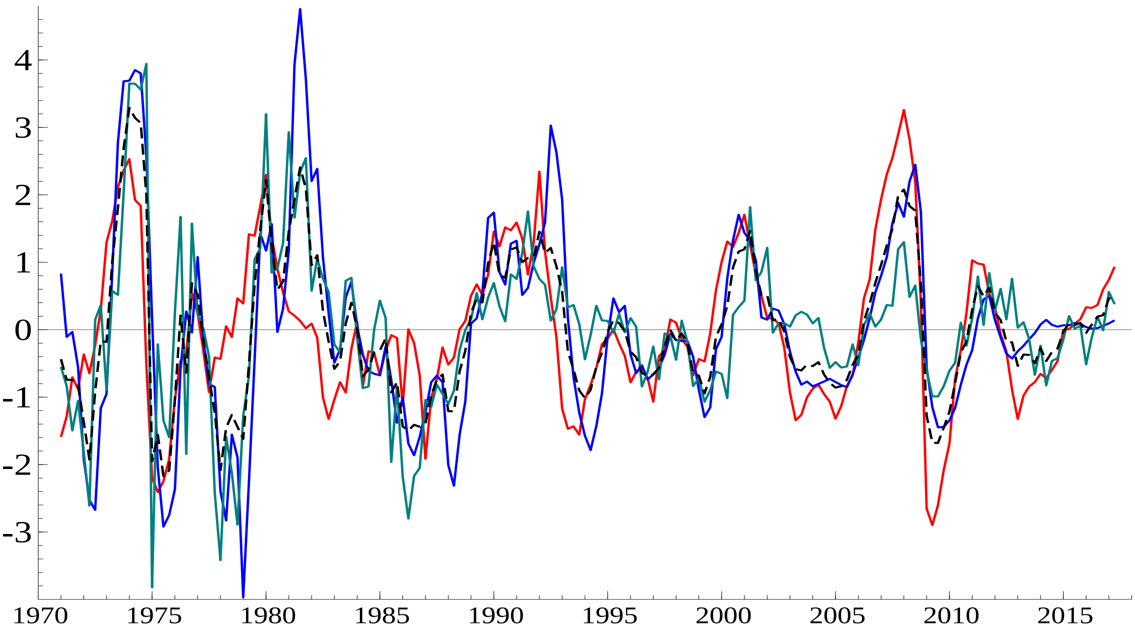

Figure 1: Cyclic components obtained from HP filter for the EA. Red: output gap; Blue: interest rate gap; Green: inflation gap; Dashed black: average of gap variables.

Furthermore, much literature mentioned in Section 2.1 that studies r-star has found the HLW model is quite sensitive to different measures of ex-ante real interest rate $r_t = i_t - \pi_t^e$ with short-term nominal rate i_t and inflation expectation π_t^e. Since any arbitrary filter gives a different expectation measure which may lead to episodic performance of the HLW model (Stock and Watson, 2007), we consider it more robust to use a model-based inflation expectation when fit our model to data of different economies. In particular, we have the following system of measurement equations in our unobserved components model:

$$\begin{aligned} y_t &= y_t^* + \psi_{y,t}, \\ i_t &= \pi_t^e + r_t^* + \psi_{r,t}, \\ \pi_t &= \pi_t^e + \psi_{\pi,t} + \epsilon_{\pi,t}, \qquad \epsilon_{\pi,t} \sim N(0, \sigma_\pi^2). \end{aligned} \tag{14}$$

The extra transitory noise term $\epsilon_{\pi,t}$ in the inflation equation of (14) is an ad-hoc choice for measurement errors in core inflation and is optional[8]. π^e_t is the unobserved trend inflation. In the presence of the cyclic component $\psi_{\pi,t}$, the trend π^e_t equals the long-run inflation expectation $\lim_{s\to\infty} E_t(\pi_{t+s})$ (Beveridge and Nelson, 1981). In the appendix we show the equivalence between the inflation equation of (14) and a hybrid New Keynesian Phillips curve. It is worth noticing that instead of taking an ad-hoc measure of inflation expectation as in the HLW model, we model a stable path, or long-run expectation, of inflation π^e_t. Explicitly modeling a stable path for inflation is necessary, because all natural rates are defined in such an environment. For example, when an economy is at its flexible price equilibrium, we have $\psi_{i,t} = 0$ for $i = y$, r and π; that is $\pi_t = \pi^e_t$ (up to a measurement error), which is the stable inflation. Should this not hold, monetary policy becomes non-neutral in equilirbium which violates the definition of r-star. As comparison, in equilibrium the HLW model gives $\pi_t = b_\pi \pi_{t-1} + (1-b_\pi)\bar{\pi}_{t-1}$ as in (3), hardly justifying the "stable inflation"[9]. π^e_t goes into the interest rate equation of system (14) such that the real interest rate subject to long-run inflation expectation consists of the r-star and the real interest rate gap.

We have the following local mean state transition equations:

$$
\begin{aligned}
y^*_{t+1} &= y^*_t + g^*_t + \epsilon_{y^*,t}, & \epsilon_{y^*,t} &\sim N(0,\sigma^2_{y^*}), \\
r^*_t &= g^*_t + z_t, & & \\
z_{t+1} &= \phi z_t + \epsilon_{z,t}, & \epsilon_{z,t} &\sim N(0,\sigma^2_z), \\
\pi^e_{t+1} &= \pi^e_t + \epsilon_{\pi^e,t}, & \epsilon_{\pi^e,t} &\sim N(0,\sigma^2_{\pi^e}).
\end{aligned}
\tag{15}
$$

The equation for potential output y^*_t and the natural rate of interest r^*_t is linked by the trend growth g^*_t, similar to the HLW model (4); but in our model, the trend growth is found by the first-stage estimation. We let the r-star specific component z_t follow a stationary AR(1) process with AR coefficient $\phi \in (-1,1)$ such that it can be initialized from its unconditional distribution $N(0,\sigma^2_z/(1-\phi))$. This specification makes maximum likelihood estimation more stable and is also applied by Garnier and Wilhelmsen (2005) who explain why one should prefer a stationary z_t over a random walk. y^*_t and π^e_t are initialized using diffuse initialization and the model is

[8] In our empirical study, this term is only modeled for the UK, because the core inflation we get from Bank of England is much more volatile than the ones for the US and EA. We find that adding this term to account for this excess volatility delivers more robust results.

[9] Among many others, literature that also introduces a stocashtic trend in Phillips curve includes Cogley and Sbordone (2008), Harvey (2011), Goodfriend and King (2012), Kim et al. (2014) and Berger et al. (2016).

Table 1: Estimation Results of Time-varying Okun's Law

First stage	Parameter vector $\hat{\theta}_1$					
	ρ	$\sigma_{\Delta y}$	σ_O	λ	LB_1	LB_2
US	$-0.194(0.082)^{**}$	$2.326(0.114)^{**}$	$0.039(0.042)$	0.042	0.757	0.166
EA	$-0.267(0.104)^{**}$	$1.600(0.107)^{**}$	$0.528(0.201)^{**}$	0.079	0.760	0.026^{**}
UK	$-0.180(0.073)^{**}$	$3.309(0.159)^{**}$	$0.046(0.082)$	0.035	0.475	0.054^{*}

The table reports a selection of estimated parameters from the first-difference Okun's law with time-varying parameters (8)-(10), which is the first stage of our modeling framework. Within brackets are standard errors with ** and * indicating statistical significance at 5% and 10% level, respectively. LB_1 and LB_2 report the p-value of Ljung-Box test of no residual serial correlation resulted from the first-step IV projection (9) and the second-step estimation of (8) and (10).

estimated by maximizing the likelihood with respect to the 13-dimensional parameter vector

$$\theta_2 = (\phi, \varphi, \omega, \rho_{IS}, \rho_{PC}, \rho_{TA}, \sigma_{\psi_y}, \sigma_{\psi_r}, \sigma_{\psi_\pi}, \sigma_{y^*}, \sigma_z, \sigma_{\pi^e}, \sigma_\pi)'.$$

The appendix details the state space representation of our model. All computations are carried out using *OxMetrics7* with the state space model package *SsfPack3.0* (Koopman et al., 1999).

3 Estimation results

In this section, we report estimation results for US, EA and UK. Section 3.1 shows the estimate of potential growth rate of output, or the trend growth rate, based on the time-varying parameter model for the first-difference version of Okun's law. Section 3.2 shows the estimation results of the proposed unobserved components model with similar cycles for natural rate of interest. Some robustness checks are provided in Section 3.3.

3.1 Estimates of trend growth rate and Okun's law

Estimation of the time-varying Okun's law model (8)-(10) takes two steps where the first step addresses potential endogeneity problem. Using the Heckman-type two-step bias correction procedure proposed by Kim (2006), we can estimate all parameters and time-varying components via Kalman filter and maximum likelihood.

Table 1 shows the estimated parameter vector $\hat{\theta}_1$ in our first stage model, where the hat symbol indicates the maximum likelihood estimate. As is seen, estimates of ρ are significant with minus sign for the three economies considered. This highlights the endogeneity problem;

Table 2: TIME-VARYING OKUN COEFFICIENT

First stage	Time-varying Okun coefficient O_t				
	1961Q1	1972Q1	1981Q1	2008Q4	2017Q2
US	(-1.92, -0.82)	(-1.84, -1.06)	(-1.82, -1.12)	(-1.54, -0.65)	(-1.63, -0.42)
EA		(-4.71, -0.41)	(-2.45, 0.16)	(-4.23, -3.21)	(-3.72, 1.98)
UK	(-2.15, -0.48)	(-1.94, -0.60)	(-1.84, -0.61)	(-1.70, -0.24)	(-1.79, -0.05)

The table shows the 95% confidence interval of the time-varying Okun coefficient from model (8) and (10) for the three economies. Five periods are selected, including initial period, end of sample period, the period after the oil crises (1981Q1) and the global financial crisis (2008Q4).

thus we should indeed apply the two-step bias correction for estimating g_t^* as well as other parameters in the model. The Ljung-Box test LB_1 for the first-step IV time-varying regression (9) suggests that the TVPM is sufficient to capture changes in Δu_t and thus able to deal with potential endogeneity via orthogonal decomposition of prediction errors. However, LB_2 rejects the null of no serial correlation in the residuals of (10) for EA at 5% level. This misspecification can be mitigated following the method of Li and Mendieta-Muñoz (2018) by allowing some parametric form of autocorrelation and stochastic volatility.

Table 1 also reports the estimated value of σ_O which shows the variation of the time-varying Okun coefficient. Only the parameter for EA suggests significant time variation of O_t, whereas US and UK are expected to have an Okun coefficient of limited variation[10]. Table 2 summarizes the 95% confidence interval of O_t, for five selected periods. It is easy to see a weakening effect of Δu_t on Δy_t for the three economies, confirming the findings in Knotek II (2007) and Zanin and Marra (2012) which attribute this weakening effect in developed countries to advancement in technology and increasing labor resource utilization. This weakening Okun's law is the most evident for the EA, as O_t becomes insignificant at the end of sample period.

We present our estimates of the trend growth rate g_t^* for the three economies in Figure 2 together with the estimates from the HLW model. For US, the Congressional Budget Office (CBO) routinely publishes an estimate of the potential output y_t^*. We fit model (4) to it and obtain the CBO estimate of g_t^*.

From Figure 2 it can be seen that the HLW estimate of g_t^* is more of a slowly decreasing linear trend. Our estimated trend growth from our model resembles the one given by CBO after 1980. In particular, it captures the trough in 1981 (the Volcker-Greenspan regime), the peak in the late 90's, and the drop of potential growth rate after the global financial crisis. However,

[10]The significance is based on likelihood ratio test, but one should notice that the test statistic is non-standard because the null is at the boundary of parameter space, i.e. zero variance.

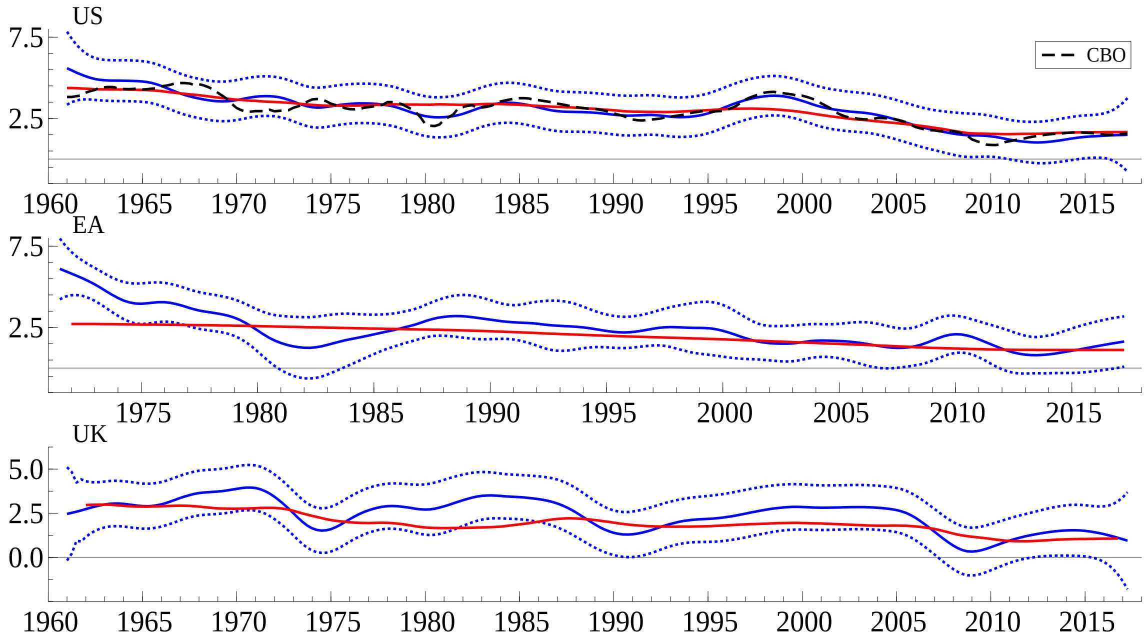

Figure 2: Estimates of trend growth rate for US, EA and UK. Blue: Estimated trend growth rate g_t^* with 95% confidence band; Red: The HLW estimate; Dashed black: Estimate of the US potential output growth rate by the Congressional Budget Office.

prior to 1980 our estimate seems to lead the CBO estimate, because CBO's data set starts two decades before ours thus the initial value effect is minor.

Similar to the US trend growth. For both economies, the HLW g_t^* for EA and UK is effectively a linear trend. Although estimation of the HLW model uses the median unbiased estimator, but it is expected to still suffer from the "pile-up" problem[11]. Our first-stage model is well suited for capturing shifts in g_t^* because the changes in the unemployment rate isolates the effect of demand shocks on changes in output (Blanchard and Quah, 1989 and Sinclair, 2009), and thus identifies changes in g_t^*. In particular, g_t^* of EA shows a sharp drop before 1980 due to the oil crises and decreasing labor productivity in the periphery countries of the then European Economic Community (Dew-Becker and Gordon, 2008); g_t^* of UK shows three drops (the mid 1970s, 1990 and 2008) on top of a decreasing trend.

3.2 Estimates of natural rate of interest and gap variables

With the trend growth rate obtained from the first stage model, we estimate the proposed unobserved components model with similar cycles (11)-(15) to the three economies. Firstly, we

[11]The first step estimation in the HLW model builds on a model without the real rate equation. As a result, the signal-to-noise ratios for g_t^* and z_t are incorrectly calibrated.

Table 3: ESTIMATION RESULTS OF UNOBSERVED COMPONENTS MODELS

Secend stage	Parameter vector $\hat{\theta}_2$ US	EA	UK
ϕ	0.968 (0.030)**	0.986 (0.012)**	0.991 (0.010)**
φ	0.911 (0.018)**	0.949 (0.023)**	0.915 (0.017)**
ω	0.179 (0.029)**	0.126	0.154 (0.051)**
period $2\pi/\omega$	35.147	50	40.893
ρ_{ry}	-0.007 (0.185)	0.000 (0.001)	0.000 (0.001)
$\rho_{\pi y}$	0.420 (0.111)**	0.578 (0.208)**	0.401 (0.165)**
$\rho_{r\pi}$	0.092 (0.088)	0.260 (0.129)*	0.146 (0.177)
β_{IS}^{ψ}	-0.039 / -0.071	-0.139 / -0.036	-0.047 / -0.009
β_{PC}^{ψ}	0.110 / 0.079	0.689 / 0.065	0.650 / 0.490
β_{TA}^{ψ}	0.538 / -	0.380 / -	0.134 / -
σ_{ψ_y}	0.591 / 0.354	0.358 / 0.290	0.644 / 0.110
σ_{ψ_r}	0.870 / -	0.582 / -	0.811 / -
σ_{ψ_π}	0.685 / 0.791	0.949 / 1.001	1.046 / 2.737
σ_{y^*}	0.356 / 0.575	0.430 / 0.400	0.557 / 0.878
σ_z	0.500 / 0.150	0.375 / 0.323	0.201 / 0.287
σ_{ψ^e}	0.422 / -	0.347 / -	0.495 / -
σ_π	- / -	- / -	3.719 / -

The table reports estimated parameters from the unobserved components model with similar cycles (11)-(15), which is the second stage of our modeling framework. Within brackets are standard errors with ** and * indicating statistical significance at 5% and 10% level, respectively. Our estimates (on the left-hand side of the slash symbol) are compared with their HLW counterparts whenever possible. Cycle period is restricted to be within $(20, 50)$ quarters. Implied beta's are calculated as in (14).

report parameter estimates in Table 3.

The period of business cycle for US and UK is estimated to be 35 and 41 quarters, respectively, which is line with the literature; however that of EA is estimated to be at the boundary, i.e. 50 quarters. This can be seen from Figure 8 that the output gap behave like a random walk locally and have a longer period than those of US and UK. Yet the damping factor φ is well within the stationary regime, so we can safely conjecture that there exists a business cycle for EA.

For the three economies, only $\rho_{\pi y}$ is found to be statistically significant (though $\rho_{r\pi}$ for UK is significant at 10% level). This suggests that the main cyclical correlation is between output and inflation. Due to our trivariate similar cycles specification, statistically near-zero ρ_{ry} or $\rho_{r\pi}$ does not necessarily imply economically near-zero IS curve, Phillips curve and Taylor principle coefficients. Using equation (13), we find that for US, the IS coefficient β_{IS}^{ψ} is smaller than the HLW estimate in absolute value, whereas for both EA and UK, EA in particular, it is found to be larger. As for the Phillips curve coefficient β_{PC}^{ψ}, our estimates are larger than the HLW estimates for all three economies. Noticeably, β_{PC}^{ψ} for EA is found to be ten times larger than the HLW counterpart. This mainly results from the contemporaneous relationship modeled by similar cycles, rather than a lagged structure. One appealing feature of our model is that, although the implied beta's differ across economies, the cycle innovation correlation coefficients are found to be rather homogeneous, ρ_{ry} and $\rho_{r\pi}$ in particular. So the difference among beta's are mainly from cycle innovation variances. Furthermore, our model also incorporates the Taylor principle while this important channel of central bank reaction is not modeled in the HLW framework. β_{TA}^{ψ} for the US is found to be 0.538, which is quite close to what John Taylor originally suggests.

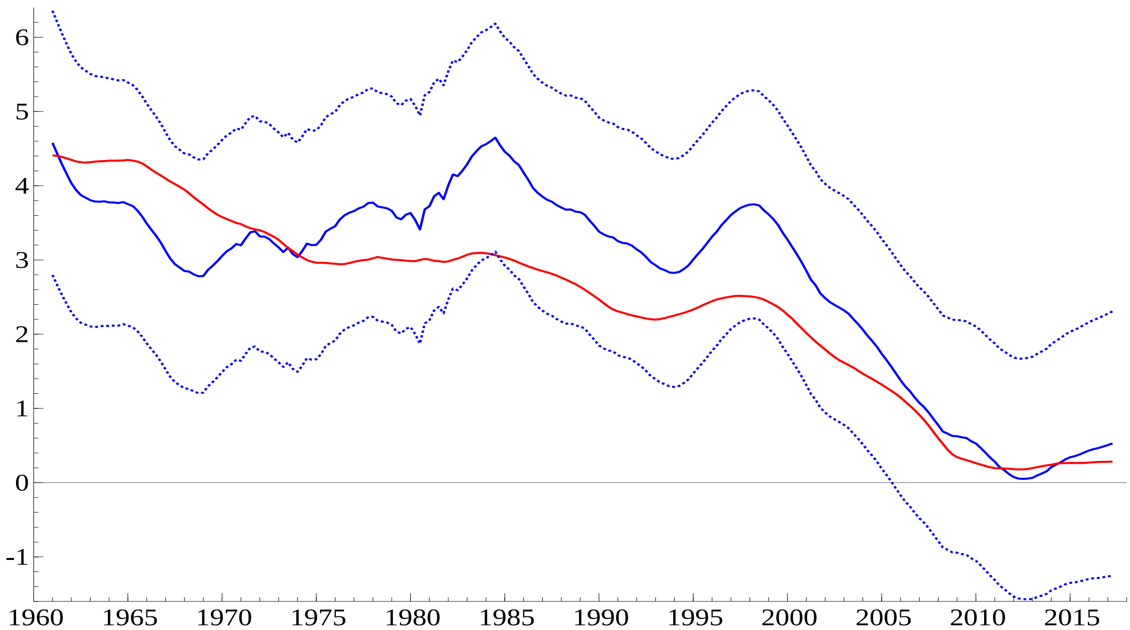

Figure 3: Estimate of US natural rate of interest. Blue: Estimated natural rate of interest r_t^* with 95% confidence band; Red: The HLW estimate.

Figure 3-5 show the estimates of natural rate of interest for the three economies. In comparison with the estimates from the HLW model (the red line), we can see that our r-star show more variation. For US, no statistical difference is present if the confidence band is taken into account. The r-star of EA from our model starts 1 percentage point (pp) higher than the HLW value, and from 1990 a drop can be spotted which prevails until recently. Even if taking uncertainty around the estimate into account, our model still suggests a level shift in r-star for EA, whereas

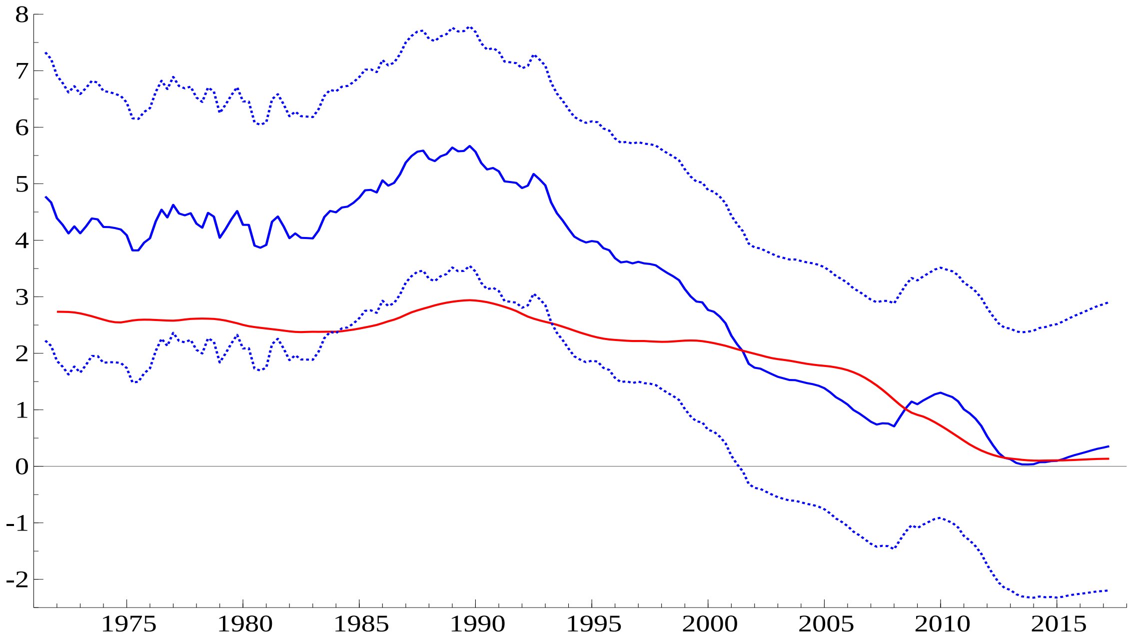

Figure 4: Estimate of EA natural rate of interest. Blue: Estimated natural rate of interest r_t^* with 95% confidence band; Red: The HLW estimate.

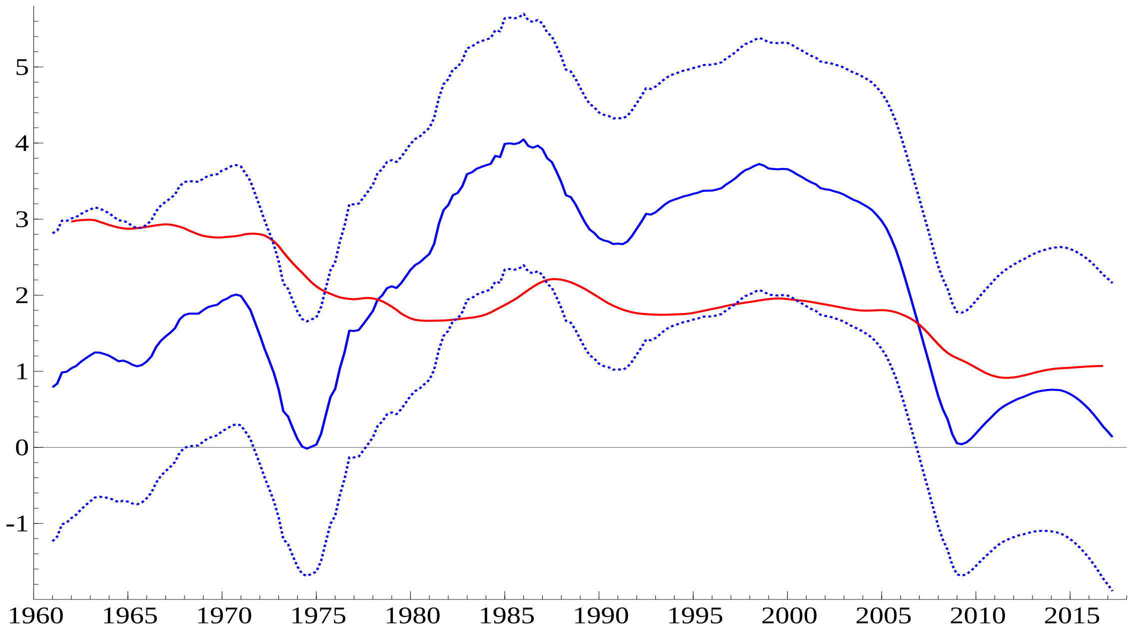

Figure 5: Estimate of UK natural rate of interest. Blue: Estimated natural rate of interest r_t^* with 95% confidence band; Red: The HLW estimate.

the HLW does not. The result of UK from our model is surprisingly different from that of the HLW model. Similar to the case of EA, UK's r-star suggested by the HLW model does not show

Table 4: Changes of Percentage Points in Natural Rates

PP change	baseline model / the HLW model sample period	US	EA	UK
Δr^*	- 1990	-1.275 / -2.076	0.505 / 0.146	1.883 / -1.105
	1990 - 2006	-2.104 / -1.330	-4.358 / -1.287	-0.891 / -0.176
	2006 - 2017	-0.547 / -0.630	-0.433 / -1.369	-1.433 / -0.543
Δg^*	- 1990	-2.915 / -1.445	-3.256 / -0.488	-1.171 / -0.542
	1990 - 2006	-0.831 / -0.944	-1.464 / -0.826	0.362 / -0.455
	2006 - 2017	-0.291 / -0.265	0.337 / -0.255	-0.514 / -0.072

This table shows the changes in terms of percentage point (pp) of estimated r_t^* and g_t^* over three periods for the US, EA and UK from our baseline model (on the left-hand side of the slash symbol) and the HLW model.

any statistical changes if one takes into account the uncertainty around the estimate. Our r-star however starts 2pp lower than the HLW value and shows a rise from 1975 to 1985, after which a big drop takes place due to the GFC.

In Table 4, we summarize percentage point changes of the natural rate of interest and output growth based on estimates from both models. We see that both model suggest that most of the fall in natural rate of output growth took place before 1990; but our model suggests that the major drop in r-star took place during 1990-2006, whereas the HLW model suggests r-star dropped the most before 1990. Our result tells that r-star of EA stayed at around 4.8% until the downward shift started in 1990, while the HLW model estimates the fall to be most profound after the GFC. Furthermore, our model suggests that most of the fall in potential output growth rate took place before 1980, amounting to a -3.2 pp change, and the HLW model simply suggests a gradual decrease of g_t^*. According to our model, the changes in UK's r-star experienced an 1.88pp-increase before 1990 and a big fall during the GFC in 2008. With the HLW mode, r-star shows a gradual fall, similar to its estimate of g_t^*. Despite these differences, both models suggest near-zero r-star for the three economies in the current period, in line with literature; however, it cannot not emphasized more that one should have a cautious take on this as the uncertainty around the estimated r-star is rather large (see extensive discussions in Matthew and Justin, 2017 and Holston et al., 2017).

The above summary can also be seen from Figure 6. Importantly, the initial value of r_t^* and g_t^* in the HLW model almost coincides, because it treats z_1 to be zero almost deterministically. As a result, together with a potentially downward-biased estimate of σ_z what we obtain is an expanding wedge between the two stars, for US and UK particularly. Additionally, the HLW

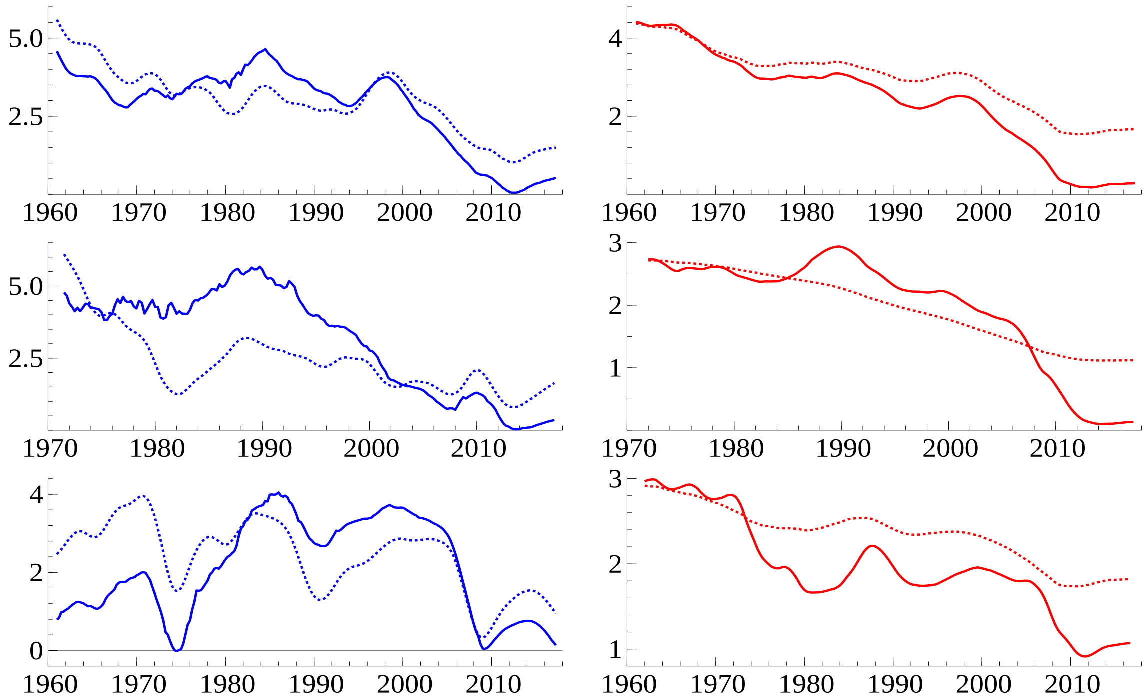

Figure 6: Baseline model and the HLW estimates. Thick line: Estimated natural rate of interest r_t^*; Dashed line: Estimated trend growth rate g_t^*. Blue: estimates from our baseline model; Red: estimated from the HLW model; Top to bottom: results for US, EA and UK, respectively.

model initializes the potential output y_t^* from their HP-filtered values almost deterministically, while we initialize all nonstationary components in our model diffusely; thus we let the data speak, which causes the big difference between our estimates of UK's natural rates and those of the HLW model at the beginning of the sample period.

Due to the differences in estimates of natural rates between our model and the HLW model, we expect to see different gap variables since both models decompose left-hand side variables into a nonstationary and a stationary component. Figure 7-9 show the estimate of output gap $\psi_{y,t}$ for the three economies, and Figure 10-12 show the estimate of real interest rate gap $\psi_{r,t}$. It can be easily seen that both models produce similar gaps for the US economy and track other institutional estimates closely, which reassures our model specification. Main differences are observed from the output gap for EA and the real interest rate gap for EA and UK.

EA's outputgap $\psi_{y,t}$ estimated by the HLW model shows a twenty-year long secular stagnation between 1980 and 2000, whereas our model finds such evidence only during the 1990s. Similar to the OECD, IMF and Oxford Economics estimates, we find a 2% peak for EA output gap in 1991. The trough in the HLW output gap in the 1980s goes into the dip in our estimate

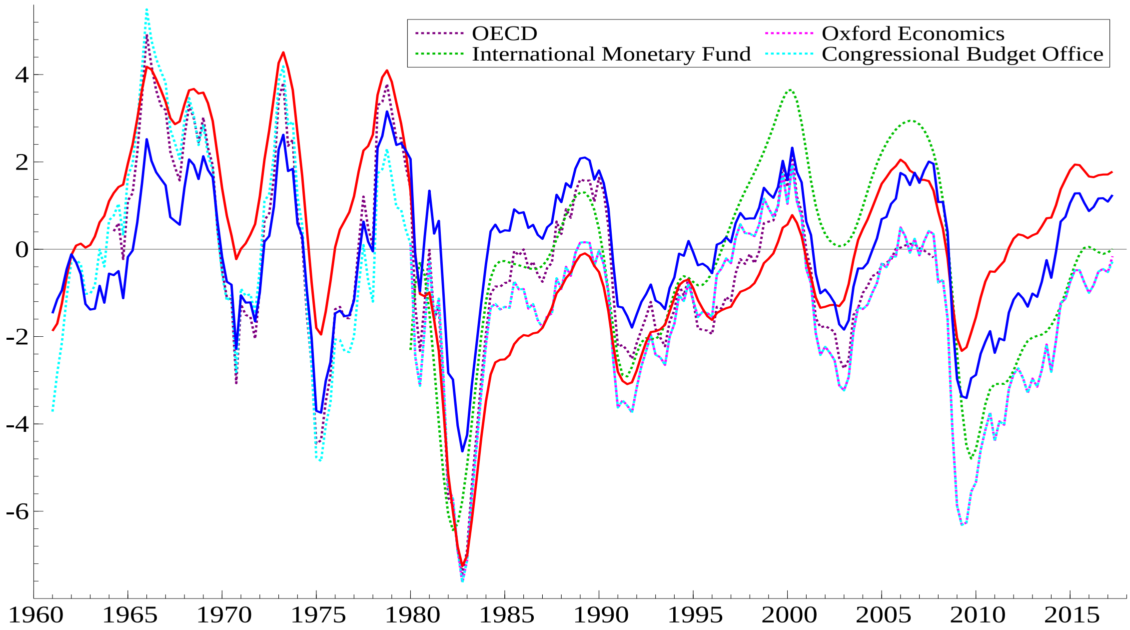

Figure 7: Estimate of US output gap. Blue: Estimated output gap $\psi_{y,t}$; Red: The HLW output gap; Colored dashed lines indicate estimates given by other institutions.

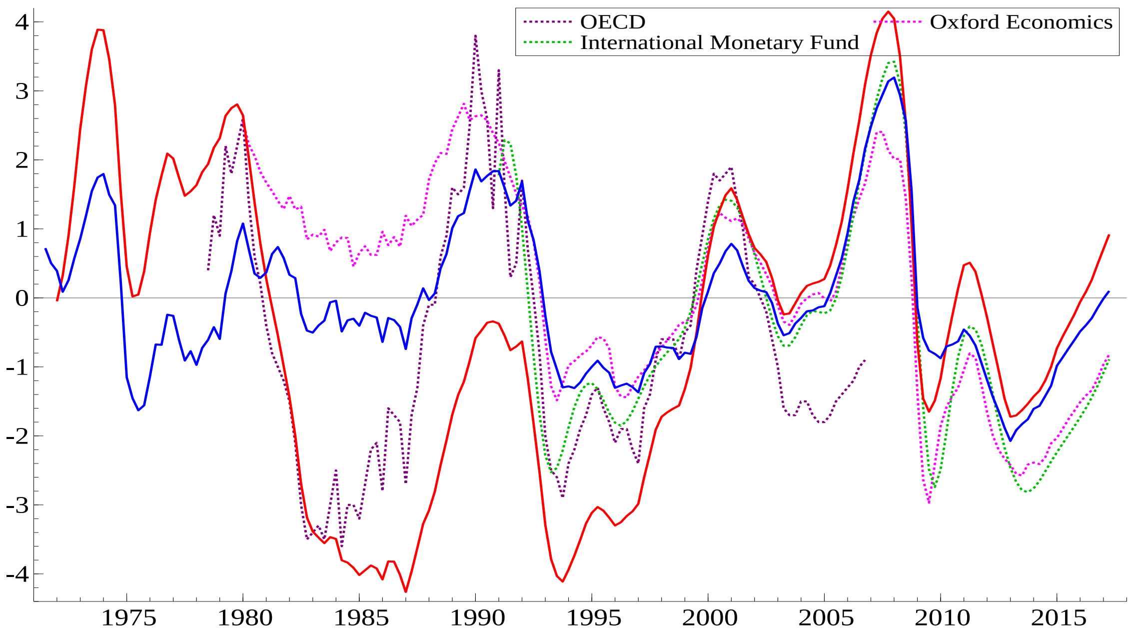

Figure 8: Estimate of EA output gap. Blue: Estimated output gap $\psi_{y,t}$; Red: The HLW output gap; Colored dashed lines indicate estimates given by other institutions.

of g_t^*; however by definition of the potential output growth rate given in section 2, this dip in g_t^* is free from output growth fluctuations explained by the changing unemployment rate of EA

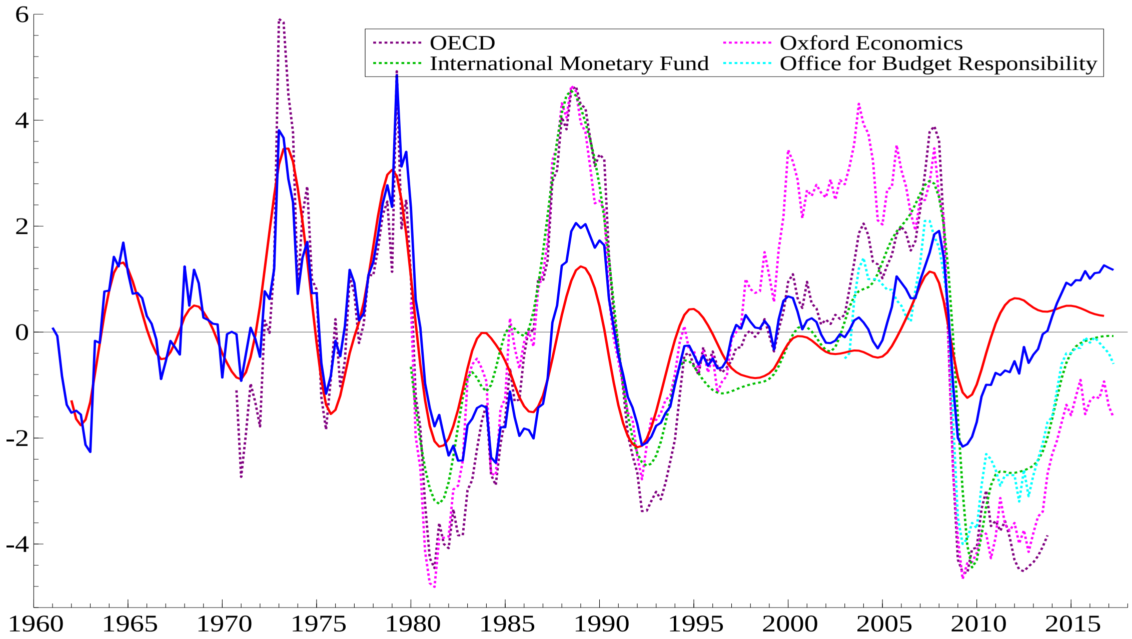

Figure 9: Estimate of UK output gap. Blue: Estimated output gap $\psi_{y,t}$; Red: The HLW output gap; Colored dashed lines indicate estimates given by other institutions.

during 1980s. Thus the drop is indeed from the potential output rather than the output gap, same as the findings in Dew-Becker and Gordon (2008) who document a significant drop in the growth rate of productivity in Europe during that period.

The estimated real interest rate gap of EA differs from the HLW estimates mostly during 1983-1993. Our estimates form a trough from 5% to -2.5% and come back to 4%, while the HLW real interest rate gap levels off at around 4%. The high real interest rate during the second half of the 1980s in the HLW model comes from low inflation expectation calculated using an ad-hoc 4-quarter moving average measure. In other words, the HLW model implicitly assumes that the representative agent always discounts four quarters in the recent past to form expectation. On the contrary, we directly treat inflation expectation as unobserved, thus is able to derive a model-consistent measure of inflation expectation. The fact that the HLW model is sensitive to different ad-hoc measures of inflation expectation renders a model-consistent inflation expectation more preferable. This can also be seen from the HLW real interest rate gap of UK which shows a nearly 27-year long positive regime between 1982 and 2009. Surprisingly, during the 1970s the UK's real interest gap is estimated to as low as -12.5% by the HLW model, whereas our model attributes these low values to the dip in the natural rate of interest r_t^* due to the drop of natural rate of output growth during that time.

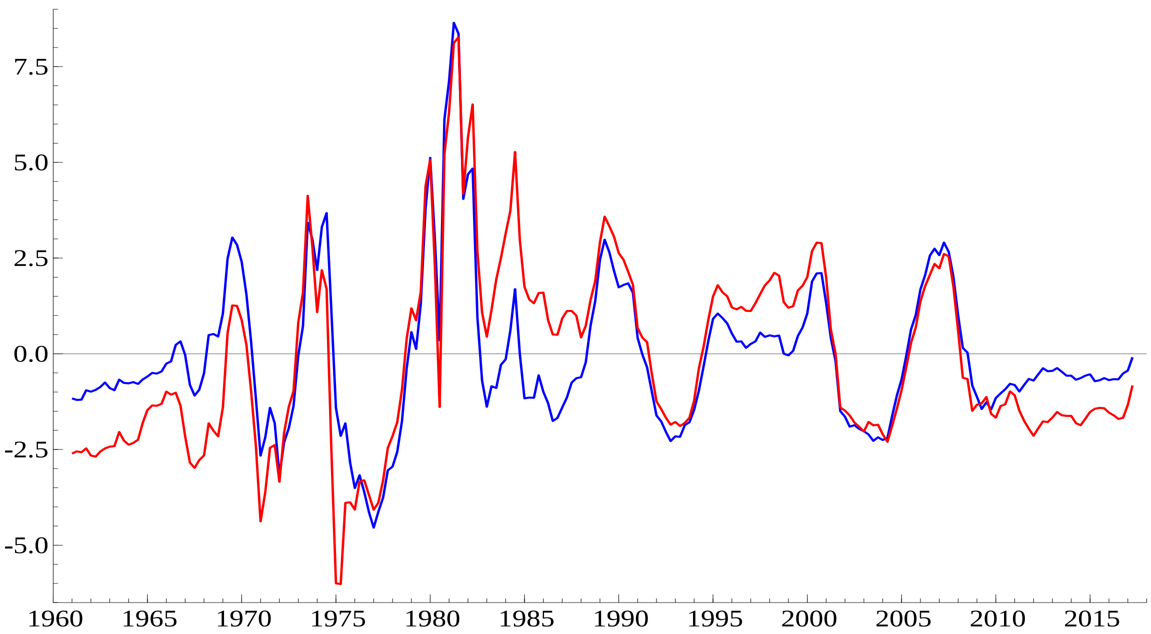

Figure 10: Estimate of US real interest rate gap. Blue: Estimated real interest rate gap $\psi_{r,t}$; Red: The HLW rate gap.

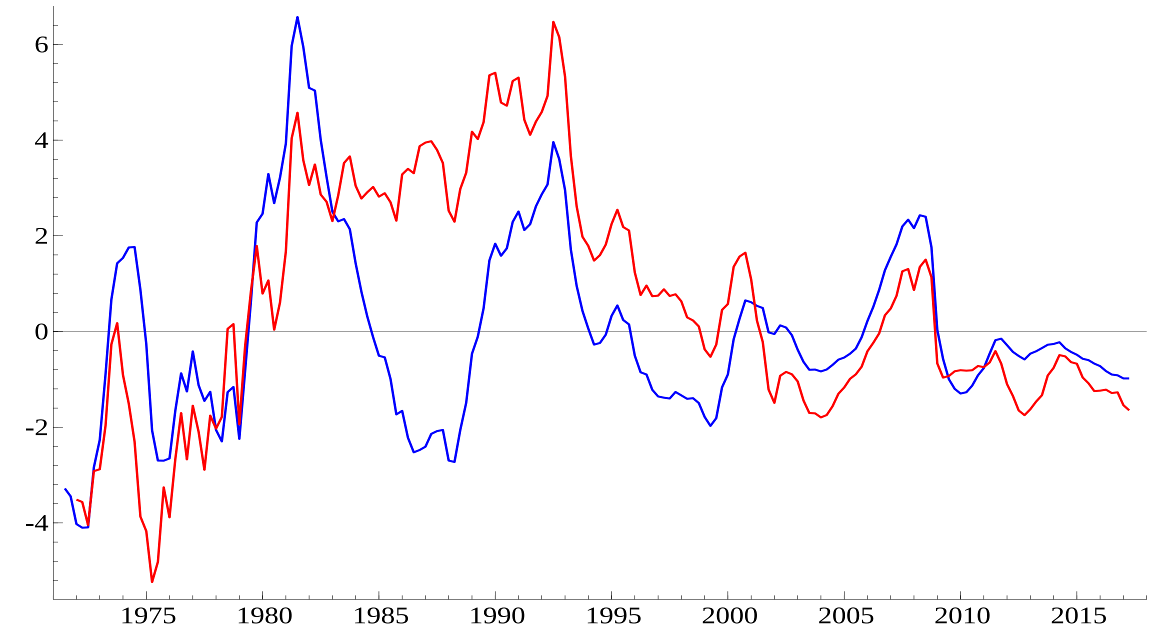

Figure 11: Estimate of EA real interest rate gap. Blue: Estimated real interest rate gap $\psi_{r,t}$; Red: The HLW rate gap.

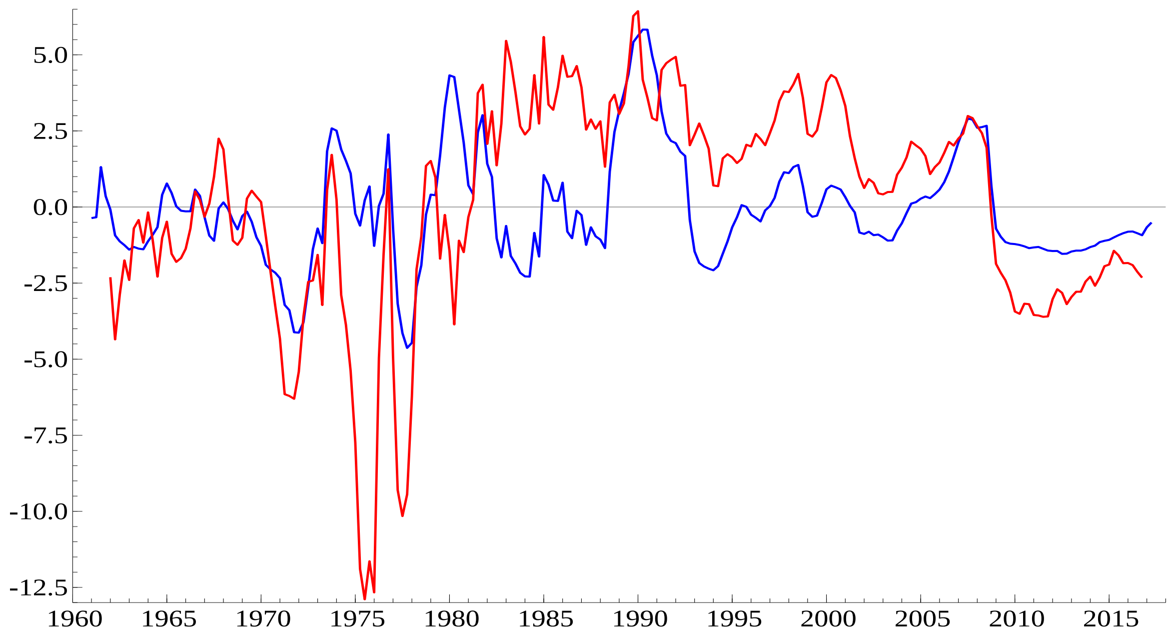

Figure 12: Estimate of UK real interest rate gap. Blue: Estimated real interest rate gap $\psi_{r,t}$; Red: The HLW rate gap.

3.3 Robustness

To check the robustness of the proposed model, we consider some basic alternatives. Table 5 shows a selection of estimated parameters. Firstly, we consider a model for US that directly uses the potential output growth rate g_t^* from CBO. Both estimated parameters and unobserved components are literally the same as the estimates obtained using our baseline model. This is expected as our first-stage model produces g_t^* that closely tracks the one given by CBO, as is seen in Figure 2. Furthermore, we can estimate the risk aversion parameter or even make it time-varying because g_t^* is obtained prior to the second-stage estimation. For all three economies, this parameter is estimated to be close to one when considered static and shows a small gradual increase over time when considered stochastic, suggesting the representative agent effectively has a log utility function. Under both cases, other estimated parameters are quite close to the ones estimated by the baseline specification, i.e. restricting $\alpha = 1$. The estimated natural rate of interest r_t^* for the three economies when either static or stochastic risk aversion is estimated does not suggest any noteworthy difference from the baseline, thus we do not show the comparisons. Yet we should notice that Garnier and Wilhelmsen (2005) estimate the risk aversion in the HLW model when fitting it to EA data and find quite different r-star. Our model thus shows robustness in terms of estimation of risk aversion. Lastly, we re-estimate the baseline model using only data

Table 5: Other Model Specifications

Other specifications	Selected parameters β^{ψ}_{PC}	β^{ψ}_{IS}	β^{ψ}_{TA}	σ_{ψ_r}	σ_z
US					
CBO	0.099	-0.055	0.454	0.846	0.517
α	0.108	-0.034	0.534	0.89	0.454
α_t	0.107	-0.032	0.535	0.876	0.486
after 1990	0.081	-0.027	0.355	0.29	0.137
EA					
α	0.709	-0.149	0.390	0.577	0.381
α_t	0.759	-0.175	0.414	0.562	0.389
after 1990	0.168	-0.262	0.751	0.414	0.174
UK					
α	0.648	-0.051	0.148	0.808	0.212
α_t	0.713	-0.058	0.138	0.753	0.197
after 1990	0.064	-0.032	0.919	0.368	0.146

The table reports some estimated parameters for the three economies under different model specifications. For US, CBO indicates we directly use the trend growth g^*_t published by the Congressional Budget Office. α indicates the model with estimated risk aversion parameter as in (1). α_t indicates the model with a time-varying risk aversion. "After 1990" uses data after 1990Q1.

after 1990Q1. The results suggest flattening Phillips curve for the three economies, EA and UK in particular, which is in line with literature on weakening response of inflation to output gap (Berger et al., 2016). IS curve also seems to be weaker after 1990 for US and UK, but stronger for EA. The implied Taylor principle coefficient shows that central banks in Europe have made more responsive monetary policy since 1990, whereas the estimate for US suggests the opposite. The difference between full sample estimates and estimates obtained using sample after 1990 leaves the room for time-varying parameter modeling, which is an important avenue for future research on r-star. Apparently, this model uncertainty attributes to the estimation uncertainty of r-star, as Holston et al. (2017) notice.

4 Conclusion

The natural rate of interest or r-star plays a central role in monetary policy. It is recognized that r-star is subject to low-frequency time-variation due to gradual shifts in potential output growth rate. Literature has devoted much effort in estimating these natural rates. Our paper complements this discussion by proposing an unobserved components model with similar cycles estimated using a two-stage procedure. In the first stage, we pin down the potential output growth rate using a first-difference version of Okun's law with time-varying parameters. In the second stage, the unobserved components model is estimated with the output gap, real interest rate gap and inflation gap identified by similar cycles through Phillips curve, IS curve and a Taylor rule. Our model is not only robust to initialization of nonstationary components in the model, but also to inflation expectation measures. Empirically, we fit our model to US, EA and UK data with comparisons to the results from the Holston et al. (2017)'s model. We find that the fall in potential output growth starts much before the GFC for the three economies, whereas the r-star of US and EA starts to fall after 1985. The UK's r-star starts low in the 1960s and 1970s, and experiences an increase from 1980s until its significant drop during the GFC. All r-stars are near-zero in the recent periods, but uncertainty suggests that policy makers should take extra caution until we can be more certain about their exact values.

Appendices

Similar cycles and the New Keynesian Phillips curve

The similar cycles model imposes identical autocorrelation functions for variables in the system. Only looking at output, we have

$$
\begin{aligned}
& y_t = y_t^* + \psi_{y,t}, \\
& \begin{bmatrix} \psi_{y,t+1} \\ \tilde{\psi}_{y,t+1} \end{bmatrix} = \varphi \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \psi_{y,t} \\ \tilde{\psi}_{y,t} \end{bmatrix} + \begin{bmatrix} \kappa_{y,t} \\ \tilde{\kappa}_{y,t} \end{bmatrix}, \qquad \kappa_{y,t}, \tilde{\kappa}_{y,t} \sim N(0, \sigma^2_{\psi_y} I_2).
\end{aligned}
$$

It can be shown that the reduced form model for the output gap $\psi_{y,t} = y_t - y_t^*$ follows an ARMA(2,1) dynamics,

$$
\psi_{y,t} = \rho_1 \psi_{y,t-1} + \rho_2 \psi_{y,t-2} + \theta \eta_{t-1} + \eta_t, \qquad \eta_t \sim N(0, \sigma^2_{\psi_y}), \tag{16}
$$

where $\rho_1 = -\varphi^2$, $\rho_2 = 2\varphi \cos\omega$ and $\theta = -\varphi(\cos\omega + \sin\omega)$.

Cogley and Sbordone (2008) and Harvey (2011) derive a New Keynesian Phillips curve when a stochastic trend inflation is present. They show that such a model with only forward-looking components provides better in-sample fit for the US data. We follow this literature and show that the similar cycles model permits a hybrid New Keynesian Phillips curve between the deviation of inflation from its trend π_t^e and output gap $\psi_{y,t}$. Define the following inflation variable with a backward-looking component

$$
\pi_t^c = \frac{(\pi_t - \pi_t^e) - a(\pi_{t-1} - \pi_{t-1}^e)}{1 - a},
$$

and consider the standard Phillips curve

$$
\pi_t^c = \gamma E_t(\pi_{t+1}^c) + \beta \psi_{y,t} + \epsilon_{\pi^c,t}, \tag{17}
$$

where a is the backward-looking weight; $\epsilon_{\pi^c,t}$ is a white noise disturbance; γ is a discount factor. The question thus becomes if there exists a pair (a, b) such that the inflation gap is proportional to the output gap, or equivalently, if for some b we have

$$
\psi_{\pi,t} = \pi_t - \pi_t^e = b\psi_{y,t} + \zeta_t, \tag{18}
$$

where ζ_t is a noise term. If the above holds true, then we know $\psi_{\pi,t}$ has the same autocorrelation function as $\psi_{y,t}$; so they are similar stochastic cycles.

Provided the usual transversality condition, iterating (17) forward gives

$$\pi_t^c = \beta \sum_{s=0}^{\infty} \gamma^s E_t \psi_{y,t+s} + \epsilon_{\pi^c,t}.$$

Inserting (16) into the above, we can solve for the inflation path as

$$\pi_t^c = \beta[1 \quad 0] \left(I_2 - \gamma \begin{bmatrix} \rho_1 & 1 \\ \rho_2 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \psi_{y,t} \\ \rho_2 \psi_{y,t-1} + \theta \eta_t \end{bmatrix} + \epsilon_{\pi^e,t},$$

or equivalently

$$\frac{1}{1-a}(\psi_{\pi,t} - a\psi_{\pi,t-1}) = \frac{\beta}{1-\gamma\rho_1-\gamma^2\rho_2}(\psi_{y,t} + \gamma\rho_2\psi_{y,t-1} + \gamma\theta\eta_t) + \epsilon_{\pi^c,t}.$$

Using (18) and by undetermined coefficients, we see

$$\begin{aligned} a &= -\gamma\rho_2, \\ b &= \frac{(1+\gamma\rho_2)\beta}{1-\gamma\rho_1-\gamma^2\rho_2}, \\ \zeta_t &= \sum_{s=1}^{t} a^s \left(\gamma\theta\eta_s + \frac{1-\gamma\rho_1-\gamma^2\rho_2}{\beta}\epsilon_{\pi^c,s} \right). \end{aligned}$$

State space representation

A linear Gaussian state space model takes the following form:

$$x_{t+1} = \delta_t + \Phi x_t + \eta_t, \qquad \eta_t \sim N(0, \Sigma_\eta), \tag{19}$$

$$z_t = \tau_t + \Lambda x_t + \epsilon_t, \qquad \epsilon_t \sim N(0, \Sigma_\epsilon). \tag{20}$$

Equation (19) and (20) are the state transition and measurement equation, respectively. η_t and ϵ_t are the vector of state innovations and idiosyncratic disturbances, respectively, and $E(\eta_t\epsilon_t') = 0$. δ_t, τ_t, Λ, Φ, Σ_η and Σ_ϵ are either fixed or predetermined system matrices which may contain unknown parameters that need to estimated.

In our model, we have $z_t = (y_t, i_t, \pi_t)'$ and

$$x_t = (y_t^*, z_t, \pi_t^e, \psi_{y,t}, \psi_{r,t}, \psi_{\pi,t}, \tilde{\psi}_{y,t}, \tilde{\psi}_{r,t}, \tilde{\psi}_{\pi,t})'.$$

The system matrices are given by

$$\delta_t = \begin{bmatrix} g_t^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi\cos\omega & 0 & 0 & \varphi\sin\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi\cos\omega & 0 & 0 & \varphi\sin\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \varphi\cos\omega & 0 & 0 & \varphi\sin\omega \\ 0 & 0 & 0 & -\varphi\sin\omega & 0 & 0 & \varphi\cos\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varphi\sin\omega & 0 & 0 & \varphi\cos\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varphi\sin\omega & 0 & 0 & \varphi\cos\omega \end{bmatrix},$$

$$\tau_t = \begin{bmatrix} 0 \\ g_t^* \\ 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma_\eta = \begin{bmatrix} \sigma_{y^*} & 0 & 0 & 0 & 0 \\ 0 & \sigma_z & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\pi^e} & 0 & 0 \\ 0 & 0 & 0 & \Sigma_\psi & 0 \\ 0 & 0 & 0 & 0 & \Sigma_\psi \end{bmatrix}, \quad \Sigma_\epsilon = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_\pi \end{bmatrix},$$

where Σ_ψ is defined as (12). Estimation of the model is based on prediction error decomposition using Kalman filter, which also produces estimate of state x_t.

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