Auction Design by an Informed Seller: The Optimality of Reserve Price Signaling*

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Abstract

This paper studies mechanism design by a seller privately informed of the quality of an indivisible object. The privacy of the seller's information matters for mechanism design: selecting a mechanism that maximizes the seller's profit when her information is public is not incentive compatible for the seller when her information is private, as a lower-quality seller has an incentive to mimic a higher-quality seller. I show that reserve prices are the least costly device to separate higher-quality sellers from lower-quality ones. In equilibria that maximize the expected profit of every type of the seller among all separating equilibria, the lowest-quality seller adopts her public-information optimal mechanism, and each higher-quality seller adopts a mechanism that differs from her public-information optimal mechanism only in that the reserve prices are higher.

Keywords: Mechanism design, informed principal, reserve price, signaling

JEL classification: D44, D82

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1 Introduction

Most of the literature assumes that a mechanism designer has no relevant private information. This assumption does not fit real-world situations well. For example, a house owner may have private information valuable to potential buyers in evaluating the house; in keyword auctions, auctioneers are usually better informed than buyers about the frequencies that the keywords are searched.

I study a problem of mechanism design in which the mechanism designer is privately informed. A seller chooses a mechanism to sell a single indivisible object to one of multiple (potentially asymmetric) buyers. The players in the mechanism are the buyers. Before choosing the mechanism, the seller receives a private signal regarding the quality of the object. This signal directly affects the seller's and buyers' valuations of the object. This mechanism-selection game involves signaling in that the seller's choice of the mechanism may (at least partially) reveal her private information to the buyers. However, unlike a typical signaling game, in which the action space of a sender is a finite-dimensional set of actions, the action space of the seller in this game is an infinite-dimensional set of mechanisms.

This game captures environments without an independent mediator to run the mechanism.[1] Unmediated trade is common in practice, especially when the gains from trade are modest (as in, for example, the transactions on customer-to-customer online shopping platforms). As Farrell (1983) argues, finding a neutral mediator can be very costly or impossible in many cases.

I show that in general the mechanism-selection game has no equilibrium in which every type of the seller chooses a mechanism that maximizes her profit in the case that her information is public (henceforth, public-information optimal mechanism). The reason is simple: a lower-quality seller wants to pretend to be a higher-quality seller, to extract more profit from the buyers. This implies that to separate lower-quality sellers from higher-quality sellers, the mechanisms adopted by higher-quality sellers should be less profitable than those

[1] In the presence of such a mediator, the seller could also be a participant in the mechanism; see the discussion at the end of this Introduction.

adopted in the public information case.

In the model, there are (at least) three ways to disincentivize lower-quality sellers from mimicking higher-quality ones: (1) increase the reserve prices of the mechanisms adopted by the higher-quality sellers, to decrease the probabilities of selling the object; (2) decrease the expected payments of the buyers to the higher-quality sellers, or have the higher-quality sellers burn money without changing the rule for allocating the object; (3) change the allocation rules of the higher-quality sellers by, for example, giving some buyers unfair advantages when bidding for the object, to reduce the competitiveness of the bidding games and induce lower bids from the buyers.

I characterize the seller-optimal separating equilibria, which maximize the (interim) expected profit of every type of the seller among all possible separating equilibria.[2] This seemingly complex problem has a simple solution: the equilibrium strategies of the seller differ from the public-information optimal mechanisms only in reserve prices. Specifically, the lowest-quality seller uses the same mechanism that she adopts under public information, while all other types of the seller adopt mechanisms having higher reserve prices than their public-information optimal mechanisms. This result, which is the main result in this paper, shows that the first approach mentioned above is the optimal way of separating the seller types.

To understand why the result holds, let us compare the first two approaches. To eliminate the incentive for a lower-quality seller to mimic a higher type, we need to reduce her revenue gain from doing so. Under the second approach, to reduce the revenue gain by a certain amount we need to force a higher-quality seller to give up that amount of revenue. Under the first approach, which decreases the probability of trade, because a higher-quality seller values the object more, her loss from the decreased trading probability is less than that of a lower-quality seller; therefore, it is less than her loss using the second approach. For any other approach, including the third one, can be proved to be no better than a combination

[2] I discuss pooling equilibria near the end of Section 3, after characterizing the seller-optimal separating equilibria.

of the first and second approaches, so is less desirable than the first one.

The setup of my model is similar to those of Jullien and Mariotti (2006) and Cai et al. (2007). These two papers study reserve-price-signaling games in which the buyers are symmetric, the auction is a second-price auction, and the privately informed seller has the freedom to set only the reserve price. Both papers characterize the unique separating equilibrium of their games, and find that the lowest-quality seller sets the same reserve price as in the public information case, whereas other types set higher reserve prices compared with the public information case. In my model, I allow the buyers to be asymmetric and endow the seller with the freedom to design every element of the mechanism, rather than the freedom to vary only the reserve prices.[3] My main result shows that even if the seller has the freedom to choose any mechanism, she ends up adopting a mechanism that differs from her public-information optimal mechanism only in the reserve prices. This result justifies the assumption in reserve-price-signaling games that the informed seller is restricted to choose only the reserve prices.

This paper is closely related to the literature on mechanism design by an informed principal. The game of mechanism selection studied in this literature differs from the main game I study in that the principal is an active participant in the mechanism, so an independent mediator is required to run the mechanism. The seminal work of Myerson (1983) lays the foundation for analyzing the informed-principal problem by developing several solution concepts with different strengths.[4] He also introduces *safe mechanisms* , which are mechanisms that are incentive compatible and individually rational for the principal and the agents regardless of the agents' beliefs about the principal's type. This class of mechanisms is used in the proof of the main result of the current paper. The analyses of Maskin and Tirole (1990, 1992) focus on the one-principal/one-agent case. Maskin and Tirole (1992) consider the environment where the principal's private information directly affects the agent's utility, which is similar to the environment considered in the current paper. They find that, un-

[3] Cai et al. (2007) consider a more general information structure than I do: they allow for the buyers' signals to be affiliated. Also see Lamy (2010) for a corrigendum to Cai et al. (2007).

[4] Myerson (1985) applies the theories in studying bilateral trading problems.

der certain conditions, the set of *Rothschild-Stiglitz-Wilson (RSW) mechanisms*, which are safe mechanisms that maximize the interim expected payoff of the principal among all safe mechanisms, coincides with the set of equilibrium strategies of the principal.

Maskin and Tirole (1990) and Mylovanov and Tröger (2012a, 2012b, 2015) focus on the case in which the principal's information does not directly enter the agents' utility functions; it affects the payoffs of the agents only through its effects on the principal's equilibrium behavior. Mylovanov and Tröger (2012a) provide a solution concept for the informed-principal problem for general private value environments, *strongly neologism-proof allocation*. Maskin and Tirole (1990) show that generically the principal is better off concealing her information from the buyers. Mylovanov and Tröger (2012b, 2015) study the conditions under which the privacy of the principal's information does not distort the selection of mechanisms.

I prove my main result using the game of mechanism selection studied in the literature on the informed-principal problem. In this auxiliary game, I characterize the set of RSW mechanisms and show that any RSW mechanism corresponds to the strategy of the seller in some seller-optimal separating equilibrium of the main game I study. Fully characterizing the set of RSW mechanisms is independently interesting, given the aforementioned finding of Maskin and Tirole (1992) regarding the relationship between the RSW mechanisms and the equilibrium strategies of the principal in the informed-principal problem.

2 Setup

An indivisible object is for sale. The owner of the object designs a selling mechanism through which she allocates the object to one of n potential buyers.

The seller *privately* observes a signal s, which determines her valuation $v_0(s)$ of the object. I assume that $v_0(s)$ is increasing in s and twice continuously differentiable. The valuation $v_i(s, t_i)$ by buyer $i = 1, 2, \ldots, n$ of the object depends on the seller's signal s and a private signal t_i. I assume that $v_i(s, t_i)$ is twice continuously differentiable in both signals and is increasing in t_i. It is common knowledge that s and t_i, $i = 1, 2, \ldots, n$, are *independently* drawn from continuous distributions $F_0 : [\underline{s}, \bar{s}] \to [0, 1]$ with density $f_0 : [\underline{s}, \bar{s}] \to \mathbb{R}_{++}$ and $F_i : [\underline{t}_i, \bar{t}_i] \to [0, 1]$ with density $f_i : [\underline{t}_i, \bar{t}_i] \to \mathbb{R}_{++}$, respectively. To simplify the notation, I define $t = (t_1, t_2, \ldots, t_n)$ and use S, T_i, and T to denote $[\underline{s}, \bar{s}]$, $[\underline{t}_i, \bar{t}_i]$, and $\prod_{i=1}^{n} [\underline{t}_i, \bar{t}_i]$, respectively.

The seller chooses a selling mechanism after learning her private signal. A mechanism consists of a message space $\Lambda = \prod_{i=1}^{n} \Lambda_i$, where $\Lambda_i \subset \mathbb{R}$ is the set of possible messages for buyer i, an allocation rule, a payment rule, and a money-burning rule.[5] After the buyers observe the mechanism, they decide whether to participate in the mechanism and, if they choose to participate, report a message. If every buyer participates, the mechanism is implemented; otherwise, the seller keeps the object and each buyer gets the payoff 0. In the rest of the analysis, I call this game the *mechanism-selection game*. I use perfect Bayesian equilibrium as the equilibrium concept.

In this game, the revelation principle allows us to restrict to direct incentive compatible and individually rational mechanisms on the equilibrium path, although not off the equilibrium path. The message space Λ of a direct mechanism is just the signal space T, with T_i being the set of possible messages for buyer i. I use $x : T \to [0, 1]^n$, $p : T \to \mathbb{R}^n$, and $b : T \to \mathbb{R}_+$ to denote the allocation rule, payment rule, and money-burning rule of a direct mechanism. For x and p, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $p(t) = (p_1(t), p_2(t), \ldots, p_n(t))$, where $x_i(t)$ and $p_i(t)$ are respectively buyer i's probability of getting the object and expected payment to the seller under t. The value $b(t)$ is the amount of money burned by the seller under t. The allocation rule x satisfies the feasibility constraints

$$0 \leq x_i(t), \sum_{i=1}^{n} x_i(t) \leq 1 \text{ for all } i \text{ and } t. \tag{1}$$

I use $x_0(t)$ to denote the probability that the seller keeps the object when the message profile

[5]In my mechanism-selection game, a mechanism is selected after the seller has learned her type, so the choice of mechanism may partially or completely reveal her information. By imbedding money-burning rules into the selling mechanisms, I allow the seller to signal by burning money. I could also allow the seller to (partially) burn the object, but as one will see from the following analysis, it is clearly suboptimal. Thus, I omit it in the model.

is t, i.e., $x_0(t) = 1 - \sum_{i=1}^{n} x_i(t)$. By abusing the notation a little, I define

$$x_i(t_i) = \int_{T_{-i}} x_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} \text{ and } p_i(t_i) = \int_{T_{-i}} p_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i},$$

where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, $T_{-i} = \prod_{j \neq i}^{n} T_j$, and $f_{-i}(t_{-i}) = \prod_{j \neq i}^{n} f_j(t_j)$. I also define $f(t) = \prod_{i=1}^{n} f_i(t_i)$.

A direct mechanism is *incentive feasible* if and only if the feasibility condition (1) and the incentive compatibility and individual rationality constraints of the buyers are all satisfied. In general, the incentive feasibility of a mechanism depends on the belief of the buyers about the seller's type. Suppose that a direct mechanism M with allocation rule x and payment rule p is selected. Let $u_i(M, s, t_i'|t_i)$ denote buyer i's expected payoff under mechanism M from reporting the type t_i' when his type is t_i and the seller's type is s, given that all other buyers report their types truthfully. Thus,

$$u_i(M, s, t_i'|t_i) = v_i(s, t_i) x_i(t_i') - p_i(t_i').$$

To simplify the notation, I replace $u_i(M, s, t_i|t_i)$ by $u_i(M, s|t_i)$ in the rest of the analysis. Suppose that the posterior belief of the buyers about the seller's type upon observing M is $f_0(\cdot|M)$. I define

$$U_i(M, t_i'|t_i) = \int_S u_i(M, s, t_i'|t_i) f_0(s|M) ds,$$

which is buyer i's expected payoff under mechanism M from reporting the type t_i' when his signal is t_i. I replace $U_i(M, t_i|t_i)$ by $U_i(M|t_i)$ to simplify the notation. The mechanism M is incentive feasible under $f_0(\cdot|M)$ if and only if it satisfies condition (1) and every buyer would like to participate in M and report his type truthfully, given that all other buyers participate and report their types truthfully, i.e., for any i and any $t_i', t_i \in T_i$ and $t_i' \neq t_i$,

$$U_i(M|t_i) \geq U_i(M, t_i'|t_i) \text{ and } U_i(M|t_i) \geq 0.$$

3 Optimal Selling Mechanisms

The sole departure of the model studied in this paper from the one studied in the standard mechanism design literature is that the seller is privately informed about her signal. Does the privacy of the seller's signal affect the design of the mechanism? I answer this question in this section and discuss why, in general, selecting a public-information optimal mechanism fails to be an equilibrium strategy of the seller in the mechanism-selection game. Then, I present the main result of this paper, which characterizes the seller-optimal separating equilibria.

3.1 Public Information Benchmark

When signal s is public, the problem of the seller is a standard mechanism design problem. The seller chooses a mechanism M with an allocation rule x and payment rule p to maximize her expected payoff. (In this subsection, I drop the money-burning rule from the analysis, as the seller never burns money in an optimal mechanism.) The problem of the seller with signal s is

$$\max_{x,p} \int_T \left\{ v_0(s)\, x_0(t) + \sum_{i=1}^{n} p_i(t) \right\} f(t)\, dt$$

$$s.t.\ u_i(M, s|t_i) \geq u_i(M, s, t_i'|t_i), \forall i, t_i, t_i' \in T_i, \tag{2}$$

$$u_i(M, s|t_i) \geq 0, \forall i, t_i \in T_i, \tag{3}$$

$$\sum_{i=1}^{n} x_i(t) \leq 1, x_i(t) \geq 0, \forall i, \forall t \in T.$$

According to Milgrom and Segal (2002), constraints (2) and (3) can be replaced by

$$x_i(t_i) \geq x_i(t_i'), \text{ for each } i, \text{ whenever } t_i > t_i', \tag{4}$$

$$u_i(M, s|t_i) = \int_{\underline{t}_i}^{t_i} \frac{\partial v_i\left(s, \tilde{t}_i\right)}{\partial t_i} x_i\left(\tilde{t}_i\right) d\tilde{t}_i + u_i(M, s|\underline{t}_i), \text{ for all } i \text{ and } t_i \tag{5}$$

$$u_i(M, s|\underline{t}_i) \geq 0, \text{ for all } i. \tag{6}$$

From (5), according to the definition of $u_i(M, s|t_i)$, for all i, all s, and all t_i, we have

$$p_i(t_i) = v_i(s, t_i) x_i(t_i) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial t_i} x_i(\tilde{t}_i) d\tilde{t}_i - u_i(M, s|\underline{t}_i).$$

Substituting $p_i(t_i)$ into the objective function, and then using integration by parts, we obtain

$$v_0(s) + \int_T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] x_i(t) f(t) dt - \sum_{i=1}^{n} u_i(M, s|\underline{t}_i), \tag{7}$$

where $J_i(s, t_i)$ is the virtual valuation of buyer i under (s, t_i):

$$J_i(s, t_i) = v_i(s, t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)} \frac{\partial v_i(s, t_i)}{\partial t_i}.$$

Thus, an optimal mechanism maximizes (7) subject to the feasibility constraint (1) and constraints (4), (5), and (6). Throughout this paper, I impose the following regularity assumption, which is satisfied for commonly used functional forms of v_i when the hazard rate $f_i(t_i) / [1 - F_i(t_i)]$ is non-decreasing in t_i.[6]

Assumption 1 *$J_i(s, t_i)$ is increasing in t_i and $J_i(s, \bar{t}_i) > v_0(s)$ for all i and $s \in S$.*

The following proposition characterizes the profit-maximizing mechanisms under this assumption. This result is standard in the literature (see Myerson, 1981); therefore, I omit its proof.

Proposition 1 *Under Assumption 1, a mechanism (x, p) is optimal for the seller of type $s \in S$ if and only if the allocation rule x satisfies*

$$x \in \arg\max_{\hat{x}} \left\{ \int_T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] \hat{x}_i(t) f(t) dt : \hat{x} \text{ satisfies } (1) \right\},$$

[6]For example, when v_i has the linear form $v_i(s, t_i) = \alpha s + \beta t_i, \alpha, \beta > 0$ or the multiplicative form $v_i(s, t_i) = u(s) t_i$, $u(s) > 0$, it satisfies Assumption 1, given $f_i(t_i) / [1 - F_i(t_i)]$ is non-decreasing in t_i.

and the payment rule p satisfies

$$p_i(t_i) = v_i(s, t_i)\, x_i(t_i) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i\left(s, \tilde{t}_i\right)}{\partial t_i} x_i\left(\tilde{t}_i\right) d\tilde{t}_i, \text{ for all } i, t_i.$$

In any optimal mechanism M for the type-s seller, the expected payoff of the lowest-type buyer i is equal to 0, i.e., $u_i(M, s|\underline{t}_i) = 0$ for all i.

This proposition indicates that in an optimal mechanism for the seller of type s, we have $\sum_{i=1}^{n} x_i(t) = 1$ if $\max_k \{J_k(s, t_k)\}$ is strictly larger than $v_0(s)$, and $x_i(t) > 0$ only if $J_i(s, t_i)$ is larger than $\max\{\max_{k \neq i}\{J_k(s, t_k)\}, v_0(s)\}$. This means that in an optimal mechanism the seller keeps the object if $v_0(s)$ is larger than the virtual valuations of all the buyers and allocates the object to a buyer if the buyer's virtual valuation is the highest among all the buyers and is greater than $v_0(s)$. An optimal allocation automatically satisfies the monotonicity constraint (4), given Assumption 1.

If the buyers are symmetric, i.e., $T_1 = T_i$, $F_1 = F_i$, and $v_1 = v_i$ for all $i > 1$, then the second-price auction with reserve price $v_1(s, r^F(s))$ is optimal for the seller of type $s \in S$, where $r^F : S \to T_1$ satisfies $\left[J_1(s, r^F(s)) - v_0(s)\right]\left[r^F(s) - \underline{t}_1\right] = 0$. The superscript F denotes the public information case, which is sometimes called the full information case in the literature on the informed-principal problem.

Let $\left(x^{F,s}, p^{F,s}\right)$ be an optimal mechanism for the type-s seller and $U_0^F(s)$ the optimal expected payoff of the type-s seller. We have

$$\begin{aligned} U_0^F(s) &= v_0(s) + \int_T \sum_{i=1}^{n} \left[J_i(s, t_i) - v_0(s)\right] x_i^{F,s}(t)\, f(t)\, dt \\ &= v_0(s)\, x_0^{F,s} + \int_T \sum_{i=1}^{n} J_i(s, t_i)\, x_i^{F,s}(t)\, f(t)\, dt, \end{aligned}$$

where $x_0^{F,s} = 1 - \int_T \sum_{i=1}^{n} x_i^{F,s}(t)\, f(t)\, dt$. Define

$$g(s, x) = v_0(s)\, x_0 + \int_T \sum_{i=1}^{n} J_i(s, t_i)\, x_i(t)\, f(t)\, dt, \tag{8}$$

where $x_0 = 1 - \int_T \sum_{i=1}^n x_i(t) f(t)\, dt$. It is easy to verify that $g(s,x)$ is absolutely continuous and differentiable with respect to s for any feasible allocation rule x. Let $g_1(s,x)$ be the partial derivative of g with respect to s. Because the derivatives are all bounded, there exists a sufficiently large number d such that for all s,

$$\sup_x |g_1(s,x)| = \sup_x \left| v_0'(s)\, x_0 + \int_T \sum_{i=1}^n \frac{\partial J_i(s,t_i)}{\partial s} x_i(t) f(t)\, dt \right| \leq d.$$

Therefore, according to Theorem 2 of Milgrom and Segal (2002), we have

$$U_0^F(s) = \int_{\underline{s}}^s g_1\left(\tilde{s}, x^{F,\tilde{s}}\right) d\tilde{s} + U_0^F(\underline{s}). \tag{9}$$

3.2 Optimal Mechanism for Privately Informed Seller

In this subsection, I switch to the case where the seller is privately informed about her signal. I first discuss how the profile of public-information optimal mechanisms $\{(x^{F,s}, p^{F,s}) : s \in S\}$ fails to be an equilibrium strategy of the seller in this private information environment. Then I characterize the strategies of the seller in the seller-optimal separating equilibria, and show the optimality of reserve prices in signaling the type of the seller.

3.2.1 Public Information Benchmark is not Implementable

As I show in (18) in the next section, if the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$ is an equilibrium strategy of the seller, then

$$U_0^F(s) = \int_{\underline{s}}^s v_0'(\tilde{s})\, x_0^{F,\tilde{s}} d\tilde{s} + U_0^F(\underline{s}). \tag{10}$$

Comparing equations (9) and (10), we have

$$\int_{\underline{s}}^s g_1\left(\tilde{s}, x^{F,\tilde{s}}\right) d\tilde{s} - \int_{\underline{s}}^s v_0'(\tilde{s})\, x_0^{F,\tilde{s}}(\tilde{s})\, d\tilde{s} = \int_{\underline{s}}^s \int_T \sum_{i=1}^n \frac{\partial J_i(\tilde{s},t_i)}{\partial s} x_i^{F,\tilde{s}}(t) f(t)\, dt d\tilde{s}. \tag{11}$$

Therefore, as long as the difference in (11) is not zero, $\left\{\left(x^{F,s}, p^{F,s}\right) : s \in S\right\}$ is not an equilibrium strategy of the seller.

If the seller's signal does not affect the buyers' valuations of the object, i.e., the model is a private value model, then $\partial J_i(s, t_i)/\partial s = 0$ for all (s, t_i), and the difference in (11) is equal to 0. In this case, the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$ is a seller-optimal separating equilibrium strategy, and the privacy of seller's information is irrelevant.

The rest of the paper focuses on the environments where the difference in (11) is positive by imposing the following assumption, which ensures that the surplus that the seller can extract from any buyer is increasing in s.[7]

Assumption 2 *$J_i(s, t_i)$ is strictly increasing in s for all $t_i \in T_i$, $i = 1, \ldots, n$.*

Under this assumption, I show in the following proposition the reason that the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$ fails to be an equilibrium strategy.[8]

Proposition 2 *Given Assumption 2, a lower-type seller has an incentive to mimic a higher-type one under $\left\{\left(x^{F,s}, p^{F,s}\right) : s \in S\right\}$ if all buyers truthfully report their types.*

The reason is simple: by deviating to the public-information optimal mechanism of a higher-type seller, a lower-type seller can extract more surplus from buyers in trade (Assumption 2), even though she may suffer from a decrease in the probability of trade, which is of second order compared with the increase in trade surplus. The proof of this proposition is in the appendix.

3.2.2 Separating through Reserve Prices

Since under the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$, lower-type sellers have an incentive to mimic higher-type ones, to separate different types of the seller we should reduce the profitability

[7] This assumption is also satisfied for some commonly adopted functional forms for v_i. For example, (1) the linear form $v_i(s, t_i) = \alpha s + \beta t_i$, $\alpha, \beta > 0$, and (2) the multiplicative form $v_i(s, t_i) = u(s) \cdot t_i$, with $t_i - (1 - F_i(t_i))/f_i(t_i) > 0$ for any t_i.

[8] In the case where the buyers are symmetric, to make a lower type seller have an incentive to mimic a higher type seller under strategy $\left\{\left(x^{F,s}, p^{F,s}\right) : s \in S\right\}$, assuming that $v_i(s, t_i)$ is increasing in s, instead of Assumption 2, is sufficient. However, for the asymmetric case, it is unclear whether Assumption 2 can be replaced.

of the mechanisms adopted by the higher-type sellers, so as to deter lower-type sellers from mimicking higher-type ones. The following theorem implies that raising the reserve prices in the public-information optimal mechanisms of higher-type sellers is the least costly way of achieving separation.

Theorem 1 *Under Assumptions 1 and 2, there exists a seller-optimal separating equilibrium, which maximizes the (interim) expected profit of every type of the seller among all possible separating equilibria. A strategy $\{(x^s, p^s, b^s) : s \in S\}$ is the equilibrium strategy of the seller in a seller-optimal separating equilibrium if and only if the allocation rule x^s satisfies*

$$x^s \in \arg\max_{\hat{x}} \left\{ \int_T \sum_{i=1}^n \left[J_i(s,t_i) - v_0(s)\right] \hat{x}_i(t) f(t)\, dt \text{: } \hat{x} \text{ satisfies (1) and } \hat{x}_0 = x_0^s \right\}, \quad (12)$$

where x_0^s is increasing in s and satisfies $x_0^{\underline{s}} = x_0^{F,\underline{s}}$, $x_0^s > x_0^{F,s}$, for all $s > \underline{s}$, and

$$v_0(s) + \int_T \sum_{i=1}^n \left[J_i(s,t_i) - v_0(s)\right] x_i^s(t) f(t)\, dt = \int_{\underline{s}}^s v_0'(\tilde{s})\, x_0^{\tilde{s}} d\tilde{s} + U_0^F(\underline{s}), \text{ for every } s, \quad (13)$$

and the payment rule p^s satisfies

$$p_i^s(t_i) = v_i(s,t_i)\, x_i^s(t_i) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i\left(s,\tilde{t}_i\right)}{\partial t_i} x_i^s\left(\tilde{t}_i\right) d\tilde{t}_i, \text{ for every } i, s, \text{and } t_i. \quad (14)$$

The lowest type of each buyer gets 0 expected payoff, and the seller never burns money, i.e.,

$$u_i\left(M^s, s|\underline{t}_i\right) = 0 \text{ and } b^s(t) = 0, \text{ for every } i, s, \text{and } t.$$

To understand this theorem, let us compare it with Proposition 1, which characterizes the public-information optimal mechanisms. In these two results, the payment rules for the seller have the same structure, and the expected payoff of the lowest type of every buyer and the amount of money burned are both equal to 0. The major difference between the mechanisms in these two results lies in the allocation rules. In the current theorem, condition

(12) indicates that the object is still allocated to a buyer who has the highest virtual valuation among all the buyers when there is trade, i.e., $x_i(t) > 0$ only if $J_i(s, t_i) \geq \max_{k \neq i} \{J_k(s, t_k)\}$. The probability of trade, however, is lower than it is in the public-information optimal mechanism if the seller has type $s > \underline{s}$, as $x_0^s > x_0^{F,s}$ for all $s > \underline{s}$. Condition (13) ensures that the strategy of the seller is incentive compatible. The decrease in the probability of trade is associated with raising the reserve prices. The following corollary for the case of symmetric buyers illustrates the increase in the reserve prices.

Corollary 1 *Under Assumptions 1 and 2, if the buyers are symmetric, there exists a seller-optimal separating equilibrium in which a seller of type s chooses the second-price auction with reserve price $v_1(s, r(s))$, where $r(s)$ is the minimal type of buyers that can get the object with positive probability. The function $r : S \to T_1$ is increasing in s and satisfies $J_1(s, r(s)) - v_0(s) \geq 0$, for all s, which holds with equality if and only if $s = \underline{s}$.*

In the public information case, as pointed out below Proposition 1, it is optimal for a seller of type s to choose the second-price auction with reserve price $v_1(s, r^F(s))$, where $r^F(s)$ satisfies $J_1(s, r^F(s)) - v_0(s) = 0$. In the private information case, the reserve price set by a seller of type $s > \underline{s}$ becomes $v_1(s, r(s)) > v_1(s, r^F(s))$, as $J_1(s, r(s)) - v_0(s) > 0$, in a seller-optimal separating equilibrium. The increased reserve prices make the strategy of the seller incentive compatible.[9]

In the equilibria characterized in the theorem, some high types of the seller may choose to sell the object with probability 0. If this is the case, then the equilibria, to be precise, are partial pooling equilibria. However, the types of the seller that pool are out of the market,

[9]In the asymmetric case, different buyers potentially face different reserve prices under the same mechanism in a seller-optimal separating equilibrium. However, all buyers' reserve prices correspond to the same virtual valuation. I illustrate how the seller sets the reserve prices when her type is $s > \underline{s}$. Define $r_i(s, \underline{J})$ as the type of buyer i having virtual valuation $\underline{J}$, so $J_i(s, r_i(s, \underline{J})) = \underline{J}$. Given Assumption 1, $r_i(s, \underline{J})$ is increasing in $\underline{J}$. If the seller keeps the object with probability x_0^s, then she chooses $\underline{J}$ such that $\prod_{i=i}^n F_i(r_i(s, \underline{J})) = x_0^s$. The resulting $r_i(s, \underline{J})$ is the minimum type of the buyer i that can get the object with positive probability, and the reserve price for buyer i is consequently $v_i(s, r_i(s, \underline{J}))$, according to the payment rule (14). In a public-information optimal mechanism, $\underline{J} = v_0(s)$, so $x_0^{F,s} = \prod_{i=i}^n F_i(r_i(s, v_0(s)))$. In Theorem 1, since $x_0^s > x_0^{F,s}$ for $s > \underline{s}$, we have $\underline{J} > v_0(s)$ and $r_i(s, \underline{J}) > r_i(s, v_0(s))$. Condition (13), which is to ensure the incentive compatibility of the seller's strategy, determines the value of $\underline{J}$ chosen by each type of the seller.

while the ones in the market still fully separate from each other. Thus, I still treat these equilibria as separating equilibria.

Now I interpret Theorem 1. Proposition 2 points out that in the private information case, if the seller adopts $\{(x^{F,s}, p^{F,s}) : s \in S\}$ as her strategy and the buyers report their types truthfully, then a lower-type seller has an incentive to pretend to be a higher type, because this allows her to sell the object at a relatively higher price, even though the probability of trade might be reduced. Given that the expected revenue of every type of seller is determined by x and $\sum_{i=1}^{n} u_i (M, s|\underline{t}_i)$ (see (7)), we can disincentivize lower-type sellers from mimicking higher-type ones by changing the higher-type ones' allocation rule x and/or $\sum_{i=1}^{n} u_i (M, s|\underline{t}_i)$. For example, (1) we increase the reserve prices of the mechanism adopted by a higher-type seller, so as to decrease the probability of selling the object; (2) we change the allocation rule from always allocating the object to the buyer with the highest virtual valuation, while maintaining the monotonicity of the allocation rule (required by (4)); (3) we can increase the expected payoffs $u_i (M, s|\underline{t}_i)$, $i = 1, 2, \ldots, n$, of the lowest types of the buyers, which is equivalent to uniformly decreasing the payment of the buyers, or burning money. The first two approaches involve changes in the allocation rule x only, and the third approach changes only $\sum_{i=1}^{n} u_i (M, s|\underline{t}_i)$. Approaches (1) and (3) are straightforward and easy to implement. Approach (2) is open-ended, as it can be done in many different ways.

Theorem 1 tells us that from the seller's perspective, the first approach outperforms all other possible approaches, including the other two mentioned above. This is because the first approach is less costly than any other approach for higher-type sellers to separate themselves from lower-type ones. The reasons are simple. To eliminate the incentive of a lower-type seller to misreport upward, we need to reduce her revenue gain from doing so. To reduce the revenue gain by a certain amount, the third approach forces a higher-type seller to give up the same amount of revenue. The first approach, which increases the probability that the seller keeps the object with no payment, induces less payoff loss to a higher-type seller, because a higher-type seller values the object more than a lower type.[10] For any other

[10] When the reserve prices are increased, the payments of the buyers are higher in the case where there is

approach of deterring lower-type sellers from mimicking higher types, I show through the proof of Proposition 3, which is an intermediate step in proving Theorem 1, that it is no better than some combination of the first and third approaches, so cannot outperform the first approach.

This theorem provides support to the literature on reserve price signaling (Jullien and Mariotti, 2006; Cai et al., 2007). In the models in that literature, the selling mechanisms have fixed formats: the seller who faces symmetric buyers is allowed to change only the reserve prices. The theorem shows that even if the seller has the freedom to vary every element of the mechanism, she would still stick to using the reserve prices to separate different types when the signals of the players are independent.

Theorem 1 characterizes only the seller-optimal separating equilibria of the mechanism selection game. It is challenging to perform standard equilibrium refinements, such as the Intuitive Criterion (Cho and Kreps, 1987) and D1 Criterion (Banks and Sobel, 1987), to rule out other non-separating equilibria, due to the complexity of the action space of the seller. It is also challenging to identify the conditions under which the seller-optimal separating equilibria are not dominated by non-separating equilibrium in terms of the seller's interim expected payoff.[11] However, the message that signaling through reserve prices is optimal is useful beyond separating equilibria. Consider a partial pooling equilibrium in which, for some $s' \in (\underline{s}, \bar{s})$, the types of the seller in $[\underline{s}, s']$ pool, while all other types separate by choosing different mechanisms. If among the separated types, there exists a type s'' that does not adopt a mechanism derived by raising only the reserve prices of her public-information optimal mechanism, then there exists another partial pooling equilibrium in which the types in $[\underline{s}, s']$ also pool and all types get higher expected payoffs, and an open interval of types in $(s'', \bar{s}]$ get strictly higher expected payoff.

only one buyer reporting a type higher than his reserve price. However, compared with the effect of reserve prices on the probability of trade, this effect is of second order.

[11]In contrast, it is easy to find sufficient conditions under which the seller-optimal separating equilibria are dominated by other equilibria. For example, if the belief of buyers is sufficiently concentrated around the highest type $\bar{s}$ of the seller, there exists a fully pooling equilibrium dominating the seller-optimal separating equilibria. The intuition can be borrowed from the analysis of Spence (1973) for the case where the proportion of high type is large.

Assumption 2 is crucial for the optimality of reserve price signaling. If instead we assume that $J_i(s, t_i)$ is decreasing in s for all i and $t_i \in T_i$, then Theorem 1 no longer holds. This alternative assumption fits cases where the object for sale is complementary to other assets owned by the seller, and s captures the degree of complementarity. Under the alternative assumption, a higher-type seller has an incentive to pretend to be a lower-type seller under the strategy that each type of the seller adopts her public-information optimal mechanism. To separate the lower-type sellers from the higher-type ones, we should make the mechanisms adopted by the lower types less profitable. Though Theorem 1 does not hold in this new environment, the intuition underlying the theorem still applies in characterizing the seller-optimal separating equilibria: the mechanism chosen by each type of the seller differs from this type's public-information optimal mechanism only in the expected payoffs of the lowest types of the buyers or the amount of money burned. This is because the third approach mentioned above in interpreting Theorem 1, which is to uniformly decrease the payment of the buyers or burn money, now becomes optimal.

I prove Theorem 1 using another game of mechanism selection in which the seller herself is an active participant in the mechanism. This auxiliary game is the standard game studied in that literature on mechanism design by an informed principal. Since implementing a mechanism in this game requires a neutral mediator, I call this game the *mediated mechanism-selection game*, or *mediated game* for simplicity, in the rest of the analysis, to distinguish it from the main mechanism-selection game I study. Introducing this mediated game facilitates the proof of the theorem by making it easier to show that other separating equilibria are dominated by the ones characterized in Theorem 1. For the mediated game, there is a class of mechanisms named *safe mechanisms*, which are incentive feasible regardless of the buyers' beliefs about the seller's type. The mechanisms that maximize the interim expected payoff of the principal among all safe mechanisms are called *Rothschild-Stiglitz-Wilson (RSW) mechanisms.* The equilibrium strategy of the seller in any separating equilibrium of the mechanism-selection game corresponds to a safe mechanism of the mediated game. I prove that a safe mechanism is an RSW mechanism if and only if it satisfies the conditions in

Theorem 1, and any RSW mechanism corresponds to the equilibrium strategy of the seller in some separating equilibrium of the mechanism-selection game. Directly proving the theorem involves a full characterization of the set of separating equilibria, which requires one to specify a belief system for each equilibrium. My indirect approach also yields an independently interesting result: a full characterization of the set of RSW mechanisms. RSW mechanisms play an important role in the equilibrium analysis of informed-principal problems where the principals' information enters agents' valuation functions (Maskin and Tirole, 1992). Fully characterizing the set of RSW mechanisms is potentially useful for analyzing this class of informed-principal problems.

4 Mediated Game and Proof of Theorem 1

In this section, I study the mediated game, in which the seller is an active participant in the selected mechanism. The specifics of this game are the same as those of the main mechanism-selection game, except that now a mechanism has an $(n+1)$-dimensional message space $\Lambda = \Lambda_0 \times \prod_{i=1}^{n} \Lambda_i$, where $\Lambda_0 \subset \mathbb{R}$ is the set of possible messages for the seller and $\Lambda_i \subset \mathbb{R}$ is the set of possible messages for buyer i, and the implementation of a mechanism requires the participation of the seller. Most of the analysis is still restricted to direct incentive feasible mechanisms, according to the revelation principle. The message space of a direct mechanism is $S \times T$. I use $x : S \times T \to [0,1]^n$, $p : S \times T \to \mathbb{R}^n$, and $b : S \times T \to \mathbb{R}_+$ to denote the allocation rule, payment rule, and money-burning rule of a direct mechanism. The allocation rule x satisfies the feasibility constraint

$$0 \leq x_i(s,t), \sum_{i=1}^{n} x_i(s,t) \leq 1 \text{ for all } i,\ s, \text{ and } t. \tag{15}$$

Let $x_0(s,t)$ denote the probability that the seller keeps the object when the message profile is (s,t), i.e., $x_0(s,t) = 1 - \sum_{i=1}^{n} x_i(s,t)$. I define

$$x_i(s,t_i) = \int_{T_{-i}} x_i(s,t_i,t_{-i}) f_{-i}(t_{-i})\, dt_{-i} \text{ and } p_i(s,t_i) = \int_{T_{-i}} p_i(s,t_i,t_{-i}) f_{-i}(t_{-i})\, dt_{-i},$$

and $x_0(s) = \int_T x_0(s,t) f(t)\, dt$, and $b(s) = \int_T b(s,t) f(t)\, dt$.

4.1 Incentive Feasible Mechanisms

Since the mechanism is proposed after the seller learns her type, the choice of mechanism may partially or completely reveal the seller's information. Suppose that a direct mechanism $M = (x,p,b)$ is proposed by the seller. Let $\bar{U}_0(M,s'|s)$ denote the type-s seller's expected payoff under mechanism M from reporting her type as s' given that all buyers report their types truthfully. Thus,

$$\bar{U}_0(M,s'|s) = \int_T \left\{ v_0(s)\, x_0(s',t) + \sum_{i=1}^{n} p_i(s',t) - b(s',t) \right\} f(t)\, dt.$$

To simplify the notation, I replace $\bar{U}_0(M,s|s)$ by $\bar{U}_0(M|s)$ in the following analysis. If M is incentive feasible, then condition (15) is satisfied by its allocation rule x, and the seller with $s \in S$ has an incentive to participate and report truthfully, i.e., for any $s' \in S$ and $s' \neq s$,

$$\bar{U}_0(M|s) \geq \bar{U}_0(M,s'|s) \text{ and } \bar{U}_0(M|s) \geq v_0(s). \tag{16}$$

Standard techniques show that constraints (16) are equivalent to

$$x_0(s) \geq x_0(s') \text{ whenever } s > s', \tag{17}$$

$$\bar{U}_0(M|s) = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + \bar{U}_0(M|\underline{s}) \text{ for all } s, \text{ and} \tag{18}$$

$$\bar{U}_0(M|\bar{s}) \geq v_0(\bar{s}). \tag{19}$$

Let $\bar{u}_i(M, s, t_i'|t_i)$ denote the expected payoff of buyer i under mechanism M from reporting his type as t_i', given that his type is t_i, the seller's type is s, and all other players report their types truthfully. Thus,

$$\bar{u}_i(M, s, t_i'|t_i) = v_i(s, t_i) \, x_i(s, t_i') - p_i(s, t_i').$$

As before, I use $\bar{u}_i(M, s|t_i)$ in place of $\bar{u}_i(M, s, t_i|t_i)$ to simplify the notation. If the posterior of the buyers about the seller's type is $f_0(\cdot|M)$, then the expected payoff $\bar{U}_i(M, t_i'|t_i)$ of buyer i under mechanism M from reporting the type t_i', given that his type is t_i and all other players report their types truthfully, is

$$\bar{U}_i(M, t_i'|t_i) = \int_S \bar{u}_i(M, s, t_i'|t_i) f_0(s|M) \, ds.$$

Again, I replace $\bar{U}_i(M, t_i|t_i)$ by $\bar{U}_i(M|t_i)$ to simplify the notation. The incentive feasibility of M requires that for any i and any $t_i', t_i \in T_i$ and $t_i' \neq t_i$,

$$\bar{U}_i(M|t_i) \geq \bar{U}_i(M, t_i'|t_i) \text{ and } \bar{U}_i(M|t_i) \geq 0. \tag{20}$$

Reformulating the constraints of the buyers is not necessary for the analysis. However, one should note that if the seller's signal is public, then (20) can be reformulated to a set of constraints analogous to (4), (5), and (6).

I construct a mechanism $M^F = \left(x^F, p^F, b^F\right)$ with

$$x^F(s, t) = x^{F,s}(t), p^F(s, t) = p^{F,s}(t), \text{ and } b^F(s, t) = 0,$$

for all $(s, t) \in S \times T$. It is easy to see that M^F satisfies the IC and IR constraints of the buyers in (20), regardless of the belief of the buyers about the seller's type, as $\left\{\left(x^{F,s}, p^{F,s}\right) : s \in S\right\}$ satisfies (2) and (3). Thus, if all other players report truthfully, it is incentive compatible for a buyer to report his type truthfully. But the IC constraints of the seller in (16) are violated, as I have shown in Proposition 2.

4.2 Inscrutability Principle

In general, different types of seller can select different mechanisms in an equilibrium of this game. The associated signaling problem potentially complicates the analysis. Thanks to the *inscrutability principle* introduced in Myerson (1983), we can assume without loss of generality that all types of the seller in equilibrium propose the same mechanism; thus the choice of mechanism reveals no private information of the seller on the equilibrium path, and the posterior of the buyers after observing the mechanism is the same as their prior. The inscrutability principle is formally stated in the following lemma. All the proofs in this section are relegated to the appendix.

Lemma 1 (Inscrutability Principle, Myerson (1983)) *In the mediated game, for any equilibrium, there exists another equilibrium in which the seller's choice of mechanism is independent of her type and each type of the seller gets the same expected payoff as in the original equilibrium.*

This lemma enables us to focus the analysis on cases where all types of the seller propose the same mechanism.

4.3 Safe Mechanisms and Proof of Theorem 1

In general, whether a proposed mechanism is incentive feasible for the buyers depends on their beliefs about the seller's type. However, there is a set of mechanisms whose incentive feasibility is independent of the buyers' beliefs. These are called *safe mechanisms*, which were first studied by Myerson (1983).

Definition 2 *A direct mechanism is safe if it is incentive feasible for the seller, and if it is incentive feasible for the buyers conditional on every possible seller's type.*

According to the definition, if mechanism M is safe, then it satisfies constraints (17), (18), and (19) for the seller, and satisfies the following constraints for the buyers: for any i, s, and any $t'_i, t_i \in T_i$ and $t'_i \neq t_i$,

$$\bar{u}_i(M, s|t_i) \geq \bar{u}_i(M, s, t'_i|t_i) \text{ and } \bar{u}_i(M, s|t_i) \geq 0. \tag{21}$$

The incentive feasibility of a safe mechanism M is independent of $f_0(\cdot|M)$, because regardless of $f_0(\cdot|M)$, (21) implies (20).

Safe mechanisms are interesting because any separating-equilibrium strategy of the seller in the main mechanism-selection game corresponds to a safe mechanism in the mediated game. If among safe mechanisms, we can find one that dominates every other safe mechanism in terms of the interim expected payoff of the seller, then this safe mechanism must weakly dominate the equilibrium strategy of the seller in any seller-optimal separating equilibrium of the mechanism-selection game. In the analysis below, I show that there indeed exists such an optimal safe mechanism.

The next lemma characterizes the set of safe mechanisms. It can be proved using standard techniques based on (17), (18) for the seller, and (21) for the buyers; therefore, I omit its proof. The set of safe mechanisms is non-empty and convex.

Lemma 2 *A direct feasible mechanism M is safe if and only if it satisfies the following conditions,*

1. $x_0(s) \geq x_0(s')$ *and, for each* i, $x_i(s, t_i) \geq x_i(s, t'_i)$, *whenever* $s > s', t_i > t'_i$.
2. $$v_0(s) + \int_T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] x_i(s, t) f(t) dt - \sum_{i=1}^{n} \bar{u}_i(M, s|\underline{t}_i) - b(s) = \int_{\underline{s}}^{s} v'_0(\tilde{s}) x_0(\tilde{s}) d\tilde{s} + \bar{U}_0(M|\underline{s}) \;\; \forall s. \tag{22}$$
3. $p_i(s, t_i) = v_i(s, t_i) x_i(s, t_i) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial t_i} x_i(s, \tilde{t}_i) d\tilde{t}_i - \bar{u}_i(M, s|\underline{t}_i) \;\; \forall i, s, t_i$.
4. $\bar{U}_0(M|\bar{s}) \geq v_0(\bar{s}), \bar{u}_i(M, s|\underline{t}_i) \geq 0 \;\; \forall i, s$.

In the following proposition, I characterize the safe mechanisms that maximize the expected payoff of every type of the seller among all safe mechanisms. Maskin and Tirole (1992) call these mechanisms *Rothschild-Stiglitz-Wilson mechanisms*. Formally, a safe mechanism M^* is an RSW mechanism if for any s,

$$\bar{U}_0(M^*|s) = \max_{M=(x,p,b)} \bar{U}_0(M|s)$$
$$s.t.\ M \text{ satisfies } (17), (18), (19), \text{ and } (21)$$

RSW mechanisms, if they exist, are of particular interest in the mediated game. First, RSW mechanisms determine the minimum equilibrium payoff of each type of the seller in the mediated game. Second, if RSW mechanisms are not dominated by any other mechanism in terms of the interim expected payoff of the seller, then they coincide with the set of equilibrium strategies of the mediated game.[12]

The following proposition shows that an RSW mechanism can be derived by raising the reserve prices of M^F. The lowest-type seller gets the same expected payoff as she obtains under M^F, but all other types of the seller are worse off than they are under M^F.

Proposition 3 *Under Assumptions 1 and 2, a safe mechanism M^* is an RSW mechanism if and only if it satisfies the following three conditions.*

1. *The allocation rule x^* has the properties that $x^*(s, \cdot)$ solves*

$$\max_{\hat{x}} \int_T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)]\, \hat{x}_i(t) f(t)\, dt$$
$$s.t.\ \hat{x} \text{ satisfies (1) and } \hat{x}_0 = x_0^*(s),$$

   and $x_0^*(\underline{s}) = x_0^F(\underline{s})$, $x_0^*(s) > x_0^F(s)$ for all $s > \underline{s}$.

[12] In the terminology of Myerson (1983), an undominated RSW mechanism is a *strong solution* to the informed-principal game. A strong solution is an *expectational equilibrium*, a *core mechanism*, and a *neutral optimum*, which are all solution concepts to the informed-principal game, but with increasing strengths.

2. *The lowest type of every buyer gets* 0 *expected payoff, and the seller never burns money, i.e.,*

$$\bar{u}_i\left(M^*, s|\underline{t}_i\right) = 0 \text{ and } b^*\left(s\right) = 0, \text{for all } s, i.$$

3. *The lowest type of the seller gets her optimal public-information payoff, i.e.,*

$$\bar{U}_0\left(M^*|\underline{s}\right) = \bar{U}_0\left(M^F|\underline{s}\right).$$

All RSW mechanisms have essentially the same allocation rule.

An RSW mechanism differs from M^F, which is characterized in Proposition 1, only in reserve prices. Given this proposition, we can prove the existence of an RSW mechanism by proving the existence of an allocation rule x satisfying the conditions in Lemma 2, with $\sum_{i=1}^{n}\bar{u}_i\left(M, s|\underline{t}_i\right) = 0$ and $\bar{U}_0\left(M|\underline{s}\right) = U_0\left(M^F|\underline{s}\right)$, and the first condition in Proposition 3.

Proposition 4 *There exists an RSW mechanism. If M^* is an RSW mechanism, then $x_0^*\left(s\right)$ is continuous and strictly increasing in s as long as $x_0^*\left(s\right) < 1$.*

RSW mechanisms are naturally connected with the seller-optimal separating equilibria in the mechanism-selection game. As mentioned above, every separating-equilibrium strategy of the seller in the mechanism-selection game corresponds to a safe mechanism, so any RSW mechanism gives the seller weakly higher interim expected payoffs than does any separating equilibrium. Proposition 4 readily shows that in any RSW mechanism, different types of the seller in the market (i.e., the types of the seller who trade with positive probabilities) choose different rules of allocating the object. Thus, RSW mechanisms are natural candidates for the strategies of the seller in seller-optimal separating equilibria. The following proposition, combined with Propositions 3 and 4, completes the proof of Theorem 1.

Proposition 5 *If the mechanism $M^* = (x^*, p^*, b^*)$ is an RSW mechanism, then $\left\{\left(x^*\left(s,\cdot\right), p^*\left(s,\cdot\right), b^*\left(s,\cdot\right)\right) : s \in S\right\}$ is the equilibrium strategy of the seller in a seller-optimal separating equilibrium of the mechanism-selection game, in which the seller of type s chooses*

$(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot))$, *and the belief of the buyers upon observing any off-equilibrium mechanism is that the seller is of type* $\underline{s}$.

To see why the three propositions above imply Theorem 1, first consider the "if" part of the theorem. If the strategy $\{(x^s, p^s, b^s) : s \in S\}$ of the seller in the mechanism-selection game satisfies the conditions in Theorem 1, then the mechanism M^* with

$$x^*(s,t) = x^s(t), p^*(s,t) = p^s(t), b^*(s,t) = b^s(t), \text{for every } s, t,$$

is an RSW mechanism, due to Lemma 2 and the "if" part of Proposition 3. Then using Proposition 5, we can conclude that $\{(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot)) : s \in S\}$—which is exactly $\{(x^s, p^s, b^s) : s \in S\}$—is the equilibrium strategy of the seller in a seller-optimal separating equilibrium of the mechanism-selection game.

Now consider the "only if" part of the theorem. Suppose that the strategy $\{(x^s, p^s, b^s) : s \in S\}$ of the seller is the equilibrium strategy in a seller-optimal separating equilibrium of the mechanism-selection game. Then Propositions 4 and 5 imply that $\{(x^s, p^s, b^s) : s \in S\}$ corresponds to an RSW mechanism of the mediated game; otherwise it cannot be a seller-optimal separating-equilibrium strategy in the mechanism-selection game. According to Lemma 2 and the "only if" part of Proposition 3, we can conclude that the profile $\{(x^s, p^s, b^s) : s \in S\}$ satisfies the conditions in Theorem 1.

5 Concluding Remarks

In this paper, I study the design of a selling mechanism by a privately informed seller. In general, the privacy of the seller's information affects mechanism design: a lower-quality seller would like to adopt the public-information optimal mechanism of a higher-quality seller. However, the privacy of the information does not lead to unfamiliar and complicated selling mechanisms in equilibrium. In characterizing the seller-optimal separating equilibria of this game, I find that the equilibrium mechanism chosen by each type of the seller differs from her public-information optimal mechanism only in reserve prices. This characterization unveils an interesting role of reserve prices: reserve prices are the optimal device for separating different types of the principal.

This finding regarding the role of reserve prices may help to simplify the analysis of some informed-principal problems by reducing the infinite-dimensional signaling problems to finite-dimensional (one-dimensional, in the case of symmetric buyers) ones in which only the reserve prices are the signals. Examples of these informed-principal problems include multi-unit auction design by an informed seller, optimal auction design with resale opportunities, and auction design by competing informed sellers.

The analysis of this paper focuses on a special interdependent value environment in which a buyer's valuation of the object depends only on the seller's information and his own private information. Allowing a buyer's valuation to depend on other buyers' information does not change the results if the dependence is linear. However, whether the results hold in more general interdependent value environments is unknown.

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Appendix

Proof of Proposition 2

Let $U_0^F(\hat{s}|s)$ denote the expected payoff of the type-s seller from choosing the public-information optimal mechanism of the type-$\hat{s}$ seller, if the buyers report their types truthfully. We have

$$\begin{aligned} U_0^F(\hat{s}|s) &= \int_T \left\{ v_0(s)\, x_0^{F,\hat{s}}(t) + \sum_{i=1}^n p_i^{F,\hat{s}}(t) \right\} f(t)\, dt \\ &= U_0^F(\hat{s}) - [v_0(\hat{s}) - v_0(s)]\, x_0^{F,\hat{s}}, \end{aligned}$$

where the second equality is due to the definition of $U_0^F(\hat{s})$. To examine the incentive of the type-s seller to deviate, we take the difference between $U_0^F(\hat{s}|s)$ and $U_0^F(s)$, and obtain

$$\begin{aligned} U_0^F(\hat{s}|s) - U_0^F(s) &= \int_s^{\hat{s}} g_1\left(\tilde{s}, x^{F,\tilde{s}}\right) d\tilde{s} - \int_s^{\hat{s}} v_0'(\tilde{s})\, x_0^{F,\hat{s}} d\tilde{s} \\ &= \int_s^{\hat{s}} \left[v_0'(\tilde{s}) \left(x_0^{F,\tilde{s}} - x_0^{F,\hat{s}} \right) + \int_T \sum_{i=1}^n \frac{\partial J_i(\tilde{s}, t_i)}{\partial s} x_i^{F,\tilde{s}}(t)\, f(t)\, dt \right] d\tilde{s}, \end{aligned}$$

where the first equality is due to (9) and the second equality is derived according to the definition of $g_1(s,x)$ in (8).

For $s \in [\underline{s}, \bar{s})$, let $\hat{s} = s + \Delta$. When Δ is an arbitrarily small positive number, due to the continuity of $x_0^{F,s}$ in s, $v_0'(\tilde{s})\,(x_0^{F,\tilde{s}} - x_0^{F,\hat{s}})$ is arbitrarily small for $\tilde{s} \in [s, \hat{s})$.[13] However, the second term in the bracket above is bounded from zero. Thus, we have $U_0^F(\hat{s}|s) - U_0^F(s) > 0$. That is, the type-$s$ seller gets better off from deviating (locally) upwards.

Proof of Lemma 1

[13] The continuity of $x_0^{F,s}$ can be easily proved. Let $r_i(s)$ be the "reserve price" set by the seller of type s for buyer i. That is, if buyer i has type $t_i < r_i(s)$, he has no chance to get this object regardless of the types of other buyers. The value of $r_i(s)$ is determined by $J_i(s, r_i(s)) - v_0(s) = 0$. The differentiability of J_i with respect to its two arguments implies that $r_i(s)$ is differentiable in s. The definition of $x_0^{F,s}$ gives that $x_0^{F,s} = \prod_{i=1}^n F_i(r_i(s))$. Distributions $F_i, i = 1, 2, \ldots, n$, are continuous in r_i, thus $x_0^{F,s}$ is continuous in s.

I prove the result only for pure-strategy equilibria. The proof for mixed-strategy equilibria can be derived easily based on the discussion below. Consider a partition $\left\{S^l\right\}_{l\in I}$ of S, where I is a set of index that can be finite or infinite. The partition $\left\{S^l\right\}_{l\in I}$ is defined in a way that different types of the seller in the same element S^l choose the same mechanism M^l in equilibrium. By choosing mechanism M^l in equilibrium, the seller signals the buyers that her type belongs to S^l. According to the revelation principle, we can assume that M^l is a direct incentive feasible mechanism, without loss of generality. That is, M^l satisfies (15), (16), and (20). In equilibrium, we have

$$\bar{U}_0\left(M^l|s\right) \geq \bar{U}_0(M^{l'}|s), \text{ for } s \in S^l, l' \neq l. \tag{23}$$

I construct an inscrutable mechanism $M = (x, p, b)$: for $s \in S^l, l \in I$

$$x\left(s,t\right) = x^l\left(s,t\right), p\left(s,t\right) = p^l\left(s,t\right), b\left(s,t\right) = b^l\left(s,t\right),$$

where $x^l\left(s,t\right)$, $p^l\left(s,t\right)$, and $b^l\left(s,t\right)$ are the allocation rule, payment rule, and money-burning rule of mechanism M^l, respectively. It is clear that if every type of the seller chooses the mechanism M, then M satisfies constraint (20) of the buyers, because for any i, t_i,

$$\begin{aligned}
\bar{U}_i(M|t_i) &= \int_I \left[\bar{U}_i(M^l|t_i)\int_{S^l} f_0\left(s\right)ds\right]dl \\
&\geq \int_I \left[\bar{U}_i\left(M^l, t_i'|t_i\right)\int_{S^l} f_0\left(s\right)ds\right]dl = \bar{U}_i\left(M, t_i'|t_i\right),
\end{aligned}$$

where the first and last equalities are from the definition of M, and the inequality is derived using the incentive compatibility of M^l, and

$$\bar{U}_i(M|t_i) = \int_I \left[\bar{U}_i(M^l|t_i)\int_{S^l} f_0\left(s\right)ds\right]dl \geq 0,$$

due to $\bar{U}_i(M^l|t_i) \geq 0$.

The construction of M immediately implies that any type of the seller has no incentive to misreport her signal due to the incentive compatibility of M^l, $l \in I$, and (23). By truthfully reporting her type, the seller gets the same payoff as in the original equilibrium strategy $\{M^l\}_{l\in I}$.

Proof of Proposition 3

I show the sufficiency and necessity of these conditions separately below.

(1) Sufficiency of the three conditions:

Suppose that M^* is a safe mechanism satisfying all the three conditions, but there exists a safe mechanism M such that for some $s \in S$, $\bar{U}_0(M|s) > \bar{U}_0(M^*|s)$. If this is true, then we have

$$\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + \bar{U}_0(M|\underline{s}) > \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right),$$

according to (22). Since $\bar{U}_0(M|\underline{s}) \le \bar{U}_0\left(M^F|\underline{s}\right)$, we have

$$\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} > \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s}.$$

This inequality holds only if there is a set $S^M \subset [\underline{s}, s]$ such that $x_0(s) > x_0^*(s)$ for $s \in S^M$. Let $s^{\sup}$ be the supremum of S^M. If $s^{\sup} \notin S^M$, then we can find a $s^{\varepsilon} \in S^M$ that is arbitrarily close to $s^{\sup}$. Define

$$s^M = \begin{cases} s^{\sup}, \text{ if } s^{\sup} \in S^M; \\ s^{\varepsilon}, \text{ if } s^{\sup} \notin S^M. \end{cases}$$

Thus, $s^M \in S^M$ and

$$x_0\left(s^M\right) > x_0^*\left(s^M\right). \tag{24}$$

Then, we have

$$\int_{\underline{s}}^{s^M} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + \bar{U}_0(M|\underline{s}) > \int_{\underline{s}}^{s^M} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right),$$

because the set $\left[s^M, s^{\sup}\right]$ is small or empty and $x_0(s') \le x_0^*(s')$ for all $s' \in (s^{\sup}, s]$. According to equation (22) and Condition 2 of this proposition, which indicates

$\sum_{i=1}^{n} \bar{u}_i \left(M^*, s^M|\underline{t}_i\right) + b^* \left(s^M\right) = 0$, we obtain

$$\int_T \sum_{i=1}^{n} \left[J_i \left(s^M, t_i\right) - v_0 \left(s^M\right)\right] x_i \left(s^M, t\right) f(t)\, dt - \sum_{i=1}^{n} \bar{u}_i \left(M, s^M|\underline{t}_i\right) - b^M \left(s^M\right)$$
$$> \int_T \sum_{i=1}^{n} \left[J_i \left(s^M, t_i\right) - v_0 \left(s^M\right)\right] x_i^* \left(s^M, t\right) f(t)\, dt.$$

Since $\sum_{i=1}^{n} \bar{u}_i \left(M, s^M|\underline{t}_i\right) + b^M \left(s^M\right)$ is nonnegative, we have

$$\int_T \sum_{i=1}^{n} \left[J_i \left(s^M, t_i\right) - v_0 \left(s^M\right)\right] x_i \left(s^M, t\right) f(t)\, dt$$
$$> \int_T \sum_{i=1}^{n} \left[J_i \left(s^M, t_i\right) - v_0 \left(s^M\right)\right] x_i^* \left(s^M, t\right) f(t)\, dt.$$

This inequality holds only $x_0 \left(s^M\right) < x_0^* \left(s^M\right)$, because M^* satisfies Condition 1 of this proposition. This contradicts (24). Therefore, M^* is an RSW mechanism.

(2) Necessity of the three conditions:

Suppose that M^* is an RSW mechanism. Without loss of generality, I assume that $b^*(s) = 0$ for all s, because if for some s', $b^*(s') > 0$, then we can construct a new mechanism M' which is the same as M^* except that at s',

$$\sum_{i=1}^{n} \bar{u}_i \left(M', s'|\underline{t}_i\right) = \sum_{i=1}^{n} \bar{u}_i \left(M^*, s'|\underline{t}_i\right) + b^*(s'), \text{ and } b'(s') = 0.$$

It is obvious that M' is also an RSW mechanism. Combining this argument with the proof below that $\sum_{i=1}^{n} \bar{u}_i \left(M^*, s|\underline{t}_i\right) = 0$ for any s, we can prove $b^*(s) = 0$ for all s.

Firstly, I show that Condition 3 is satisfied by M^*. To proceed, I construct a mechanism $\hat{M}$ as follows: $\hat{x}_0(\underline{s}) = x_0^F(\underline{s})$, $\hat{x}_0(s) = \sup_{\tilde{s}\in[\underline{s},s]} \left\{x_0^F(\tilde{s})\right\}$ for $s > \underline{s}$, and $\hat{x}(s,\cdot)$ solves

$$\max_x \int_T \sum_{i=1}^{n} \left[J_i(s,t_i) - v_0(s)\right] x_i(t) f(t)\, dt \tag{25}$$
$$s.t.\ x \text{ satisfies (1) and } x_0 = \hat{x}_0(s),$$

and the payment rule satisfies

$$\hat{p}_i(s,t) = v_i(s,t_i)\,\hat{x}_i(s,t) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i\left(s,\tilde{t}_i\right)}{\partial t_i}\hat{x}_i\left(s,\tilde{t}_i,t_{-i}\right)d\tilde{t}_i - \bar{u}_i(\hat{M},s|\underline{t}_i), \text{ for any } i,\ s, \text{ and } t,$$

where

$$\bar{u}_i(\hat{M},s|\underline{t}_i) = \frac{1}{n}\{v_0(s) + \int_T \sum_{i=1}^n \left[J_i(s,t_i) - v_0(s)\right]\hat{x}_i(s,t)\,f(t)\,dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\,d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right)\}.$$

The following lemma shows that $\hat{M}$ is safe.

Lemma 3 *Mechanism $\hat{M}$ is a safe mechanism.*

Proof. It is clear that $\hat{M}$ immediately satisfies all the constraints for a safe mechanism, except that $\sum_{i=1}^n \bar{u}_i(\hat{M},s|\underline{t}_i) \geq 0$. The definition of $\hat{x}$ implies that for $s_1 < s_2 \in S$, either there exists a $s' \in (s_1, s_2]$ such that $x_0^F(s') = \hat{x}_0(s_2)$ or $\hat{x}_0(s_1) = \hat{x}_0(s_2)$. In the first case,

$$\begin{aligned}
&\sum_{i=1}^n \bar{u}_i(\hat{M},s_2|\underline{t}_i) - \sum_{i=1}^n \bar{u}_i(\hat{M},s_1|\underline{t}_i) \\
&\geq v_0(s_2)\,x_0^F(s') + \int_T \sum_{i=1}^n J_i(s_2,t_i)\,x_i^F(s',t)\,f(t)\,dt - v_0(s_1)\,\hat{x}_0(s_1) \\
&\quad - \int_T \sum_{i=1}^n J_i(s_1,t_i)\,\hat{x}_i(s_1,t)\,f(t)\,dt - \int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\,d\tilde{s} \\
&\geq v_0(s_2)\,x_0^F(s') + \int_T \sum_{i=1}^n J_i(s',t_i)\,x_i^F(s',t)\,f(t)\,dt - v_0(s_1)\,\hat{x}_0(s_1) \\
&\quad - \int_T \sum_{i=1}^n J_i(s_1,t_i)\,\hat{x}_i(s_1,t)\,f(t)\,dt - \int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\,d\tilde{s} \\
&\geq U_0\left(M^F|s'\right) - U_0\left(M^F|s_1\right) + \left[v_0(s_2) - v_0(s')\right]x_0^F(s') - \int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\,d\tilde{s} \\
&= \int_{s_1}^{s'} g_1'\left(\tilde{s},q^{\tilde{s}}\right)d\tilde{s} - \int_{s_1}^{s'} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\,d\tilde{s} \\
&= \int_{s_1}^{s'} \left\{v_0'(\tilde{s})\left[x_0^F(\tilde{s}) - \hat{x}_0(\hat{s})\right] + \int_T \sum_{i=1}^n J_{i1}'(\tilde{s},t_i)\,x_i^F(\tilde{s},t)\,f(t)\,dt\right\}d\tilde{s}.
\end{aligned}$$

The first inequality uses the definition of $\hat{x}$ in (25) and the fact that $x_0^F(s') = \hat{x}_0(s_2)$. The second inequality is derived using the monotonicity of J_i and the non-negativity of $x_i^F(s', t)$. The third inequality is based on the definition of $U_0\left(M^F|s'\right)$ and the optimality of x^F. The first equality uses the result of (9) and the fact that $\hat{x}_0(s) = x_0^F(s')$ for $s \in [s', s_2]$. The last equality is obtained by substituting the expression of $g_1\left(\tilde{s}, q^{\tilde{s}}\right)$ into the first equality. It is clear that if s_1 and s_2 are arbitrarily close to each other, then this difference is positive.

In the second case, i.e., $\hat{x}_0(s_1) = \hat{x}_0(s_2)$, it is obvious that

$$\sum_{i=1}^{n} \bar{u}_i(\hat{M}, s_2|\underline{t}_i) - \sum_{i=1}^{n} \bar{u}_i(\hat{M}, s_1|\underline{t}_i) \geq 0.$$

Thus, no matter which case happens, $\sum_{i=1}^{n} \bar{u}_i(\hat{M}, s|\underline{t}_i)$ is non-decreasing in s. Since

$$\sum_{i=1}^{n} \bar{u}_i(\hat{M}, \underline{s}|\underline{t}_i) = 0,$$

$\sum_{i=1}^{n} \bar{u}_i(\hat{M}, s|\underline{t}_i)$ is never negative. ■

It is clear that $\bar{U}_0\left(M^F|\underline{s}\right) \geq \bar{U}_0\left(M^*|\underline{s}\right)$, given that

$$\bar{U}_0\left(M^*|\underline{s}\right) = v_0\left(\underline{s}\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(\underline{s}, t_i\right) - v_0\left(\underline{s}\right)\right] x_i^*\left(\underline{s}, t\right) f\left(t\right) dt - \sum_{i=1}^{n} \bar{u}_i\left(M^*, \underline{s}|\underline{t}_i\right),$$

$$\text{and } \sum_{i=1}^{n} \bar{u}_i\left(M^*, \underline{s}|\underline{t}_i\right) \geq 0,$$

according to Lemma 2. According to the definition of the RSW mechanism, we have $\bar{U}_0\left(M^*|\underline{s}\right) \geq \bar{U}_0(\hat{M}|\underline{s}) = \bar{U}_0\left(M^F|\underline{s}\right)$. Thus, $\bar{U}_0\left(M^*|\underline{s}\right) = \bar{U}_0\left(M^F|\underline{s}\right)$.

I prove Condition 1 and Condition 2 together in the rest of the proof. Suppose that Condition 1 is not satisfied by M^*, then for some s, either $x^*(s, \cdot)$ is not a solution of

$$\max_{\hat{x}} \int_T \sum_{i=1}^{n} \left[J_i\left(s, t_i\right) - v_0\left(s\right)\right] \hat{x}_i\left(t\right) f\left(t\right) dt \tag{26}$$

$$s.t.\ \hat{x} \text{ satisfies (1) and } \hat{x}_0 = x_0^*\left(s\right),$$

or $x_0^*(s) \leq x_0^F(s)$ or both. If the first case happens, then we can construct a new RSW mechanism $\bar{M}^*$ by adjusting the allocation rule of M^* such that $\bar{x}^*(s,\cdot)$ solves (26), so $\bar{x}_0^*(s) = x_0^*(s)$. The expected payoff to the lowest type of each buyer under $\bar{M}^*$ is defined in the following way so that the equation (22) holds,

$$\sum_{i=1}^{n} \bar{u}_i\left(\bar{M}^*, s|\underline{t}_i\right) = \sum_{i=1}^{n} \bar{u}_i\left(M^*, s|\underline{t}_i\right) + \int_T \sum_{i=1}^{n} J_i(s,t_i)\,\bar{x}_i^*(s,t)\,f(t)\,dt - \int_T \sum_{i=1}^{n} J_i(s,t_i)\,x_i^*(s,t)\,f(t)\,dt$$

According to the definition of $\bar{x}^*$ and the supposition that M^* is an RSW mechanism, we have

$$\sum_{i=1}^{n} \bar{u}_i\left(\bar{M}^*, s|\underline{t}_i\right) > \sum_{i=1}^{n} \bar{u}_i\left(M^*, s|\underline{t}_i\right) \geq 0.$$

This inequality implies that the failure of the first part of Condition 1 is equivalent to the failure of Conditions 2. Thus, in the rest of the proof, I assume that the M^* satisfies the first part of Condition 1, and then show that $x_0^*(s) \geq x_0^F(s)$ and Condition 2 must hold.

Suppose for some s_1, $x_0^*(s_1) < x_0^F(s_1)$. According to the continuity of $x_0^F(s)$, there exists $s_2 < s_1$ (s_2 close to s_1) such that $x_0^*(s_1) < x_0^F(s_2)$. I construct a safe mechanism M with allocation rule

$$x_i^M(s,t) = \begin{cases} x_i^*(s,t), \text{ for } s < s_2, \\ x_i^F(s_2,t), \text{ for } s \geq s_2, \end{cases} \tag{27}$$

and payment rule

$$p_i^M(s,t) = v_i(s,t_i)\,x_i^M(s,t) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i\left(s,\tilde{t}_i\right)}{\partial t_i} x_i^M\left(s,\tilde{t}_i,t_{-i}\right) d\tilde{t}_i - \bar{u}_i(M,s|\underline{t}_i), \text{ for any } i,\ s, \text{ and } t,$$

where $\bar{u}_i(M,s|\underline{t}_i)$ satisfies that if $s < s_2$,

$$\bar{u}_i(M,s|\underline{t}_i) = \bar{u}_i(M^*,s|\underline{t}_i);$$

if $s \geq s_2$,

$$\bar{u}_i\left(M, s|\underline{t}_i\right) = \frac{1}{n}\{v_0\left(s\right)x_0^M\left(s\right) + \int_T \sum_{i=1}^n J_i\left(s, t_i\right)x_i^M\left(s, t\right)f\left(t\right)dt$$

$$- \int_{\underline{s}}^s v_0'\left(\tilde{s}\right)x_0^M\left(\tilde{s}\right)d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right)\}$$

I show that M is a safe mechanism in the following lemma.

Lemma 4 *Mechanism M is a safe mechanism.*

Proof. Since M^* is safe, the construction of M immediately implies that it satisfies equation (22) and the feasibility condition. According to (27), $x_i^M\left(s, t_i\right)$ and $x_0^M\left(s\right)$ satisfy the monotonicity conditions. Now I show that $\bar{u}_i\left(M, s|\underline{t}_i\right)$ is always nonnegative. For $s < s_2$, we have $\bar{u}_i\left(M, s|\underline{t}_i\right) = \bar{u}_i\left(M^*, s|\underline{t}_i\right) \geq 0$. For $s = s_2$,

$$\begin{aligned}
\sum_{i=1}^n \bar{u}_i\left(M, s_2|\underline{t}_i\right) &= v_0\left(s\right)x_0^F\left(s_2\right) + \int_T \sum_{i=1}^n J_i\left(s_2, t_i\right)x_i^F\left(s_2, t\right)f\left(t\right)dt \\
&\quad - \int_{\underline{s}}^{s_2} v_0'\left(\tilde{s}\right)x_0^*\left(\tilde{s}\right)d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right) \\
&\geq v_0\left(s\right)x_0^*\left(s_2\right) + \int_T \sum_{i=1}^n J_i\left(s_2, t_i\right)x_i^*\left(s_2, t\right)f\left(t\right)dt \\
&\quad - \int_{\underline{s}}^{s_2} v_0'\left(\tilde{s}\right)x_0^*\left(\tilde{s}\right)d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right) \\
&= \sum_{i=1}^n \bar{u}_i\left(M^*, s_2|\underline{t}_i\right) \geq 0.
\end{aligned}$$

The weak inequality is due to the optimality of $x^F\left(s_2, \cdot\right)$ at s_2.

For $s > s_2$, we have

$$\begin{aligned}
\sum_{i=1}^n \bar{u}_i\left(M, s|\underline{t}_i\right) &= v_0\left(s\right)x_0^M\left(s\right) + \int_T \sum_{i=1}^n J_i\left(s, t_i\right)x_i^M\left(s, t\right)f\left(t\right)dt \\
&\quad - \int_{\underline{s}}^s v_0'\left(\tilde{s}\right)x_0^M\left(\tilde{s}\right)d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right) \\
&= v_0\left(s\right)x_0^F\left(s_2\right) + \int_T \sum_{i=1}^n J_i\left(s, t_i\right)x_i^F\left(s_2, t\right)f\left(t\right)dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} - \int_{s_2}^{s} v_0'(\tilde{s})\, x_0^F(s_2)\, d\tilde{s} - \bar{U}_0\left(M^F | \underline{s}\right) \\
&= v_0(s_2)\, x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s, t_i)\, x_i^F(s_2, t)\, f(t)\, dt \\
&\quad - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} - \bar{U}_0\left(M^F | \underline{s}\right) \\
&> v_0(s_2)\, x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s_2, t_i)\, x_i^F(s_2, t)\, f(t)\, dt \\
&\quad - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} - \bar{U}_0\left(M^F | \underline{s}\right) \\
&= \sum_{i=1}^n \bar{u}_i\left(M, s_2 | \underline{t}_i\right) \geq 0.
\end{aligned}
$$

Therefore, M is a safe mechanism. ■

In mechanism M, the seller of type $s \in [s_2, s_1]$ gets a higher expected payoff than under M^*, because for $s \in [s_2, s_1]$,

$$
\begin{aligned}
\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^M(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right) &= \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \int_{s_2}^{s} v_0'(\tilde{s})\, x_0^F(s_2)\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right) \\
&> \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \int_{s_2}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right) \\
&= \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right),
\end{aligned}
$$

where the first equality is from the definition of x^M, the inequality is based on that $x_0^*(s_1) < x_0^F(s_2)$. This contradicts that M^* is an RSW mechanism.

Let us turn to Condition 2. Suppose that for some $\hat{s} > \underline{s}$, $\sum_{i=1}^n \bar{u}_i\left(M^*, \hat{s} | \underline{t}_i\right) > 0$. Then we have the following lemma.

Lemma 5 *There exists some $\underline{\delta} > 0$ such that over the interval $[\hat{s} - \underline{\delta}, \hat{s}]$,*

$$
\inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^n \bar{u}_i\left(M^*, s | \underline{t}_i\right) > 0.
$$

Proof. Suppose that for some $\hat{s}$, $\sum_{i=1}^n \bar{u}_i\left(M^*, \hat{s} | \underline{t}_i\right) > 0$, then there must exist $\delta > 0$, such that for all $s \in [\hat{s} - \delta, \hat{s}]$, $\sum_{i=1}^n \bar{u}_i\left(M^*, s | \underline{t}_i\right) > 0$. Suppose this is not true, i.e., for any $\delta > 0$, there exists $s \in [\hat{s} - \delta, \hat{s}]$ such that $\sum_{i=1}^{n} \bar{u}_i (M^*, s|\underline{t}_i) = 0$, then we can find a sequence $\{s_n\}_{n=1}^{\infty}$ converging to $\hat{s}$ with $\sum_{i=1}^{n} \bar{u}_i (M^*, s_n|\underline{t}_i) = 0$ for every n. According to equation (22), we have

$$v_0 (s_n) + \int_T \sum_{i=1}^{n} [J_i (s_n, t_i) - v_0 (s_n)] \, x_i^* (s_n, t) \, f (t) \, dt = \int_{\underline{s}}^{s_n} v_0' (\tilde{s}) \, x_0^* (\tilde{s}) \, d\tilde{s} + \bar{U}_0 (M^*|\underline{s}) \, .$$

By continuity,

$$\lim_{n \to \infty} v_0 (s_n) + \int_T \sum_{i=1}^{n} [J_i (s_n, t_i) - v_0 (s_n)] \, x_i^* (s_n, t) \, f (t) \, dt$$
$$= \int_{\underline{s}}^{\hat{s}} v_0' (\tilde{s}) \, x_0^* (\tilde{s}) \, d\tilde{s} + \bar{U}_0 (M^*|\underline{s}) \, . \quad (28)$$

However, since $x_0^* (s_n) \leq x_0^* (\hat{s})$ and Condition 1 of Proposition 3 is satisfied, there is

$$v_0 (s_n) + \int_T \sum_{i=1}^{n} [J_i (s_n, t_i) - v_0 (s_n)] \, x_i^* (s_n, t) \, f (t) \, dt$$
$$\geq v_0 (s_n) + \int_T \sum_{i=1}^{n} [J_i (s_n, t_i) - v_0 (s_n)] \, x_i^* (\hat{s}, t) \, f (t) \, dt.$$

The expression on the RHS of the inequality is continuous. Using (28), taking limits on both sides of this inequality yields

$$\int_{\underline{s}}^{\hat{s}} v_0' (\tilde{s}) \, x_0^* (\tilde{s}) \, d\tilde{s} + \bar{U}_0 (M^*|\underline{s}) \geq v_0 (\hat{s}) + \int_T \sum_{i=1}^{n} [J_i (\hat{s}, t_i) - v_0 (\hat{s})] \, x_i^* (\hat{s}, t) \, f (t) \, dt.$$

This contradicts equation (22), given $\sum_{i=1}^{n} \bar{u}_i (M^*, \hat{s}|\underline{t}_i) > 0$. This completes the proof that there exists $\delta > 0$, such that for all $s \in [\hat{s} - \delta, \hat{s}]$, $\sum_{i=1}^{n} \bar{u}_i (\underline{t}_i|s, M^*) > 0$.

Now I show that there exists $\underline{\delta} \in (0, \delta)$ such that $\inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^{n} \bar{u}_i (M^*, s|\underline{t}_i) > 0$. I again prove this by contradiction. Suppose this is not the case, then for any $\underline{\delta} \in (0, \delta)$,

$$\inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^{n} \bar{u}_i (M^*, s|\underline{t}_i) = 0.$$

If so, for decreasing positive sequences $\{\delta_m\}_{m=1}^{\infty}$ and $\{\varepsilon_m\}_{m=1}^{\infty}$ with $\delta_1 < \delta$, $\lim_{m\to\infty} \delta_m = 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$, we can construct a sequence $\{s_m\}_{m=1}^{\infty}$ such that

$$s_m \in [\hat{s} - \delta_m, \hat{s}] \text{ and } \sum_{i=1}^{n} \bar{u}_i \left(M^*, s_m | \underline{t}_i\right) < \varepsilon_m.$$

This implies that

$$\lim_{m\to\infty} s_m = \hat{s} \text{ and } \lim_{m\to\infty} \sum_{i=1}^{n} \bar{u}_i \left(M^*, s_m | \underline{t}_i\right) = 0.$$

According to equation (22),

$$\begin{aligned}
&\lim_{m\to\infty} v_0\left(s_m\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_m, t_i\right) - v_0\left(s_m\right)\right] x_i^*\left(s_m, t\right) f\left(t\right) dt \\
&= \lim_{m\to\infty} \int_{\underline{s}}^{s_m} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + \lim_{m\to\infty} \sum_{i=1}^{n} \bar{u}_i\left(M^*, s_m | \underline{t}_i\right) + \bar{U}_0\left(M^* | \underline{s}\right) \\
&= \int_{\underline{s}}^{\hat{s}} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + \bar{U}_0\left(M^* | \underline{s}\right). \qquad (29)
\end{aligned}$$

However, due to that $x_0^*\left(s_m\right) \leq x_0^*\left(\hat{s}\right)$ and Condition 1,

$$\begin{aligned}
&v_0\left(s_m\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_m, t_i\right) - v_0\left(s_m\right)\right] x_i^*\left(s_m, t\right) f\left(t\right) dt \\
&\qquad \geq v_0\left(s_m\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_m, t_i\right) - v_0\left(s_m\right)\right] x_i^*\left(\hat{s}, t\right) f\left(t\right) dt.
\end{aligned}$$

Taking limits on both sides, we obtain

$$\int_{\underline{s}}^{\hat{s}} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + \bar{U}_0\left(M^* | \underline{s}\right) \geq v_0\left(\hat{s}\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(\hat{s}, t_i\right) - v_0\left(\hat{s}\right)\right] x_i^*\left(\hat{s}, t\right) f\left(t\right) dt,$$

according to (29). This contradicts equation (22), given $\sum_{i=1}^{n} \bar{u}_i\left(M^*, \hat{s} | \underline{t}_i\right) > 0$. Therefore, I have proved that there exists $\underline{\delta} \in (0, \delta)$ such that $\inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^{n} \bar{u}_i\left(M^*, s | \underline{t}_i\right) > 0$. ■

Given this lemma, we can construct a safe mechanism M^{ε} making all types of the seller in the set $[\hat{s} - \underline{\delta}, \hat{s}]$ strictly better off. Specifically, the allocation rule of M^{ε} is defined as follows: for $s < \hat{s} - \underline{\delta}$, $x^{\varepsilon}(s,t) = x^{*}(s,t)$; for $s \in [\hat{s} - \underline{\delta}, \hat{s}]$, $x^{\varepsilon}(s,\cdot)$ solves

$$\max_{\hat{x}} \left\{ \int_T \sum_{i=1}^{n} [J_i(s,t_i) - v_0(s)] \hat{x}_i(t) f(t) \, dt : \hat{x} \text{ satisfies (1) and } \hat{x}_0 = x_0^{*}(s) + \varepsilon \right\};$$

and for $s > \hat{s}$, $x^{\varepsilon}(s,t) = x^{\varepsilon}(\hat{s},t)$. The $\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i)$ of M^{ε} is constructed as follows: if $s < \hat{s} - \underline{\delta}$, $\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) = \sum_{i=1}^{n} \bar{u}_i(M^{*}, s|\underline{t}_i)$; if $s \in [\hat{s} - \underline{\delta}, \hat{s}]$,

$$\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) = \sum_{i=1}^{n} \bar{u}_i(M^{*}, s|\underline{t}_i) - \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{*}(t) f(t) \, dt + \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{\varepsilon}(t) f(t) \, dt - \varepsilon v_0(\hat{s} - \underline{\delta});$$

if $s > \hat{s}$,

$$\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) = v_0(s) x_0^{\varepsilon}(s) + \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{\varepsilon}(s,t) f(t) \, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s}) x_0^{\varepsilon}(\tilde{s}) \, d\tilde{s} - \bar{U}_0\left(M^F|\underline{s}\right)$$

Lemma 6 *When ε is small enough, M^{ε} is a safe mechanism.*

Proof. Verifying that M^{ε} satisfies equation (22), monotonicity condition, feasibility condition is straightforward, so I need only to show that for all s, $\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) \geq 0$. For $s < \hat{s} - \underline{\delta}$, $\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) = \sum_{i=1}^{n} \bar{u}_i(M^{*}, s|\underline{t}_i) \geq 0$. For $s \in [\hat{s} - \underline{\delta}, \hat{s}]$,

$$\begin{aligned} & \sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) \\ & \geq \inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^{n} \bar{u}_i(M^{*}, s|\underline{t}_i) - \left\{ \begin{array}{l} \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{*}(t) f(t) \, dt \\ - \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{\varepsilon}(t) f(t) \, dt - \varepsilon v_0(\hat{s} - \underline{\delta}) \end{array} \right\}, \end{aligned}$$

which is positive when ε is small enough. For $s > \hat{s}$,

$$\sum_{i=1}^{n} \bar{u}_i(M^{\varepsilon}, s|\underline{t}_i) = v_0(s) x_0^{\varepsilon}(\hat{s}) + \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^{\varepsilon}(\hat{s},t) f(t) \, dt$$

$$
\begin{aligned}
&- \int_{\underline{s}}^{s} v_0' (\tilde{s}) \, x_0^{\varepsilon} (\tilde{s}) \, d\tilde{s} - \bar{U}_0 \left(M^F | \underline{s} \right) \\
&= v_0 (\hat{s}) \, x_0^{\varepsilon} (\hat{s}) + \int_T \sum_{i=1}^{n} J_i (s, t_i) \, x_i^{\varepsilon} (\hat{s}, t) \, f (t) \, dt \\
&\quad - \int_{\underline{s}}^{s} v_0' (\tilde{s}) \, x_0^{\varepsilon} (\tilde{s}) \, d\tilde{s} - \bar{U}_0 \left(M^F | \underline{s} \right) \\
&\geq v_0 (\hat{s}) \, x_0^{\varepsilon} (\hat{s}) + \int_T \sum_{i=1}^{n} J_i (\hat{s}, t_i) \, x_i^{\varepsilon} (\hat{s}, t) \, f (t) \, dt \\
&\quad - \int_{\underline{s}}^{s} v_0' (\tilde{s}) \, x_0^{\varepsilon} (\tilde{s}) \, d\tilde{s} - \bar{U}_0 \left(M^F | \underline{s} \right) \\
&= \int_T \sum_{i=1}^{n} \bar{u}_i \left(M^{\varepsilon}, \hat{s} | \underline{t}_i \right) \geq 0,
\end{aligned}
$$

in which the first equality and second equality are from the definition of $x^{\varepsilon} (s, \cdot)$ for $s > \hat{s}$, the inequality is due to that $J_i (s, t_i)$ is increasing in s, the last equality holds because at $s = \hat{s}$, M^{ε} satisfies equation (22). ■

Here I show that the seller with $s \in [\hat{s} - \underline{\delta}, \hat{s}]$ gets better off. For $s \in [\hat{s} - \underline{\delta}, \hat{s}]$, the difference of the seller's payoffs under M^{ε} and M^* is

$$
\int_{\underline{s}}^{s} v_0' (\tilde{s}) \, x_0^{\varepsilon} (\tilde{s}) \, d\tilde{s} - \int_{\underline{s}}^{s} v_0' (\tilde{s}) \, x_0^{*} (\tilde{s}) \, d\tilde{s} = \int_{\hat{s} - \underline{\delta}}^{s} v_0' (\tilde{s}) \, \varepsilon d\tilde{s} > 0.
$$

This again contradicts that M^* is an RSW mechanism. Thus, Condition 2 must be satisfied for M^*.

Given that Condition 2 is satisfied by M^*, it is easy to show that $x_0^* (s) > x_0^F (s)$ for $s > \underline{s}$. Suppose that this is not true for $s' > \underline{s}$, then the type- s' seller gets her public-information optimal payoff. For the types of the seller lower than s', their payoffs are weakly lower than their public-information optimal payoffs. Thus, some of these types would have an incentive to report their types being s', due to Proposition 2. This contradicts that M^* is safe. This completes the proof.

Proof of Proposition 4

The conditions in Proposition 1 and Lemma 2 imply that to prove the existence of an

RSW mechanism, the key is to prove the existence of a function $x_0^*: S \to [0,1]$ that is greater than $x_0^F: S \to [0,1]$ for all $s \in S$, increasing in s, and satisfies

$$\begin{aligned}&v_0(s) + \max_x \left\{ \int_T \sum_{i=1}^n [J_i(s,t_i) - v_0(s)]\, x_i(t)\, f(t)\, dt : x \text{ satisfies (1) and } x_0 = x_0^*(s) \right\} \\ &= \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right). \end{aligned} \tag{30}$$

Once there exists such a x_0^*, we can easily derive a pair of x^* and p^* using the conditions in Proposition 1 and Lemma 2, thus derive an RSW mechanism. Specifically, given $x_0^*(s)$, $x^*(s,\cdot)$ solving the maximization problem in (30) allocates the object to a buyer with the highest virtual surplus; if the maximum virtual surplus under profile t' is higher than that under t, then $\sum_{i=1}^n x_i^*(s,t') \geq \sum_{i=1}^n x_i^*(s,t)$. Thus, characterizing $x^*(s,\cdot)$ is equivalent to finding a value $\underline{J}$ of virtual valuation such that the seller keeps the object if and only if the maximum virtual valuation under a profile t is lower than $\underline{J}$. The value $\underline{J}$ should satisfy

$$\max_k \{J_k(s,\underline{t}_k) : 1 \leq k \leq n\} = J^{\min}(s) \leq \underline{J} \leq J^{\max}(s) = \max_k \{J_k(s,\bar{t}_k) : 1 \leq k \leq n\}. \tag{31}$$

In line with this intuition, I define $r_i(s,\underline{J})$ as the minimum type of buyer i that has positive probability to get the object given s and $\underline{J}$. Thus, if $J_i(s,\underline{t}_i) \leq \underline{J} \leq J_i(s,\bar{t}_i)$, $r_i(s,\underline{J})$ satisfies $J_i(s, r_i(s,\underline{J})) = \underline{J}$; if $J_i(s,\bar{t}_i) < \underline{J}$, $r_i(s,\underline{J}) = \bar{t}_i$.[14] The continuity and monotonicity of $J_i(s,t_i)$ in s and t_i imply that $r_i(s,\underline{J})$ is continuously decreasing in s and continuously increasing in $\underline{J}$. The probability that the seller keeps the object given $\underline{J}$ is $\prod_{i=i}^n F_i(r_i(s,\underline{J}))$, because the object is left unsold if and only if the virtual valuations of the buyers are all smaller than $\underline{J}$. Condition (30) can be rewritten as

$$v_0(s) + \sum_{i=1}^n \int_{r_i(s,\underline{J}^*(s))}^{\bar{t}_i} \int_{T_{-i}(s,t_i)} [J_i(s,t_i) - v_0(s)]\, f(t)\, dt = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right), \tag{32}$$

[14] According to condition (31), $J_i(s,\underline{t}_i) > \underline{J}$ will not happen.

where $\underline{J}^*(s)$ is defined as

$$\prod_{i=i}^{n} F_i\left(r_i\left(s, \underline{J}^*(s)\right)\right) = x_0^*(s),$$

and $T_{-i}(s, t_i)$ is defined to be the set

$$\left\{t_{-i} \in T_{-i} : J_i(s, t_i) \geq \max_k \left\{J_k(s, t_k) : k < i\right\},\ J_i(s, t_i) > \max_k \left\{J_k(s, t_k) : k > i\right\}\right\}.$$

For any type profile $(t_i, \tilde{t}_{-i})$ with $\tilde{t}_{-i} \in T_{-i}(s, t_i)$, buyer i is the highest indexed agent with the maximum virtual valuation. Thus, equation (32) corresponds to an allocation rule that allocates the object to the highest indexed member among the buyers with the maximum virtual valuation.

Proving the existence of an RSW mechanism is reduced to proving the existence of a function $x_0^* : S \to [0, 1]$ that is increasing in s, bounded by x_0^F and 1, i.e., $x_0^F(s) \leq x_0^*(s) \leq 1$ for all s, and satisfies equation (32) for all s. To proceed, I define:

$$\begin{aligned} D\left(\underline{J}, s, x_0\right) = v_0(s) + \sum_{i=1}^{n} \int_{r_i(s,\underline{J})}^{\bar{t}_i} \int_{T_{-i}(s,t_i)} \left[J_i(s, t_i) - v_0(s)\right] f(t)\, dt \\ - \left[\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right)\right] \end{aligned}$$

So $D(\underline{J}, s, x_0)$ is the difference between the LHS and RHS of equation (32), given x_0, s, and $\underline{J}$. Then I define function $\underline{J}^{x_0} : S \to R$ with

$$\underline{J}^{x_0}(s) = \arg\min_{\underline{J}} \left\{\left|D\left(\underline{J}, s, x_0\right)\right| : \max\left\{J^{\min}(s), v_0(s)\right\} \leq \underline{J} \leq J^{\max}(s)\right\}.$$

That is, $\underline{J}^{x_0}(s)$ is the value of $\underline{J}$ that minimizes $\left|D\left(\underline{J}, s, x_0\right)\right|$, given s and function x_0. Let $C(S)$ denote the set of bounded continuous functions $h : S \to R$ endowed with sup norm, $\|h\| = \sup_{s \in S} |h(s)|$. Thus, $C(S)$ is a Banach space. I use $X_0(S)$ to represent the set of continuous functions $x_0 : S \to R$ that are increasing and bounded by x_0^F and 1, so

$X_0(S) \subset C(S)$. I define a mapping $\Gamma : X_0(S) \to C(S)$, with

$$\Gamma x_0(s) = \prod_{i=1}^{n} F_i\left(r_i\left(s, \underline{J}^{x_0}(s)\right)\right).$$

Once I show that Γ has a fixed point x_0^*, and x_0^* satisfies $D\left(\underline{J}^{x_0^*}(s), s, x_0^*\right) = 0$ for all s, then the existence of an RSW mechanism is proved.

I use Schauder's fixed point theorem to prove that Γ has a fixed point. To use this theorem, we need $X_0(S)$ to be non-empty, convex, and compact. This is proved in the following lemma. After that, I show that Γ is a continuous mapping.

Lemma 7 *The set $X_0(S)$ is non-empty, convex, and compact.*

Proof. Non-emptiness and convexity are obvious. Here I show its compactness. It is obvious that $X_0(S)$ is bounded, as each of its element is bounded. I show that $X_0(S)$ is closed to complete the proof of compactness. Let $\{x_0^n\}_{n=1}^{\infty}$ be a sequence in $X_0(S)$ converging to x_0, so

$$\lim_{n\to\infty} x_0^n(s) = x_0(s), \forall s \in S. \tag{33}$$

Since for any n, $x_0^n(s)$ belongs to the closed interval $\left[x_0^F(s), 1\right]$, we have $x_0(s) \in \left[x_0^F(s), 1\right]$. Thus, x_0 is bounded by x_0^F and 1. Also, x_0 is increasing in s. Suppose not, then there must exist $s < s'$ with $x_0(s) > x_0(s')$. Due to (33), for any $\varepsilon/2 > 0$, there exists N, for any $n > N$,

$$\left|x_0^n(s) - x_0(s)\right| < \varepsilon/2,$$

or equivalently,

$$x_0(s) - \varepsilon/2 < x_0^n(s) < x_0(s) + \varepsilon/2,$$

and there exists N', for any $n > N'$,

$$\left|x_0^n(s') - x_0(s')\right| < \varepsilon/2,$$

or equivalently,

$$x_0(s') - \varepsilon/2 < x_0^n(s') < x_0(s') + \varepsilon/2.$$

For $n > \max\{N, N'\}$ and $\varepsilon < x_0(s) - x_0(s')$, we have

$$x_0^n(s) - x_0^n(s') > x_0(s) - x_0(s') - \varepsilon > 0,$$

which contradicts that x_0^n is an increasing function. Since x_0^n is continuous for any n, then Uniform Limit Theorem implies that x_0 is continuous. Therefore, $x_0 \in X_0(S)$, and $X_0(S)$ is compact. ■

The mapping Γ maps $X_0(S)$ into a subset $\hat{C}(S)$ of $C(S)$, which includes continuous functions bounded by x_0^F and 1. To prove this, I show that for any $x_0 \in X_0(S)$, Γx_0 is bounded by x_0^F and 1 and is continuous. Given that $\max\left\{J^{\min}(s), v_0(s)\right\} \leq \underline{J} \leq J^{\max}(s)$, we have

$$x_0^F(s) = \prod_{i=1}^{n} F_i\left(r_i\left(s, \max\left\{J^{\min}(s), v_0(s)\right\}\right)\right) \leq \Gamma x_0(s) \leq \prod_{i=1}^{n} F_i\left(r_i\left(s, J^{\max}(s)\right)\right) = 1, \forall s.$$

Thus, Γx_0 is bounded by x_0^F and 1. Since $D(\underline{J}, s, x_0)$ is continuous in $\underline{J}$ and s, and the interval $\left[\max\left\{J^{\min}(s), v_0(s)\right\}, J^{\max}(s)\right]$ is compact and continuous in s, $\underline{J}^{x_0}(s)$ is continuous in s according to the Theorem of the Maximum. This consequently implies that $\Gamma x_0(s) = \prod_{i=i}^{n} F_i(r_i(s, \underline{J}(s)))$ is continuous in s.

Lemma 8 *The mapping Γ is a continuous mapping.*

Proof. Consider a converging sequence $\{x_0^n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_0^n = x_0$ in sup norm. That is, for any $\varepsilon > 0$, there exists $N > 0$, for any $n > N$,

$$\|x_0^n - x_0\| < \varepsilon.$$

Since $\|\cdot\|$ is sup norm, we have

$$\left|\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^n(\tilde{s})\, d\tilde{s} - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s}\right| = \left|\int_{\underline{s}}^{s} v_0'(\tilde{s})\left[x_0^n(\tilde{s}) - x_0(\tilde{s})\right] d\tilde{s}\right|$$
$$< \varepsilon\left[v_0(\bar{s}) - v_0(\underline{s})\right],$$

and for any $s \in S$,

$$\left|D\left(\underline{J}, s, x_0^n\right) - D\left(\underline{J}, s, x_0\right)\right| < \varepsilon\left[v_0(\bar{s}) - v_0(\underline{s})\right]. \tag{34}$$

Now I show that $\underline{J}^{x_0^n} \to \underline{J}^{x_0}$ in sup norm. Let

$$A = \left\{(s, \underline{J}) : s \in S, \text{ and } \underline{J} \in \left[\max\left\{J^{\min}(s), v_0(s)\right\}, J^{\max}(s)\right]\right\}.$$

It is obvious that A is compact. I define a subset A_ε of A by

$$A_\varepsilon = \left\{(s, \underline{J}) \in A : \left|\underline{J} - \underline{J}^{x_0}(s)\right| \geq \varepsilon\right\}. \tag{35}$$

The set A_ε is compact, and for ε small enough, it is non-empty. The result is trivial when A_ε is empty. For any ε, let

$$\delta_\varepsilon = \min_{(s,\underline{J}) \in A_\varepsilon} \left|\left|D\left(\underline{J}, s, x_0\right)\right| - \left|D\left(\underline{J}^{x_0}(s), s, x_0\right)\right|\right|. \tag{36}$$

The continuities of D in $\underline{J}$ and s and the continuity of $\underline{J}^{x_0}(s)$ in s ensure the existence of δ_ε. According to (34), for any $\delta > 0$, there exists N_δ, for any $n > N_\delta$,

$$\left|D\left(\underline{J}, s, x_0^n\right) - D\left(\underline{J}, s, x_0\right)\right| < \delta/2, \tag{37}$$

so

$$\left|\left|D\left(\underline{J}^{x_0^n}(s), s, x_0\right)\right| - \left|D\left(\underline{J}^{x_0}(s), s, x_0\right)\right|\right|$$

$$
\begin{aligned}
&= \left|D\left(\underline{J}^{x_0^n}(s), s, x_0\right)\right| - \left|D\left(\underline{J}^{x_0}(s), s, x_0\right)\right| \\
&\leq \left|D\left(\underline{J}^{x_0^n}(s), s, x_0\right)\right| - \left|D\left(\underline{J}^{x_0^n}(s), s, x_0^n\right)\right| + \left|D\left(\underline{J}^{x_0}(s), s, x_0^n\right)\right| - \left|D\left(\underline{J}^{x_0}(s), s, x_0\right)\right| \\
&\leq \left|D\left(\underline{J}^{x_0^n}(s), s, x_0\right) - D\left(\underline{J}^{x_0^n}(s), s, x_0^n\right)\right| + \left|D\left(\underline{J}^{x_0}(s), s, x_0^n\right) - D\left(\underline{J}^{x_0}(s), s, x_0\right)\right| \\
&< \delta.
\end{aligned}
$$

The equality and first inequality are based on the definitions of $\underline{J}^{x_0^n}(s)$ and $\underline{J}^{x_0}(s)$, the second inequality is using the triangle inequality of absolute values. The last inequality is from (37). Thus, from (35) and (36), for $n > N_{\delta_\varepsilon}$,

$$\left|\underline{J}^{x_0^n}(s) - \underline{J}^{x_0}(s)\right| < \varepsilon, \forall s \in S.$$

This is equivalent to that $\underline{J}^{x_0^n} \to \underline{J}^{x_0}$ in sup norm. Thus, $\lim_{n\to\infty} \Gamma x_0^n = \Gamma x_0$ in sup norm, because

$$\|\Gamma x_0^n - \Gamma x_0\| = \sup_{s \in S} \left| \prod_{i=1}^{n} F_i\left(r_i\left(s, \underline{J}^{x_0^n}(s)\right)\right) - \prod_{i=1}^{n} F_i\left(r_i\left(s, \underline{J}^{x_0}(s)\right)\right) \right|.$$

Therefore, the mapping Γ is continuous. ■

The mapping Γ does not map $X_0(S)$ into itself, as we cannot guarantee that Γx_0 is increasing in s. Here I define another mapping Ψ over $\hat{C}(S)$, with $\Psi h(s) = \sup_{\hat{s} \in [\underline{s}, s]} h(\hat{s})$, for $h \in \hat{C}(S)$. It is obvious that $\Psi h(s)$ is increasing in s. So Ψ maps $\hat{C}(S)$ into $X_0(S)$, and the compound mapping $\Psi \circ \Gamma$ maps $X_0(S)$ into itself.

Lemma 9 *The mapping Ψ is a continuous mapping.*

Proof. Consider a converging sequence $\{h^n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} h^n = h$ in sup norm, then for any $\varepsilon > 0$, there exists N_ε, for any $n > N_\varepsilon$,

$$\|h^n - h\| < \varepsilon,$$

which is equivalent to

$$|h^n(s) - h(s)| < \varepsilon, \forall s \in S. \tag{38}$$

According to the definition of Ψ, for $s \in S$

$$\begin{aligned} |\Psi h^n(s) - \Psi h(s)| &= \left| \sup_{\hat{s} \in [\underline{s}, s]} h^n(\hat{s}) - \sup_{\hat{s} \in [\underline{s}, s]} h(\hat{s}) \right| \\ &= |h^n(s') - h(s'')|, \end{aligned}$$

where s' and s'' are values in $[\underline{s}, s]$ that maximize h^n and h, respectively. If $h^n(s') - h(s'') > 0$, then

$$|h^n(s') - h(s'')| = h^n(s') - h(s'') \leq h^n(s') - h(s');$$

if $h^n(s') - h(s'') \leq 0$, then

$$|h^n(s') - h(s'')| = h(s'') - h^n(s') \leq h(s'') - h^n(s'').$$

Thus, given (38), for $n > N_\varepsilon$, we have

$$|\Psi h^n(s) - \Psi h(s)| < \varepsilon, \forall s \in S.$$

That is,

$$\|\Psi h^n - \Psi h\| < \varepsilon.$$

This completes the proof that Ψ is continuous. ■

Given all the results above, we have that the compound mapping $\Psi \circ \Gamma$ is continuous and maps from $X_0(S)$, which is non-empty, convex, and compact, into itself. According to Schauder's fixed point theorem, $\Psi \circ \Gamma$ has a fixed point on $X_0(S)$, that is, there exists a $x_0^* \in X_0(S)$ satisfying

$$x_0^* = \Psi \circ \Gamma x_0^*.$$

It is obvious that $x_0^*(\underline{s}) = x_0^F(\underline{s})$, according to (30).

The function x_0^* is also a fixed point of Γ. To prove this, I first show that if for some $s \in S$, $x_0^*(s) = 1$, then $x_0^*(s') = \Gamma x_0^*(s') = 1$ for $s' \geq s$. If $x_0^*(s) = 1$, then there exists $\hat{s} \leq s$,

$\Gamma x_0^*(\hat{s}) = 1$ which implies that

$$D\left(J^{\max}(\hat{s}), \hat{s}, x_0^*\right) = v_0(\hat{s}) - \left[\int_{\underline{s}}^{\hat{s}} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right)\right] \geq 0,$$

so for all $\hat{s}' \in (\hat{s}, \bar{s}]$,

$$\begin{aligned} D\left(J^{\max}(\hat{s}'), \hat{s}', x_0^*\right) &= v_0(\hat{s}') - \left[\int_{\underline{s}}^{\hat{s}'} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right)\right] \\ &= D\left(J^{\max}(\hat{s}), \hat{s}, x_0^*\right) + \int_{\hat{s}}^{\hat{s}'} v_0'(\tilde{s})\left[1 - x^*(\tilde{s})\right] d\tilde{s} \geq 0, \end{aligned}$$

Hence, $\Gamma x_0^*(\hat{s}') = 1 = x_0^*(\hat{s}')$.

Now I show that if $x_0^*(s) < 1$, $x_0^*(s)$ is strictly increasing in s. I prove this by contradiction. Suppose that for some $s' < s''$, $x_0^*(s') = x_0^*(s'') < 1$. Let $\check{s} = \inf\left\{\hat{s} \in S : x_0^*(\hat{s}) = x_0^*(s'')\right\}$. The continuity and monotonicity of x_0^* guarantee the existence of $\check{s}$. For $s < \check{s}$, $x_0^*(s) < x_0^*(\check{s})$, and for all $s \in [\check{s}, s'']$, $x_0^*(s) = x_0^*(s'')$. Moreover, $x_0^*(\check{s}) = \Gamma x_0^*(\check{s}) \geq x_0^F(\check{s})$, because $x_0^*(s) < x_0^*(\check{s})$ for $s < \check{s}$. It is not possible to have $D\left(\underline{J}^{x_0^*}(\check{s}), \check{s}, x_0^*\right) = 0$, because if so, for $s \in (\check{s}, s'']$,

$$\begin{aligned} & v_0(s)\, x_0^*(s) + \int_T \sum_{i=1}^n J_i(s, t_i)\, x_i^*(s, t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right) \\ & \geq v_0(s)\, x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(s, t_i)\, x_i^*(\check{s}, t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right) \\ & > v_0(s)\, x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(\check{s}, t_i)\, x_i^*(\check{s}, t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F|\underline{s}\right) \\ & = D\left(\underline{J}^{x_0^*}(\check{s}), \check{s}, x_0^*\right) + \left[v_0(s) - v_0(\check{s})\right] x_0^*(\check{s}) - \int_{\check{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} \\ & = 0. \end{aligned} \tag{39}$$

In the first line, $x^*(s, \cdot)$ denote the allocation rule maximizing the virtual surplus given $x_0^*(s)$. The first inequality is due to the optimality of $x^*(s, \cdot)$. The second inequality is because $J_i(s, t_i)$ is strictly increasing in s. The first equality is based on the definition of $D\left(\underline{J}^{x_0^*}(\check{s}), \check{s}, x_0^*\right)$ and $x_0^*(\check{s}) = \Gamma x_0^*(\check{s})$, and the last equality is resulted from $D\left(\underline{J}^{x_0^*}(\check{s}), \check{s}, x_0^*\right) = 0$ and $x_0^*(s) = x_0^*(s'')$ for all $s \in [\check{s}, s'']$. This sequence of inequalities implies that $\Gamma x_0^*(s) > x_0^*(s)$. This contradicts that x_0^* is a fixed point of $\Psi \circ \Gamma$. Thus, it is only possible to have $D\left(\underline{J}^{x_0^*}(\check{s}), \check{s}, x_0^*\right) < 0$, which indicates that $\Gamma x_0^*(\check{s}) = x_0^F(\check{s})$. Since I have shown that for $s < \check{s}$, $x_0^*(s) < x_0^*(\check{s}) = \Gamma x_0^*(\check{s})$, and $x_0^F(s) < x_0^*(s)$, $\Gamma x_0^*(\check{s}) = x_0^F(\check{s})$ implies $x_0^F(\check{s}) = \Psi x_0^F(\check{s})$.

I define

$$D(s, x_0^*) = v_0(s)\, x_0^*(s) + \int_T \sum_{i=1}^n J_i(s, t_i)\, x_i^*(s, t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right),$$

$$\underline{\check{s}} = \sup\left\{s \in [\underline{s}, \check{s}] : D(s, x_0^*) = 0\right\}.$$

The set $\{s \in [\underline{s}, \check{s}] : D(s, x_0^*) = 0\}$ is non-empty, as it includes $\underline{s}$, so $\underline{\check{s}}$ exists. The continuity of $x_0^*(s)$ implies that $\underline{\check{s}} < \check{s}$ and $D(\underline{\check{s}}, x_0^*) = 0$, $D(s, x_0^*) > 0$, for $s \in (\underline{\check{s}}, \check{s}]$.

I prove that $x_0^*(s') = \Psi x_0^F(s')$, for all $s' \in (\underline{\check{s}}, \check{s}]$. Suppose not, i.e., $x_0^*(s') > \Psi x_0^F(s')$ for some $s' \in (\underline{\check{s}}, \check{s}]$. Given that $x_0^*(s') = \Psi \circ \Gamma x_0^*(s')$, there exists $s'' \leq s'$, such that $x_0^*(s') = \Gamma x_0^*(s'')$, thus $\Gamma x_0^*(s'') > \Psi x_0^F(s') \geq x_0^F(s'')$, which implies that $D(s'', x_0^*) = 0$ and $s'' \leq \underline{\check{s}}$, and $x_0^*(s) = x_0^*(s'')$ for all $s \in (s'', s']$. The argument in the paragraph containing (39) already shows that this is impossible. Therefore, $x_0^*(s') = \Psi x_0^F(s')$, for all $s' \in (\underline{\check{s}}, \check{s}]$.

The continuities of $x_0^*(s)$ and $\Psi x_0^F(s)$ imply that $x_0^*(\underline{\check{s}}) = \Psi x_0^F(\underline{\check{s}})$. Then we obtain

$$\begin{aligned}
D(\check{s}, x_0^*) - D(\underline{\check{s}}, x_0^*) &= v_0(\check{s})\, x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(\check{s}, t_i)\, x_i^*(\check{s}, t)\, f(t)\, dt \\
&\quad - v_0(\underline{\check{s}})\, x_0^*(\underline{\check{s}}) + \int_T \sum_{i=1}^n J_i(\underline{\check{s}}, t_i)\, x_i^*(\underline{\check{s}}, t)\, f(t)\, dt - \int_{\underline{\check{s}}}^{\check{s}} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} \\
&\geq 0,
\end{aligned}$$

according to (25) and Lemma 3. This contradicts the supposition that

$$D(\check{s}, x_0^*) = D(\underline{J}(\check{s}), \check{s}, x_0^*) < 0.$$

This completes the proof that $x_0^*(s)$ is strictly increasing in s if $x_0^*(s) < 1$. Hence, $x_0^*(s) = \Gamma x_0^*(s)$ for $x_0^*(s) < 1$. Combining with the result above that $x_0^*(s) = \Gamma x_0^*(s)$ for $x_0^*(s) = 1$, x_0^* is a fixed point of Γ.

In the rest of the proof, I show that for x_0^*,

$$D\left(\underline{J}^{x_0^*}(s), s, x_0^*\right) = 0, \forall s \in S.$$

I still prove this by contradiction. First, suppose that there exists s', $D\left(\underline{J}^{x_0^*}(s'), s', x_0^*\right) > 0$, which implies that $x_0^*(s') = 1$. The continuities of x_0^* and $D\left(\underline{J}^{x_0^*}(s), s, x_0^*\right)$ in s indicate that there exists $\hat{s} < s'$, such that $x_0^*(\hat{s}) = 1$ and $D\left(\underline{J}^{x_0^*}(\hat{s}), \hat{s}, x_0^*\right) = 0$. The monotonicity of x_0^* gives us

$$\begin{aligned} D\left(\underline{J}^{x_0^*}(s'), s', x_0^*\right) &= v_0(s') - \left[\int_{\underline{s}}^{s'} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \bar{U}_0\left(M^F | \underline{s}\right)\right] \\ &= D\left(\underline{J}^{x_0^*}(\hat{s}), \hat{s}, x_0^*\right) + \int_{\hat{s}}^{s'} v_0'(\tilde{s})\left[1 - x_0^*(\tilde{s})\right] d\tilde{s} \\ &= 0, \end{aligned}$$

which violates the supposition that $D\left(\underline{J}^{x_0^*}(s'), s', x_0^*\right) > 0$. Second, suppose that there exists s'', $D\left(\underline{J}^{x_0^*}(s''), s'', x_0^*\right) < 0$, which implies that $x_0^*(s'') = x_0^F(s'')$. Still, using the continuity of x_0^* and $D\left(\underline{J}^{x_0^*}(s), s, x_0^*\right)$, we obtain that there exists $\hat{s} < s''$, such that $x_0^*(s) = x_0^F(s)$ for $s \in [\hat{s}, \bar{s}]$, $D\left(\underline{J}^{x_0^*}(\hat{s}), \hat{s}, x_0^*\right) = 0$, and

$$\begin{aligned} &D\left(\underline{J}^{x_0^*}(s''), s'', x_0^*\right) - D\left(\underline{J}^{x_0^*}(\hat{s}), \hat{s}, x_0^*\right) \\ &= v_0(s'')\, x_0^F(s'') + \int_T \sum_{i=1}^{n} J_i(s'', t_i)\, x_i^F(s'', t)\, f(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&- v_0(\hat{s})\, x_0^F(\hat{s}) + \int_T \sum_{i=1}^n J_i(\hat{s}, t_i)\, x_i^F(\hat{s}, t)\, f(t)\, dt \\
&- \int_{\underline{\check{s}}}^{\check{s}} v_0'(\tilde{s})\, x_0^F(\tilde{s})\, d\tilde{s} \\
> 0.
\end{aligned}
$$

The inequality is due to the fact that the profile $\left\{\left(x^{F,s}, p^{F,s}\right) : s \in S\right\}$ is not an incentive compatible strategy of the seller. (See equation (11)). Therefore, $D\left(\underline{J}^{x_0^*}(s), s, x_0^*\right) = 0$ for all $s \in S$.

Proof of Proposition 5

Suppose that the seller with type s deviates from $(x^*(s, \cdot), p^*(s, \cdot), b^*(s, \cdot))$ to an off-equilibrium mechanism M in the mechanism-selection game. The mechanism M may not be a direct mechanism. (There is always an equilibrium for the continuation game following mechanism M under belief $\Pr(\underline{s}) = 1$, in which no buyer participates in the mechanism. To make the analysis non-trivial, I assume that upon observing M, all the buyers choose to participate if M has an equilibrium played by the buyers under belief $\Pr(\underline{s}) = 1$.) According to the revelation principle, we can find a direct mechanism $\hat{M} = (\hat{x}, \hat{p}, \hat{b})$ that is incentive feasible under belief $\Pr(\underline{s}) = 1$ and gives every player the same interim expected payoff as in the equilibrium of the continuation game following M. In $\hat{M}$, the domain of $\hat{x}$, $\hat{p}$, and $\hat{b}$ is T. The expected payoff of the type-s seller under $\hat{M}$ can be expressed as follows,

$$
v_0(s) + \int_T \sum_{i=1}^n \left[J_i(\underline{s}, t_i) - v_0(s)\right] \hat{x}_i(t)\, f(t)\, dt - \sum_{i=1}^n U_i(\hat{M}|\underline{t}_i) - \hat{b},
$$

where $\hat{b} = \int_T \hat{b}(t)\, f(t)\, dt$. From this expression of expected payoff, we can see that if $\hat{M}$ maximizes the type-s seller's expected payoff under the most pessimistic belief, then $\hat{M}$ satisfies the following conditions.

1. Allocation rule:

$$\begin{cases} \hat{x}_i(t) > 0 \text{ only if } J_i(\underline{s}, t_i) \geq \max\{v_0(s), \max_{k\neq i}\{J_k(\underline{s}, t_k)\}\}, \text{ and} \\ \sum_{i=1}^n \hat{x}_i(t) = 1 \text{ if } \max_k\{J_k(\underline{s}, t_k)\} > v_0(s), \forall i, t, s. \end{cases}$$

2. Envelope condition:

$$U_i(\hat{M}|t_i) = \int_{\underline{t}_i}^{t_i} \frac{\partial v_i(\underline{s}, \tilde{t}_i)}{\partial t_i} \hat{x}_i(\tilde{t}_i)\, d\tilde{t}_i + U_i(\hat{M}|\underline{t}_i), \forall i, t_i.$$

3. Expected payoff to the lowest type of a buyer and money burning:

$$\hat{b} = 0, U_i(\hat{M}|\underline{t}_i) = 0, \forall i.$$

It is not immediately clear whether the optimal $\hat{M}$ gives a lower expected payoff to the type-s seller than $(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot))$, given that the buyers report their types truthfully. I take an indirect approach to prove this. Let $\hat{M}(s) = (\hat{x}(s,\cdot), \hat{p}(s,\cdot), \hat{b}(s,\cdot))$ denote the optimal direct incentive feasible mechanism for the type-s seller under belief $\Pr(\underline{s}) = 1$. According to Milgrom and Segal (2002), for all s,

$$v_0(s) + \int_T \sum_{i=1}^n [J_i(\underline{s}, t_i) - v_0(s)]\, \hat{x}_i(s,t) f(t)\, dt = \int_{\underline{s}}^s v_0'(\tilde{s})\, \hat{x}_0(\tilde{s})\, d\tilde{s} + U_0(M^F|\underline{s}). \quad (40)$$

The left-hand side of equation (40) can be rewritten as below,

$$v_0(s) + \int_T \sum_{i=1}^n [J_i(s, t_i) - v_0(s)]\, \hat{x}_i(s,t) f(t)\, dt \\ - \int_T \sum_{i=1}^n [J_i(s, t_i) - J_i(\underline{s}, t_i)]\, \hat{x}_i(s,t) f(t)\, dt,$$

in which the term $\int_T \sum_{i=1}^n [J_i(s, t_i) - J_i(\underline{s}, t_i)]\, \hat{x}_i(s,t) f(t)\, dt$ is positive, due to Assumption 2. Below I construct a safe mechanism $\breve{M} = (\breve{x}, \breve{p}, \breve{b})$ in the mediated game, which gives any type s of the seller the same payoff as in $\hat{M}(s)$.

1. Allocation rule:

$$\breve{x}(s,t) = \hat{x}(s,t)\,,\ \forall s,t.$$

2. Expected payoff to the lowest type of a buyer and money burning:

$$\begin{aligned}\bar{u}_i(\breve{M}, s|\underline{t}_i) &= \int_T \left[J_i(s,t_i) - J_i(\underline{s},t_i)\right] \breve{x}_i(s,t)\, f(t)\, dt, \text{ and} \\ \breve{b}(s,t) &= 0,\ \forall i,s,t.\end{aligned}$$

3. Payment rule:

$$\breve{p}_i(s,t) = v_i(s,t_i)\,\breve{x}_i(s,t) - \int_{\underline{t}_i}^{t_i} v'_{i2}\left(s,\tilde{t}_i\right) \breve{x}_i\left(s,\tilde{t}_i,t_{-i}\right) d\tilde{t}_i - \bar{u}_i(\breve{M}, s|\underline{t}_i),\ \forall i,s,t.$$

It is easy to check that the mechanism $\breve{M}$ is safe. Since no safe mechanism can give any type of the seller a higher expected payoff than does an RSW mechanism, $\breve{M}(s)$ gives type s of the seller a lower payoff than does $(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot))$, given that the buyers report their types truthfully. Thus, any type $s \in S$ of the seller has no incentive to deviate from mechanism $(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot))$ to any other mechanism, given the off-equilibrium being $\Pr(\underline{s}) = 1$. This implies that $\{(x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot)) : s \in S\}$ is the equilibrium strategy of the seller in a separating equilibrium of the mechanism-selection game. This separating equilibrium is seller optimal, because otherwise (x^*, p^*, b^*) fails to be an RSW mechanism in the mediated game, which raises a contradiction.