

A Class of Markovian Models for the Term Structure of Interest Rates Under Jump-Diffusions

A Thesis Submitted for the Degree of
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by

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Abstract

Jump-Diffusion processes capture the standardized empirical statistical features of interest rate dynamics, thus providing an improved setting to overcome some of the mispricing of derivative securities that arises with the extensively developed pure diffusion models. A combination of jump-diffusion models with state dependent volatility specifications generates a class of models that accommodates the empirical statistical evidence of jump components and the more general and realistic setting of stochastic volatility.

For modelling the term structure of interest rates, the Heath, Jarrow & Morton (1992) (hereafter HJM) framework constitutes the most general and adaptable platform for the study of interest rate dynamics that accommodates, by construction, consistency with the currently observed yield curve within an arbitrage free environment. The HJM model requires two main inputs, the market information of the initial forward curve and the specification of the forward rate volatility. This second requirement of the volatility specification enables the model builder to generate a wide class of models and in particular to derive within the HJM framework a number of the popular interest rate models.

However, the general HJM model is Markovian only in the entire yield curve, thus requiring an infinite number of state variables to determine the future evolution of the yield curve. By imposing appropriate conditions on the forward rate volatility, the HJM model can admit finite dimensional Markovian structures, where the generality of the HJM models coexists with the computational tractability of Markovian structures.

The main contributions of this thesis include:

- ▷ *Markovianisation of jump-diffusion versions of the HJM model - Chapters 2 and 3.* Under a specific formulation of state and time dependent forward rate volatility specifications, Markovian representations of a generalised Shirakawa (1991) model are developed. Further, finite dimensional affine realisations of the term structure in terms of forward rates are obtained. Within this framework, some specific classes of jump-diffusion term structure models are examined such as extensions of the Hull & White (1990), (1994) class of models and the Ritchken & Sankarasubramanian (1995) class of models to the jump-diffusion case.

- ▷ *Markovianisation of defaultable HJM models - Chapters 4.* Suitable state dependent volatility specifications, under deterministic default intensity, lead to Markovian defaultable term structures under the Schönbucher (2000), (2003) general HJM framework. The state variables of this model can be expressed in terms of a finite number of benchmark defaultable forward rates. Moving to the more general setting of stochastic intensity of defaultable term structures, we discuss model limitations and an approximate Markovianisation of the system is proposed.
- ▷ *Bond option pricing under jump-diffusions - Chapter 5.* Within the jump-diffusion framework, the pricing of interest rate derivative securities is discussed. A tractable Black-Scholes type pricing formula is derived under the assumption of constant jump volatility specifications and a viable control variate method is proposed for pricing by Monte Carlo simulation under more general volatility specifications.

CHAPTER 1

Introduction

1.1. Literature Review and Motivation

The geometric Brownian Motion process has been extensively used in the finance literature as the fundamental process driving the underlying asset dynamics. However, as is well known this process fails to adequately capture many observed features of such dynamics.

In financial markets, as Merton (1976) argued, “bursts of information” are often reflected in price behavior as jumps. The existence of jump component in the stock market and foreign exchange market has been verified by Ball & Torous (1985), Bates (1996), Jorion (1988) and Pan (2002). More specifically in bond markets, information bursts such as real world phenomena that lead to interventions by central banks, or more local factors such as supply and demand shocks, economic news announcements, cause jumps to interest rate levels. Empirical studies such as Hardouvelis (1988), Balduzzi, Green & Elton (1998) and Green (1998) support the evidence of the impact of information on the Treasury bond market. Additionally, models accommodating skewness and kurtosis such as stochastic volatility or jump-diffusion models, as Chan, Karolyi, Longstaff & Sanders (1992), Das (1998) and Ait-Sahalia (1996) show, appear to fit better the observed interest rates. Thus, drawing motivation from the fact that the observed short rate trajectories exhibit jumps from time to time and the considerable skewness and kurtosis of the empirical distributions of the short rate, a number of jump-diffusion models for the spot rate has been developed, including Ahn & Thompson (1988), Babbs & Webber (1994), Naik & Lee (1995), Das & Foresi (1996), Piazzesi (1999), Das (2002) and Chacko & Das (2002). Within the HJM framework, where the focus is on the forward rate dynamics, Shirakawa (1991) was the first to attempt to incorporate discontinuous forward rate dynamics. Subsequently a very general framework for term structure modelling under marked point processes (of which jump-diffusion processes are a special case) was developed by Björk, Kabanov & Runggaldier (1997). More recent work on jump-diffusion versions of the HJM framework

include Glasserman & Kou (2003), who consider the market model,¹ and Das (2000) who treats a discrete time version of the HJM model.

A combination of jump-diffusion models with stochastic volatility models, as the empirical study of Bakshi, Cao & Chen (1997) on the stock market demonstrates, provides a class of models that fit better empirical facts and internalize the desired return distributions (negative skewness and excess kurtosis implicit in option prices), especially at short time horizons. Regarding term structure models, there are empirical studies such as Chan et al. (1992) and Flesaker (1993) that indicate that deterministic volatility structures of spot rate models driven solely by diffusion processes provide an unsatisfactory fit to real market data. Moreover, more generalised volatility models, such as those analysed in Amin & Morton (1995), fail to incorporate the empirical feature of increasing leptokurtosis as the time interval decreases. Thus, motivated by the above literature, the focus of this thesis is on the jump-enhancement of diffusion term structure models,² and more specifically, a combination of jump-diffusion models under state dependent volatility specifications. The proposed models demonstrate an ability to capture the standardized empirical statistical features of interest rates providing a more accurate and flexible setting for pricing and hedging derivative securities as well as credit risk sensitive instruments.

For modelling the term structure of interest rates, the HJM framework constitutes the most general and adaptable platform for the study of interest rate dynamics that accommodates, by construction, consistency with the currently observed yield curve within an arbitrage free environment. The HJM model requires the market information of the initial forward curve and the specification of the forward rate volatility. This second requirement of the volatility specification enables the model builder to generate a wide class of models, to derive a number of popular interest rate models and to easily extend to the multi-factor version of the HJM model.

However, the HJM model in general is only Markovian in the entire yield curve thus in principle requiring an infinite number of state variables to determine the current state. By

¹Examples of market models (term structure models based on simple forward rates such as LIBOR rates) under diffusions include those of Miltersen, Sandmann & Sondermann (1997), Brace, Gatarek & Musiela (1997) and Jamshidian (1997). Jump-diffusion extensions of market models have been pursued in Glasserman & Merener (2003) and Glasserman & Kou (2003).

²The, now almost classical, term structure models under Wiener diffusions include Vasicek (1977), Cox, Ingersoll & Ross (1985), Heath et al. (1992), Hull & White (1990),(1994), Ritchken & Sankarasubramanian (1995) and Inui & Kijima (1998).

imposing appropriate conditions on the volatility structure, HJM models can admit finite dimensional Markovian representations - a procedure that is called Markovianisation of HJM models - where the generality of the HJM models coexists with the tractability of Markovian representations. This feature makes the model suitable for numerical evaluation techniques such as Monte Carlo simulations, finite difference or tree methods. Early papers on the Markovianisation of HJM models under Wiener diffusions include Cheyette (1992), Carverhill (1994), Ritchken & Sankarasubramanian (1995), Jeffrey (1995) and Bhar & Chiarella (1997), where the conditions on the volatility structure for the spot rate process to be Markovian are examined for one factor HJM models. Inui & Kijima (1998) and de Jong & Santa-Clara (1999) extend these conditions to multi-factor HJM models. Duffie & Kan (1996) developed a square root volatility model. Further, Björk & Landèn (2002), Björk & Svensson (2001) and Chiarella & Kwon (2001b),(2003) generalise the above results in various directions by assuming more general forward rate volatility specifications. Focusing on the Markovianisation of the jump-diffusion version of the HJM class of models, Björk & Gombani (1999) allow forward rates to be driven by a multi-dimensional Wiener process as well as by a marked point process and give the necessary and sufficient conditions on a deterministic volatility structure, for the existence of finite dimensional realizations. They also showed that the state variables constitute a minimal set of benchmark forward rates. Motivated by Björk & Gombani (1999), the research work reported in this thesis deals with the extension that incorporates state dependent volatility structures.

1.1.1. Defaultable term structure models. In the literature that aims to price financial instruments subject to default risk, there are two main modelling approaches: the structural approach, where default is triggered (at maturity or any time during the lifetime of the contract) when the value of the firm falls below a barrier value, and the reduced form approach, where the time of default is modelled directly using jump (e.g. Cox, marked point) processes. The reduced form approach provides a more realistic framework for credit risk modelling since default is triggered by exogenous sources in an unpredictable

manner.³ The defaultable term structure models studied in this thesis belong to the reduced form class of models. Jarrow & Turnbull (1995) first introduced default driven by a single Poisson process with constant intensity and known payoff at default. The more general set-up of stochastic intensities and stochastic payoffs at default was pursued in Madan & Unal (1998). Duffie & Singleton (1997), (1999) and references therein developed an approach based on the assumption that the losses at default are expressed in terms of the fractional reduction in the value of the defaultable security due to the default. Within the HJM framework, Schönbucher (1998), (2000), (2003) considers a model allowing for multiple defaults and restructuring upon defaults. In addition, Bielecki & Rutkowski (2002) and references cited therein treat defaultable HJM type models with multiple ratings for corporate bonds.

However, the empirical implementation of reduced form models is still rather limited, due to the fact that the representations of these models are not in a form that is most convenient for numerical and/or econometric estimation. The class of defaultable models developed in this thesis achieves the goal of providing a flexible and tractable setting within the reduced-form class of models, representations of which are in a form suitable for calibration, parameter estimation and numerical implementations. This thesis, more specifically, handles defaultable term structure models within the HJM framework and discusses the model restrictions that lead to Markovian structures.

1.1.2. Pricing of Interest Rate Derivatives. The evaluation of interest rate derivatives represents another research direction on term structure models that has significant implications for financial markets due to the rapidly increasing trading volume of this type of instruments. The value of an interest-rate option is substantially affected by the presence of skewness and kurtosis on the interest rates. The kurtosis explains the smile effect and results in fat-tailed distributions, while the skewness results in asymmetric interest rate distributions that matches with the empirically observed distributional profile of interest rates. Jump-diffusion and/or stochastic volatility models demonstrate an ability to accommodate these features, thus these classes of models should provide more accurate

³Reduced form models are capable of adapting the well-established methodologies developed within a default-free environment to the pricing of credit risk derivatives. The structural approach remains of fundamental importance, especially for pricing and managing bond portfolios, however, the advantageous modelling settings of the reduced form models make this class of models more attractive. See Duffie & Lando (2001) for an example of the reduced form representation of structural type models.

pricing evaluations. However, these classes of models come at the expense of an increasing complexity that makes it impossible in most of the cases to derive computationally tractable solutions.

Most of the jump-diffusion models of interest rates do not admit tractable closed form solutions even when the jump sizes considered are constant or drawn from a distribution. Studies such as Ahn & Thompson (1988), Ahn (1988), Mercurio & Runggaldier (1993), Naik & Lee (1995), Baz & Das (1996) and Das (1999) are more in the spirit of approximate numerical analysis of the evaluation of interest rate instruments. More advanced term structure models of stochastic volatility with jumps have been studied by Bates (1996), Duffie & Kan (1996) and Chacko & Das (2002), however the pricing of derivative instruments within their frameworks is in general not computationally tractable.⁴ Research work on obtaining closed form evaluation formulas for bonds and bond options is quite limited and includes work from Jamshidian (1989), Shirakawa (1991), Heston (1993) (stochastic volatility setting), Das & Foresi (1996) and Glasserman & Kou (2003). The contribution of this thesis is to present two classes of term structure models that combine jump behavior of interest rates and more general volatility specifications and also maintain tractability in the pricing of interest rate derivative securities.

1.2. Thesis Structure

The thesis pursues three main research directions involving term structure modelling of interest rates using jump-diffusion processes. The first component of the thesis, covered in Chapters 2 and 3, analyses the issue of constructing finite dimensional Markovian term structure models, within the extended jump-diffusion HJM framework, by imposing appropriate conditions on the state dependent forward rate volatility structure. The second component, discussed in Chapter 4, considers the adaptation of the Markovianisation procedure to a defaultable term structure environment. The nature of the default intensities (deterministic or stochastic) determines the level of the restrictions on the forward rate volatility processes that would provide finite dimensional realizations. The final component, Chapter 5, deals with the pricing of interest rate derivatives under the jump-diffusion

⁴Numerical studies under diffusions within the HJM framework has been undertaken by Jarrow (1996), Carr & Yang (1998) for deterministic volatility specifications and Amin & Morton (1995), Li, Ritchken & Sankarasubramanian (1995) and Chiarella, Clewlow & Musti (2003) for stochastic volatility specifications.

setting developed in the earlier chapters. Chapter 6 concludes and suggests further directions for research.

1.2.1. Jump-Diffusion versions of the HJM model. There is a great deal of evidence from empirical studies strongly supporting the hypothesis that financial asset prices exhibit jumps that cannot be adequately captured solely by the use of Wiener processes. More specifically, within the HJM term structure modelling framework, where the focus is on the forward rate dynamics, Shirakawa (1991), Björk et al. (1997) and more recently Das (2000) and Glasserman & Kou (2003) developed jump-diffusion term structure models under different jump specifications and in most of these cases under deterministic volatility processes. In this thesis, we deal with two issues related to the HJM model with jump-diffusions. First, we consider the HJM model incorporating state dependent volatilities with jumps, since both of these features capture the empirical characteristics of interest rates. Second, and more importantly, the state space of general HJM models is usually infinite-dimensional and so in general we are dealing with non-Markovian models. One path to tractable HJM models is the procedure known as Markovianisation. As pointed out earlier, Markovianisation of HJM models under Wiener diffusions has been studied extensively, with the extensions to the jump-diffusion case having been studied by Björk & Gombani (1999).

For reasons of completeness, we start in Chapter 2 with a deterministic volatility jump-diffusion model of the term structure of interest rates and discuss the conditions on the deterministic volatility structure that would lead to a finite dimensional Markovian structure. Chapter 2 continues with a multi-factor jump-diffusion model of the term structure of interest rates under a state dependent volatility structure, where the forward rate dynamics are driven by multi-dimensional Wiener and Poisson processes. Working within the HJM framework, as extended to the jump-diffusion case by Shirakawa (1991), we obtain bond prices in an arbitrage free environment which are consistent with the currently observed forward rate curve (by construction) and Markovian in the entire yield curve. By imposing the restrictions of state dependent Wiener volatility processes and time dependent Poisson volatility functions on the volatility structure, finite dimensional Markovian realizations are obtained. Further, the finite number of the state variables of this model can be expressed as functions of a finite number of benchmark forward rates or yields

providing an exponential affine bond price formula in terms of these state variables. The model that we thereby develop provides a fairly broad tractable class of jump-diffusion term structure models that blends the generality of a stochastic volatility jump-diffusion HJM model and the tractability of the Markovian property which makes the model suitable for both calibration and econometric estimation.

Chapter 3 provides an investigation of some particular classes of jump-diffusion term structure models and in order to validate that these models conform to observed stylised facts, some numerical experiments are carried out. Within the framework developed in Chapter 2 and by imposing appropriate volatility specifications, we develop what we believe is the natural extension to the jump-diffusion case of the multi-factor Hull & White (1990), (1994) class of models and the multi-factor Ritchken & Sankarasubramanian (1995) class of models. As a numerical exercise, the effect on the forward rate levels for different Wiener and Poisson volatility specifications is examined and the two classes of models are compared demonstrating the extra flexibility of the state dependent volatility models compared to the deterministic volatility models. Taking advantage of the Markovian spot rate dynamics that these models display and using an Euler-Maruyama approximation of the Markovian system, the simulated distributions of the instantaneous spot rate are estimated. Summarizing the results, the instantaneous spot rate distribution displays the empirically observed skewness that increases as the jump volatility increases, a feature that is again more pronounced with the state dependent volatility models.

1.2.2. Defaultable HJM models. The past decade has witnessed a rapidly increasing interest in research on pricing and hedging financial instruments subject to default risk, which has inevitably changed the way that financial institutions and security traders deal with investment and risk management. The main two credit risk approaches include structural models where default is triggered (at maturity or any time during the lifetime of the contract) when the value of the firm falls below a barrier value, and intensity models where the time of default is modelled directly using jump (e.g. Cox, marked point) processes. The reduced form class of models seems to provide a more realistic platform for credit risk modelling since default is triggered by exogenous sources in an unpredictable manner, however the empirical implementation of such models is still quite limited. This is

in fact partly due to data availability considerations, but also partly to the difficulty in casting the models in a form suitable for econometric estimation.

The extension of the jump-diffusion versions of the HJM framework to the defaultable case may be regarded as a convenient set-up within the reduced form class of models that should generate a tractable class of defaultable models appropriate for numerical applications. In addition, while the Markovianisation of default-free term structure models has been extensively studied, there is a relatively limited literature on the development of Markovian representations of defaultable interest rate models. Chapter 4 analyzes a multi-factor jump-diffusion model of the defaultable term structure of interest rates under a specific volatility structure that leads to a Markovian defaultable term structure set-up. For general defaultable forward rate volatility specifications, the defaultable term structure is infinite dimensional. Schönbucher (2000), (2003) extends the HJM framework and the conditions for the absence of arbitrage to include the term structure of defaultable bond prices. Jumps and defaults are linked by the fact that at times of default, there is a jump in defaultable forward rates. The setting considered in this chapter is slightly different as jumps in the defaultable term structure cause jumps and/or defaults to the defaultable bond prices. Thus, working within the Schönbucher (2000) framework and under appropriate volatility restrictions, Markovian defaultable spot rate dynamics and defaultable bond pricing formulas, in an arbitrage free environment, are obtained. In addition, the state variables of this model can be expressed as functions of a finite number of benchmark defaultable forward rates or yields. However, the resulting model limits the credit spreads (measured by the difference between the defaultable and default free instantaneous spot rates) to evolve in a deterministic manner.

Since empirical research suggests that credit spreads evolve stochastically, we extend the model by allowing for stochastic jump intensities. Then the defaultable term structure model is non-Markovian in what seems to be an essential way, forcing us to settle on an “approximate” Markovian structure. Alternatively, another way to restore path independence is to consider constant Poisson volatilities. The last section of Chapter 4 deals with numerical examples that illustrate the statistical properties of the models developed. By performing Monte Carlo simulations on the Markovian defaultable spot rate dynamics, the effect of parameters such as the jump magnitude and the sign of the jump on the

defaultable spot rate distribution is measured for both deterministic and stochastic intensity models. Concluding, we compare these two classes of models and we find that the stochastic intensity model in general adds skewness to the defaultable spot rate.

1.2.3. Bond Option Pricing under Jump-Diffusions. Jump-diffusion models of interest rates provide an appealing modelling framework within which to price interest-rate derivative instruments as they accommodate the realistic economic features of the smile effect in option prices and of the leptokurtic distributions of the interest rates. Stochastic volatility models of interest rates contribute to the explanation of the smile, however they fail to capture the empirical feature of the increasing leptokurtosis as the time interval decreases. It turns out that the combination of these classes of models provide a better fit to the data, as empirical studies (see Bakshi et al. (1997)) demonstrate, although in terms of derivative pricing evaluations, they imply results that in general are not computationally tractable.

The contribution of Chapter 5 is to present two classes of term structure models that combine jump behavior of interest rates and more general volatility specifications and also maintain tractability in the pricing of interest rate derivatives. More specifically, closed form solutions for bond options are obtained under deterministic volatility specifications and a numerical approximate solution based on Monte Carlo simulations is proposed under the more general stochastic volatility case, which is, however numerically tractable and efficient due in large part to the fact that the term structure model developed admits finite dimensional Markovian representations.

For the deterministic volatility setup, we consider a parameterisation of the Shirakawa (1991) model of the term structure of interest rates where the forward rate dynamics are driven by a multidimensional Poisson-Gaussian process. The pricing of bond options is studied and by making use of Fourier transform techniques, a representation of its solution is obtained. In particular, a tractable Black-Scholes type pricing formula is derived under the assumption of constant jump volatility specifications. An extension of the Shirakawa (1991) framework is considered in the second part of this chapter, in which the volatility evolves stochastically, by the means of state dependent volatility specifications. Again under an appropriate equivalent probability measure, we study the pricing of bond options, however it now becomes very difficult to obtain closed form valuation formulas.

Taking appropriate state dependent volatility specifications, the interest rate dynamics become Markovian in a finite dimensional state variable. By employing these particular Markovian structures, a numerical approximation based on Monte Carlo analysis is undertaken to obtain bond option prices. Finally taking into account the closed form solutions obtained in the deterministic volatility case, a control variate technique is proposed that significantly improves the accuracy and efficiency of the Monte Carlo method.

Markovianisation of the Heath-Jarrow-Morton Model under Jump-Diffusions

This chapter presents a class of models for the term structure of (default-free) interest rate that is a jump-diffusion extension of the HJM model. The forward rate volatility restrictions leading to a finite dimensional Markovian spot rate structure for two versions of the model are discussed. The first one assumes solely deterministic volatility specifications while the second one incorporates state dependent Wiener volatility processes and time dependent Poisson volatility functions. The corresponding Markovian affine term structure of interest rates is obtained and it turns out that the state variables of these models can be expressed as functions of a finite number of benchmark forward rates.

2.1. Introduction

Skewness and kurtosis are two well-established aspects of the spot interest rate distribution in several markets. Table 2.1.1 shows the descriptive statistics for the spot rate for a number of different countries that demonstrate the persistence of these two effects. Of course the instantaneous spot rate is not a market observed quantity. Empirical studies such as Chapman, Long & Pearson (1999) however show that certain one-month rates can be a good proxy for the instantaneous spot rate. The first stylised fact of the empirical behavior is the presence of leptokurtosis in spot rates and spot rate changes. Another stylised fact is the manner in which kurtosis changes. From the Table 2.1.1, we observe that as the sample interval increases (from daily to monthly), the kurtosis of the changes in interest rates decreases. Finally, the maximum and minimum values of the spot rate changes provide us with an indication on realistic jump sizes occurring in spot rates that (for US data) can reach 2% for a positive jump and -1.7% for a negative jump.

Two modelling approaches accommodate skewness and kurtosis: jump-diffusion models and stochastic volatility models. Empirical studies such as Chan et al. (1992), Das (1998) and Ait-Sahalia (1996) demonstrate the ability of such models to provide a better fit to

Descriptive Statistics for the Spot Rate r and Changes in the Spot Rate dr .					
		US daily	UK daily	AUS daily	AUS monthly
Mean	r	5.29	9.03	5.67	5.66
	dr	-0.0003	-0.0015	-0.0013	-0.0284
Variance	r	2.5923	10.0832	1.1737	1.1939
	dr	0.0589	0.0112	0.0014	0.0509
Skewness	r	0.2059	0.3516	0.6847	0.7403
	dr	0.7163	0.9850	-1.1058	-0.2827
Excess Kurtosis	r	-0.6388	-1.1635	-0.8602	-0.7556
	dr	13.4955	36.8401	75.1381	4.5010
Max	r	10.08	15.05	8.35	8.35
	dr	2.15	1.16	0.58	0.86
Min	r	2.17	4.78	4.16	4.18
	dr	-1.74	-1.0313	-0.49	-0.87

TABLE 2.1.1. The US one-month T-bill rate, the UK one-month T-bill rate and the Australian 30-day Treasury Note rate (as percentage) are the data used. The US one-month T-bill rate data span the period April 1986 to May 1996. The UK one-month T-bill rate data span the period May 1986 to April 1997. The Australian 30-days Treasury Note rate span the period November 1991 to June 2002. The data were collected from Datastream.

observed interest rates. Although, with stochastic volatility models, considerable levels of skewness and kurtosis are achieved, the way that kurtosis changes does not accord with the empirical facts, a feature that it is better captured by jump-diffusion models. For jump-diffusion spot interest rate models, we cite Ahn & Thompson (1988), Babbs & Webber (1994), Naik & Lee (1995), Das & Foresi (1996), Piazzesi (1999), Das (2002) and Chacko & Das (2002). Within the HJM term structure modelling framework, where the focus is on the forward rate dynamics, Shirakawa (1991) introduced discontinuous forward rate dynamics, Björk et al. (1997) considered a very general framework for term structure modelling under marked-point processes, Glasserman & Kou (2003) developed jump-diffusion versions of the market model, and Das (2000) treated a discrete time version of the HJM model. Stochastic volatility models for stock prices and exchange rates,¹ have been studied extensively by a number of researchers including Scott (1987), Wiggins (1987), Hull & White (1987), Melino & Turnbull (1990), Stein & Stein (1991), Heston (1993), Amin & Morton (1995) and Melino & Turnbull (1995). Relatively little research

¹Deterministic volatility structures of diffusion spot rate models provide an unsatisfactory fit to real market data as a number of empirical studies has shown including Chan et al. (1992) and Flesaker (1993).

has been carried out on the stochastic volatility jump-diffusion models, apart from the articles of Bates (1996) on foreign exchange markets and Scott (1997) on stock markets. In this chapter, jump-diffusion versions of the HJM model under deterministic forward rate volatilities and further extensions to state dependent forward rate volatilities are considered, as an attempt to merge jump-diffusion models and stochastic volatility models.² In particular, the model is based on the Shirakawa (1991) framework, which assumes only a finite number of possible jump sizes and that there exists a sufficient number of traded bonds to hedge away all of the jump risks, in this way guaranteeing market completeness. The forward rate dynamics are driven by multi-dimensional Wiener and Poisson processes and within the Heath et al. (1992) framework, we obtain bond prices in an arbitrage free environment, even though the spot rate dynamics are non-Markovian. By imposing restrictions on the forward rate volatility structure, a Markovian representation of the stochastic dynamic system driving instantaneous spot rates is obtained. Essentially we extend to the jump-diffusion case the approach of the Markovianisation of HJM models developed by a number of authors.³ In our model, which may be viewed as providing an extension to the framework of Björk & Gombani (1999),⁴ the state variables can be expressed as functions of a finite number of benchmark forward rates or yields. The model that is thereby developed provides a fairly broad tractable class of jump-diffusion term structure models that would be suitable for both calibration and econometric estimation. The structure of this chapter is as follows. In Section 2.2 the Shirakawa jump-diffusion term structure framework is reviewed with a focus on an economic interpretation of the underlying hedging argument. In Section 2.3, under specific volatility structures, first deterministic and second state dependent, the corresponding Markovian representation of the spot rate and bond price dynamics in terms of a finite number of state variables is obtained. In Section 2.4, these state variables are expressed as finite dimensional affine realisations in terms of economic quantities observed in the market, such as discrete tenor

²Bakshi et al. (1997) show that a combination of jump-diffusion models with stochastic volatility models, in the stock market provides a class of models that better fit empirical data.

³We cite the research work by Cheyette (1992), Carverhill (1994), Jeffrey (1995), Ritchken & Sankarasubramanian (1995), Bhar & Chiarella (1997), Inui & Kijima (1998), de Jong & Santa-Clara (1999), Björk & Gombani (1999), Björk & Landèn (2002), Björk & Svensson (2001) and Chiarella & Kwon (2001b),(2003).

⁴Using ideas from state space theory, Björk & Gombani (1999) allow forward rates to be driven by a multi-dimensional Wiener process as well as by a marked point process and give the necessary and sufficient conditions on a deterministic volatility structure, for the existence of finite dimensional realizations.

forward rates and yields. In Section 2.5, the case in which the Poisson volatilities are also state dependent is considered and some insight into the reason why a Markovian representation may not be possible in this case is discussed. However, an approximate Markovian representation may be developed. Section 2.6 discusses some model limitations. More specifically, interest rates may become negative under the current framework and a suitable state dependent volatility structure to deal with this is proposed. Section 2.7 concludes and discusses future directions for research.

2.2. The Model

In this section, some fundamental relationships of the bond market and the main features of the Shirakawa (1991) model are reviewed. The model exposition is in a less abstract setting than that of Shirakawa, as we wish to emphasize more the economic intuition of the underlying hedging argument.

Using $f(t, T)$ to denote the instantaneous forward interest rate at time t for instantaneous borrowing at time $T (\geq t)$, we define as $P(t, T)$, the price at time t of a default-free discount zero-coupon bond with maturity T , i.e.,

$$P(t, T) = \exp \left(- \int_t^T f(t, s) ds \right), \quad (2.2.1)$$

so that $P(T, T) = 1$.

Generalising the basic assumption of Shirakawa (1991), on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$,⁵ the stochastic differential equation for the instantaneous forward rate $f(t, T)$ driven by both Wiener and Poisson risk is given by

$$df(t, T) = \alpha(t, T)dt + \sum_{i=1}^{n_w} \sigma_i(t, T) dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t, T) [dQ_i(t) - \lambda_i dt], \quad (2.2.2)$$

where $\alpha : [0, T] \rightarrow R_+$ is the drift function, $W_i(t)$ are standard Wiener processes ($i = 1, 2, \dots, n_w$), $Q_i(t)$ is a Poisson process with constant intensity λ_i ($i = 1, 2, \dots, n_p$). The

⁵In more formal notation, we assume that $(\Omega, \mathcal{F}, (\mathcal{F})_{0 \leq t \leq T}, \mathbb{P})$ is the probability space equipped with the natural filtration of a vector of standard Wiener processes $W_i(t)$ ($i = 1, 2, \dots, n$) and the Poisson processes $Q_i(t)$ with intensity λ_i ($i = 1, 2, \dots, n_p$), indexed on the time interval $[0, T]$.

Poisson process Q_i models the arrival time of the jump events. Recall that, by definition

$$dQ_i(t) = \begin{cases} 1, & \text{if a jump occurs in the time interval } (t, t + dt) \text{ (with probability } \lambda_i dt), \\ 0, & \text{otherwise (with probability } 1 - \lambda_i dt), \end{cases}$$

and $\mathbb{E}[dQ_i(t) \mid \mathcal{F}_{t-}] = \lambda_i dt$, $\mathbb{E}[(dQ_i)^2(t) \mid \mathcal{F}_{t-}] = \lambda_i dt$. At the Poisson jump times, the jump size is equal to $\beta_i(t, T)$. Under these assumptions, the jump feature is modelled by a multivariate point process, allowing for a finite number of jumps.⁶

The volatility specifications allow for $\sigma_i : [0, T] \rightarrow R_+$, the volatility functions associated with the Wiener noise processes, to be state dependent. Here we consider a specification of the general form

$$\sigma_i(t, T) = \sigma_i(t, T, \bar{f}(t)), \quad \text{for } i = 1, \dots, n_w, \quad (2.2.3)$$

where σ_i are well-defined functions that depend on time, maturity and $\bar{f}(t)$ is a vector of path dependent variables such as the instantaneous spot rate and/or instantaneous forward rates of different fixed maturities. It is assumed that the uncertainty driving the dynamics of the $\bar{f}(t)$ is captured entirely by the Wiener and Poisson processes in equation (2.2.2). It is in this sense that we mean that $\bar{f}(t)$ is path dependent. By omitting this level dependence, we would obtain the special case of time deterministic Wiener volatility functions. The $\beta_i : [0, T] \rightarrow R_+$, the volatility functions associated with the Poisson noise processes are assumed to be only time and maturity dependent (deterministic functions). These volatility specifications generalise the original Shirakawa framework by allowing the Wiener noise and Poisson noise to have separate volatility structures. Such a framework is appropriate if one believes that these different types of shocks impact differently across the forward curve. The empirical study of jump-diffusion interest rate models by Chiarella & Tô (2003) suggests that this may in fact be the case in some markets.

In stochastic integral equation form, equation (2.2.2) may be written

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, T, \bar{f}(s)) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, T) [dQ_i(s) - \lambda_i ds]. \quad (2.2.4)$$

⁶See Runggaldier (2001) for a good survey of jump-diffusion models.

Setting $T = t$ in equation (2.2.4), the stochastic integral equation for the instantaneous spot rate is given by

$$r(t) \equiv f(t, t) = f(0, t) + \int_0^t \alpha(s, t) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t, \bar{f}(s)) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, t) [dQ_i(s) - \lambda_i ds]. \quad (2.2.5)$$

The corresponding stochastic differential equation for the instantaneous spot rate is

$$dr(t) = \vartheta(t) dt + \sum_{i=1}^{n_w} \sigma_i(t, t, \bar{f}(t)) dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t, t) [dQ_i(t) - \lambda_i dt]. \quad (2.2.6)$$

where $\vartheta(t)$ is defined as

$$\begin{aligned} \vartheta(t) = & \frac{\partial}{\partial t} f(0, t) + \alpha(t, t) + \int_0^t \frac{\partial}{\partial t} \alpha(s, t) ds \\ & + \sum_{i=1}^{n_w} \int_0^t \frac{\partial}{\partial t} \sigma_i(s, t, \bar{f}(s)) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \frac{\partial}{\partial t} \beta_i(s, t) [dQ_i(s) - \lambda_i ds]. \end{aligned} \quad (2.2.7)$$

With application of the jump-diffusion version of Ito's lemma, the dynamics for the bond price driven by Wiener and Poisson risk, may be expressed as⁷

$$\frac{dP(t, T)}{P(t^-, T)} = [r(t) + H(t, T, \bar{f}(t))] dt - \sum_{i=1}^{n_w} \zeta_i(t, T, \bar{f}(t)) dW_i(t) - \sum_{i=1}^{n_p} (1 - e^{-\xi_i(t, T)}) dQ_i(t), \quad (2.2.8)$$

where

$$\zeta_i(t, T, \bar{f}(t)) \equiv \int_t^T \sigma_i(t, u, \bar{f}(t)) du, \quad (2.2.9)$$

$$\xi_i(t, T) \equiv \int_t^T \beta_i(t, u) du, \quad (2.2.10)$$

$$H(t, T, \bar{f}(t)) \equiv - \int_t^T \alpha(t, u) du + \sum_{i=1}^{n_w} \frac{1}{2} \zeta_i^2(t, T, \bar{f}(t)) + \sum_{i=1}^{n_p} \lambda_i \xi_i(t, T). \quad (2.2.11)$$

In this economy we have $n_w + n_p$ sources of risk, n_w due to the Wiener processes $W_i(t)$ ($i = 1, \dots, n_w$), and n_p due to the Poisson processes Q_i ($i = 1, \dots, n_p$). Using the classical hedging portfolio argument of Vasicek (1977), which maps to interest rate models

⁷See Proposition 2.2 of Björk et al. (1997). For completeness, we summarise in Appendix 2.1 the main steps in the derivation of (2.2.8).

the original Black-Scholes hedging approach, we thus place bonds of $n_w + n_p + 1$ maturities in the hedging portfolio.⁸ By taking an appropriate position in the $n_w + n_p + 1$ bonds it is possible to eliminate both Wiener and Poisson risks and after some manipulations⁹ to derive the forward rate drift restriction that extends the HJM forward rate drift restriction to now incorporate the jump feature, namely,

$$\begin{aligned} \alpha(t, T) = & \sum_{i=1}^{n_w} \sigma_i(t, T, \bar{f}(t))(-\phi_i(t) + \zeta_i(t, T, \bar{f}(t))) \\ & - \sum_{i=1}^{n_p} \beta_i(t, T)(\psi_i(t)e^{-\xi_i(t, T)} - \lambda_i). \end{aligned} \quad (2.2.12)$$

In equation (2.2.12) the ϕ_i ($i = 1, 2, \dots, n_w$) can be interpreted as the market prices of diffusion risk associated with the Wiener process sources of uncertainty W_i , whilst the ψ_i ($i = 1, 2, \dots, n_p$) are the market prices of jump risk associated with the Poisson process sources of uncertainty Q_i .

2.2.1. The Risk Neutral Dynamics under a General Volatility Specification. By an application of Girsanov's theorem appropriate to the jump-diffusion case (Bremaud (1981)), for every fixed finite time horizon T , we can obtain a unique equivalent probability measure $\tilde{\mathbb{P}}$ ¹⁰, under which the processes $\tilde{W}_i(t) = -\int_0^t \phi_i(s)ds + W_i(t)$ are standard Wiener processes (for $i = 1, \dots, n$) and the processes Q_i are Poisson processes associated with intensity $\psi_i(t)$ such that \tilde{W}_i and Q_i are mutually independent.

Substitution of (2.2.12) into (2.2.8) reduces the stochastic differential equation for the bond price under $\tilde{\mathbb{P}}$ in the now arbitrage free economy to

$$\frac{dP(t, T)}{P(t^-, T)} = r(t)dt - \sum_{i=1}^{n_w} \zeta_i(t, T, \bar{f}(t))d\tilde{W}_i(t) - \sum_{i=1}^{n_p} (1 - e^{-\xi_i(t, T)})[dQ_i(t) - \psi_i(t)dt]. \quad (2.2.13)$$

⁸The subtle issue in the hedging argument concerns whether or not the set of bonds in the hedging portfolio remains fixed over time. The Shirakawa analysis only established the existence of a set of bonds that would possibly change over time. Björk et al. (1997) established that the set of hedging bonds can in fact remain fixed over time.

⁹See Appendix 2.2 for full details of the hedging portfolio argument in the current context. The reader may refer to Björk et al. (1997) for the most general approach to deriving the arbitrage free dynamics for interest rate models under marked point processes.

¹⁰The Wiener processes $\tilde{W}_i(t)$ ($i = 1, \dots, n_w$) and the Poisson processes $Q_i(t)$ ($i = 1, \dots, n_p$) with intensity Ψ_i generate the $\tilde{\mathbb{P}}_t$ -augmentation of the filtration \mathcal{F}_t .

In addition, by obtaining the dynamics of the bond price measured in units of the money market account, the bond price can be expressed as

$$P(t, T) = \tilde{\mathbb{E}} \left[\frac{B(t)}{B(T)} \mid \mathcal{F}_t \right] = \tilde{\mathbb{E}} \left[\exp \left(- \int_t^T r(s) ds \right) \mid \mathcal{F}_t \right], \quad (2.2.14)$$

where $\tilde{\mathbb{E}}$ is expectation (given information at time t) with respect to the equivalent probability (risk neutral) measure $\tilde{\mathbb{P}}$ and $B(t)$ is the accumulated money market account

$$B(t) = \exp \left(\int_0^t r(s) ds \right).$$

Furthermore, by substitution of the drift restriction (2.2.12) for $\alpha(s, t)$ into the equation (2.2.5), we obtain the dynamics of the spot interest rate $r(t)$ under the risk neutral measure $\tilde{\mathbb{P}}$, in the integral form

$$\begin{aligned} r(t) = & f(0, t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds + \sum_{i=1}^{n_p} \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \\ & + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t, \bar{f}(s)) d\tilde{W}_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds]. \end{aligned} \quad (2.2.15)$$

The stochastic differential equations under the risk neutral measure, satisfied by the spot interest rate $r(t)$ is

$$\begin{aligned} dr(t) = & \frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \left(\int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds \right) + \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \right) \\ & + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \left(\int_0^t \sigma_i(s, t, \bar{f}(s)) d\tilde{W}_i(s) \right) + \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds] \right). \end{aligned} \quad (2.2.16)$$

Under a general specification for $\sigma_i(s, t, \bar{f}(s))$ and $\beta_i(s, t)$, the dynamics for $r(t)$ implied by (2.2.16) are non-Markovian due to the path dependence of some or all of the integral terms on the right-hand side of (2.2.16).

Similarly, the risk neutral dynamics of the instantaneous forward rate are obtained by substitution of condition (2.2.12) into (2.2.2)

$$\begin{aligned} df(t, T) = & \left(\sum_{i=1}^{n_w} \sigma_i(t, T, \bar{f}(t)) \zeta_i(t, T, \bar{f}(t)) + \sum_{i=1}^{n_p} \psi_i(t) \beta_i(t, T) [1 - e^{-\xi_i(t, T)}] \right) dt \\ & + \sum_{i=1}^{n_w} \sigma_i(t, T, \bar{f}(t)) d\tilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_i(t, T) [dQ_i(t) - \psi_i(t) dt]. \end{aligned} \quad (2.2.17)$$

2.3. A Specific Volatility Structure

In order to generate specific term structure models and to be able to obtain Markovian representations of the spot rate dynamics (2.2.15), we shall consider more specific volatility structures. Two volatility structures are developed. The first volatility structure is deterministic while the second is stochastic by the means of state dependent Wiener volatility functions.

In many of the common models, the instantaneous spot rate itself is included in the set of state variables. One of our aims is to derive jump-diffusion versions of some of the common interest rate models, thus in the following propositions, we derive the spot rate dynamics in both integral and differential form in terms of a number of stochastic factors and the instantaneous spot rate.

2.3.1. A Deterministic Volatility Structure. First, we assume the case of deterministic volatility specifications given by

ASSUMPTION 2.3.1. *For $i = 1, \dots, n_p$, the deterministic Wiener volatility structure (2.2.3) is of the form*

$$\sigma_i(s, t) = \sigma_{0i}(s)e^{-\int_s^t \kappa_{\sigma_i}(u)du}, \quad (2.3.1)$$

and for $i = 1, \dots, n_w$, the time dependent Poisson volatility functions are of the form

$$\beta_i(s, t) = \beta_{0i}(s)e^{-\int_s^t \kappa_{\beta_i}(u)du}, \quad (2.3.2)$$

where $\kappa_{\sigma_i}(t)$, $\kappa_{\beta_i}(t)$, $\sigma_{0i}(t)$ and $\beta_{0i}(t)$ are time deterministic functions.

We recall that in the no jump situation, the functional form (2.3.1) for the forward rate volatility results in the extended Vasicek model of Hull-White (one-factor model) (see Baxter & Rennie (1996), Chiarella & El-Hassan (1996)) and the Hull-White two-factor and multi-factor models (see Chiarella & Kwon (2001a)). We shall now show that this case gives a Markovian representation of (2.2.15) that may be viewed as a generalisation of the Markovian multi-factor models to the jump-diffusion case.

The derivatives with respect to the second argument (maturity) of the volatility functions (2.3.1) and (2.3.2) are expressed as

$$\frac{\partial}{\partial t} \sigma_i(s, t) = -\kappa_{\sigma_i}(t) \sigma_i(s, t), \quad (2.3.3)$$

for $i = 1, \dots, n_w$, and

$$\frac{\partial}{\partial t} \beta_i(s, t) = -\kappa_{\beta_i}(t) \beta_i(s, t), \quad (2.3.4)$$

for $i = 1, \dots, n_p$. This is a natural consequence of their functional forms, that allows the separation of the time dependent component from the maturity dependent component. As pointed out by Chiarella & Kwon (2003), this is in fact the key property of the volatility functions that leads to the Markovianisation results. The spot rate dynamics in terms of the stochastic factors are derived in the following proposition.

PROPOSITION 2.3.1. *Let $\sigma_i(s, t)$ and $\beta_i(s, t)$, satisfy Assumption 2.3.1. Then the dynamics for the spot rate can be expressed as*

$$r(t) = f(0, t) + \sum_{i=1}^{n_w} \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}(t), \quad (2.3.5)$$

in stochastic integral equation form, or as

$$\begin{aligned} dr(t) = & \left[D(t) - \sum_{i=2}^{n_w} \hat{\kappa}_{\sigma_i}(t) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} \hat{\kappa}_{\beta_i}(t) \mathcal{D}_{\beta_i}(t) - \kappa_{\sigma_1}(t) r(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\tilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (2.3.6)$$

in stochastic differential equation form, where

$$D(t) \equiv \kappa_{\sigma_1}(t) f(0, t) + \frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta_i}(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma_i}(t), \quad (2.3.7)$$

$$\hat{\kappa}_{\sigma_i}(t) \equiv \kappa_{\sigma_i}(t) - \kappa_{\sigma_1}(t), \quad (2.3.8)$$

$$\hat{\kappa}_{\beta_i}(t) \equiv \kappa_{\beta_i}(t) - \kappa_{\sigma_1}(t), \quad (2.3.9)$$

and

$$\mathcal{E}_{\sigma_i}(t) \equiv \int_0^t \sigma_i^2(s, t) ds, \quad (2.3.10)$$

$$\mathcal{E}_{\beta_i}(t) \equiv \int_0^t \psi_i(s) \beta_i^2(s, t) e^{-\xi_i(s, t)} ds, \quad (2.3.11)$$

$$\mathcal{D}_{\sigma_i}(t) \equiv \int_0^t \sigma_i(s, t) \zeta_i(s, t) ds + \int_0^t \sigma_i(s, t) d\widetilde{W}_i(s), \quad (2.3.12)$$

$$\mathcal{D}_{\beta_i}(t) \equiv \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds + \int_0^t \beta_i(s, t) (dQ_i(s) - \psi_i(s) ds). \quad (2.3.13)$$

Proof. Equation (2.3.5) is the result of using definitions (2.3.12) and (2.3.13) in (2.2.15). Taking the stochastic differential of (2.3.5) and making use of properties (2.3.3) and (2.3.4), the stochastic differential equation for the instantaneous spot rate under the risk neutral measure becomes

$$\begin{aligned} dr(t) = & \left[\frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \int_0^t \sigma_i(s, t) \zeta_i(s, t) ds + \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \right) \right. \\ & \left. - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i(s, t) d\widetilde{W}_i(s) - \sum_{i=1}^{n_p} \kappa_{\beta_i}(t) \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds] \right] dt \quad (2.3.14) \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned}$$

which, by using the results of Appendix 2.3, may be expressed as

$$\begin{aligned} dr(t) = & \left[\frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i^2(s, t) ds - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i(s, t) \zeta_i(s, t) ds \right. \\ & + \sum_{i=1}^{n_p} \int_0^t \psi_i(s) \beta_i^2(s, t) e^{-\xi_i(s, t)} ds - \kappa_{\beta_i}(t) \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \\ & \left. - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i(s, t) d\widetilde{W}_i(s) - \sum_{i=1}^{n_p} \kappa_{\beta_i}(t) \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds] \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt]. \quad (2.3.15) \end{aligned}$$

By using relation (2.3.5), one of the stochastic factors may be expressed in terms of the spot rate and the remaining stochastic factors. For instance, here we take

$$\mathcal{D}_{\sigma_1}(t) = r(t) - f(0, t) - \sum_{i=2}^{n_w} \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}(t).$$

By using definitions (2.3.12) and (2.3.13) and substitution of this last expression into equation (2.3.15) leads to the dynamics (2.3.6). \square

The $\mathcal{E}_{\beta i}(t)$ and $\mathcal{E}_{\sigma i}(t)$ are deterministic functions of time, whereas the $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ are stochastic quantities depending on the path history up to time t . These stochastic quantities satisfy stochastic differential equations with drifts and diffusion terms that depend on the current value of these quantities, as the next Proposition shows.

PROPOSITION 2.3.2. *Given the forward rate volatility specifications of Assumption 2.3.1 and assuming that the market prices of jump risk are deterministic, the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ satisfy the stochastic differential equations,*

$$d\mathcal{D}_{\sigma i}(t) = [\mathcal{E}_{\sigma i}(t) - \kappa_{\sigma i}(t)\mathcal{D}_{\sigma i}(t)]dt + \sigma_{0i}(t)d\widetilde{W}_i(t), \quad (2.3.16)$$

and

$$d\mathcal{D}_{\beta i}(t) = [\mathcal{E}_{\beta i}(t) - \kappa_{\beta i}(t)\mathcal{D}_{\beta i}(t)]dt + \beta_{0i}(t)[dQ_i(t) - \psi_i(t)dt]. \quad (2.3.17)$$

Proof. Taking the differential of the quantities (2.3.11) and (2.3.12), the stated results follow. \square

Since the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ display Markovian dynamics, the instantaneous spot rate dynamics (2.3.6) are Markovian under the forward rate volatility specifications of Assumption 2.3.1. In the following section, an exponentially affine term structure of interest rates in terms of these state variables is obtained.

2.3.1.1. Affine Term Structure of Interest Rates. As the Inui & Kijima (1998) approach indicates, by substitution of the risk neutral forward rate dynamics and the volatility specifications (2.3.1) and (2.3.2) into the fundamental relationship between bond prices and forward rates in equation (2.2.1) and manipulating the resulting integrals, we obtain the multi-factor bond price formula in terms of the instantaneous spot rate $r(t)$ and the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$.

PROPOSITION 2.3.3. *Under Proposition 2.3.2, the bond price assumes the multi-factor exponential affine form given by*

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \mathcal{M}(t, T) - \mathcal{N}_{\sigma_1}(t, T)r(t) - \sum_{i=2}^{n_w} (\mathcal{N}_{\sigma_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T))\mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} (\mathcal{N}_{\beta_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T))\mathcal{D}_{\beta_i}(t) \right\}, \quad (2.3.18)$$

where,

$$\begin{aligned} \mathcal{M}(t, T) \equiv & \mathcal{N}_{\sigma_1}(t, T)f(0, t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T)\mathcal{E}_{\sigma_i}(t) \\ & - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s)\beta_i(s, y)[1 - e^{-\xi_i(s, y)}]dyds \\ & + \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \int_0^t \psi_i(s)\beta_i(s, t)[1 - e^{-\xi_i(s, t)}]ds, \end{aligned} \quad (2.3.19)$$

and

$$\mathcal{N}_x(t, T) \equiv \int_t^T e^{-\int_t^y \kappa_x(u)du} dy, \quad x \in \{\sigma_i, \beta_i\}. \quad (2.3.20)$$

Proof. See Appendix 2.4 for details. \square

The bond price formula (2.3.18) displays a finite dimensional affine structure in terms of a number of state variables ($n_w + n_p$ in the deterministic volatility case) that are driven by diffusion processes and jump processes. In particular, the state variables $\mathcal{D}_{\sigma_i}(t)$ are driven by pure diffusion processes, whereas the state variables $\mathcal{D}_{\beta_i}(t)$ are driven by pure jump processes.

In the following sections, the extended model incorporating state dependent volatilities is developed, a model that under a stochastic volatility setting allows interest rates to experience jumps. These class of models aim to capture more effectively the empirically observed behavior of interest rates and provide a more accurate framework for the pricing of interest rate derivative instruments.

2.3.2. A State Dependent Volatility Structure. To make the discussion explicit, the state vector considered is $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_k))^T$, where T_1, T_2, \dots, T_k are a set of fixed maturities.

ASSUMPTION 2.3.2. For $i = 1, \dots, n_p$, the state dependent Wiener volatility structure (2.2.3) is of the form

$$\sigma_i(s, t, \bar{f}(s)) = \sigma_{0i}(s, \bar{f}(s)) e^{-\int_s^t \kappa_{\sigma_i}(u) du}, \quad (2.3.21)$$

and for $i = 1, \dots, n_w$, the time dependent Poisson volatility functions continue to be of the form (2.3.2), namely

$$\beta_i(s, t) = \beta_{0i}(s) e^{-\int_s^t \kappa_{\beta_i}(u) du},$$

where $\kappa_{\sigma_i}(t)$, $\kappa_{\beta_i}(t)$ and $\beta_{0i}(t)$ are deterministic functions of time and $\sigma_{0i}(t, \bar{f}(t))$ are time and state dependent functions.

Similar to the volatility specifications of Assumption 2.3.1, since the dependence on the state variable $\bar{f}(s)$ does not affect the derivative of the volatility functions with respect to maturity. Thus we retain the crucial property of the volatility functions that their derivatives with respect to the second argument (maturity) are given by

$$\frac{\partial}{\partial t} \sigma_i(s, t, \bar{f}(s)) = -\kappa_{\sigma_i}(t) \sigma_i(s, t, \bar{f}(s)), \quad (2.3.22)$$

for $i = 1, \dots, n_w$, and

$$\frac{\partial}{\partial t} \beta_i(s, t) = -\kappa_{\beta_i}(t) \beta_i(s, t), \quad (2.3.23)$$

for $i = 1, \dots, n_p$.

PROPOSITION 2.3.4. Let $\sigma_i(s, t, \bar{f}(s))$ and $\beta_i(s, t)$, satisfy Assumption 2.3.2. Then the dynamics for the spot rate can be expressed as

$$r(t) = f(0, t) + \sum_{i=1}^{n_w} \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}(t), \quad (2.3.24)$$

in stochastic integral equation form, or as

$$\begin{aligned} dr(t) = & \left[\bar{D}(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma_i}(t) - \sum_{i=2}^{n_w} \hat{\kappa}_{\sigma_i}(t) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} \hat{\kappa}_{\beta_i}(t) \mathcal{D}_{\beta_i}(t) - \kappa_{\sigma_1} r(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (2.3.25)$$

in stochastic differential equation form, where

$$\bar{D}(t) \equiv \kappa_{\sigma 1} f(0, t) + \frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta i}(t), \quad (2.3.26)$$

and $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ in the current context are defined as

$$\mathcal{E}_{\sigma i}(t) \equiv \int_0^t \sigma_i^2(s, t, \bar{f}(s)) ds, \quad (2.3.27)$$

$$\mathcal{D}_{\sigma i}(t) \equiv \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds + \int_0^t \sigma_i(s, t, \bar{f}(s)) d\widetilde{W}_i(s), \quad (2.3.28)$$

whilst $\mathcal{E}_{\beta i}(t)$ and $\mathcal{D}_{\beta i}(t)$ continue to be defined as in equations (2.3.11) and (2.3.13) respectively, and $\hat{\kappa}_{\sigma i}(t)$, $\hat{\kappa}_{\beta i}(t)$ continue to be defined as in (2.3.8) and (2.3.9) respectively.

Proof. Substitution of the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ into (2.2.15) yields (2.3.24). Taking the stochastic differential of (2.3.24) and making use of the properties (2.3.22) and (2.3.23), the stochastic differential equation for the instantaneous spot rate under the risk neutral measure becomes

$$\begin{aligned} dr(t) = & \left[\frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds + \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \right) \right. \\ & \left. - \sum_{i=1}^{n_w} \kappa_{\sigma i}(t) \int_0^t \sigma_i(s, t, \bar{f}(s)) d\widetilde{W}_i(s) + \sum_{i=1}^{n_p} \kappa_{\beta i}(t) \int_0^t \beta_i(s, t) (dQ_i(s) - \psi_i(s) ds) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (2.3.29)$$

which, by use of the results of Appendix 2.3, may be expressed as

$$\begin{aligned} dr(t) = & \left[\frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i^2(s, t, \bar{f}(s)) ds - \sum_{i=1}^{n_w} \kappa_{\sigma i}(t) \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds \right. \\ & + \sum_{i=1}^{n_p} \int_0^t \psi_i(s) \beta_i^2(s, t) e^{-\xi_i(s, t)} ds - \kappa_{\beta i}(t) \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \\ & \left. - \sum_{i=1}^{n_w} \kappa_{\sigma i}(t) \int_0^t \sigma_i(s, t, \bar{f}(s)) d\widetilde{W}_i(s) - \sum_{i=1}^{n_p} \kappa_{\beta i}(t) \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds] \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt]. \end{aligned} \quad (2.3.30)$$

Relation (2.3.24) allows us to express one of the stochastic factors in terms of the spot rate and the remaining stochastic factors. Doing so for the first stochastic factor yields

$$\mathcal{D}_{\sigma_1}(t) = r(t) - f(0, t) - \sum_{i=2}^{n_w} \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}(t). \quad (2.3.31)$$

By substituting (2.3.31) into the stochastic differential equation (2.3.30), leads to the dynamics (2.3.25). \square

Under the current set-up that follows from Assumption 2.3.2, the $\mathcal{E}_{\beta_i}(t)$ are deterministic functions of time, whereas the $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ are stochastic quantities depending on the path history up to time t . These stochastic quantities satisfy stochastic differential equations with drifts and diffusion terms that depend on the current values of these quantities and the state variables $\bar{f}(t)$, as the next Proposition shows. Note that under the deterministic volatility structure of Assumption 2.3.1, the $\mathcal{E}_{\sigma_i}(t)$ were time dependent quantities, so that there the set of the stochastic variables was smaller.

PROPOSITION 2.3.5. *Given the forward rate volatility specifications of Assumption 2.3.2 and assuming that the market prices of jump risk are time deterministic, the stochastic quantities $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ satisfy the stochastic differential equations,*

$$d\mathcal{E}_{\sigma_i}(t) = [\sigma_{0i}^2(t, \bar{f}(t)) - 2\kappa_{\sigma_i}(t)\mathcal{E}_{\sigma_i}(t)]dt, \quad (2.3.32)$$

$$d\mathcal{D}_{\sigma_i}(t) = [\mathcal{E}_{\sigma_i}(t) - \kappa_{\sigma_i}(t)\mathcal{D}_{\sigma_i}(t)]dt + \sigma_{0i}(t, \bar{f}(t))d\widetilde{W}_i(t), \quad (2.3.33)$$

and

$$d\mathcal{D}_{\beta_i}(t) = [\mathcal{E}_{\beta_i}(t) - \kappa_{\beta_i}(t)\mathcal{D}_{\beta_i}(t)]dt + \beta_{0i}(t) [dQ_i(t) - \psi_i(t)dt]. \quad (2.3.34)$$

Proof. Taking the differential of the quantities (2.3.27), (2.3.11) and (2.3.28), the stated results follow. \square

Thus, the instantaneous spot rate dynamics (2.3.25) are Markovian under the forward rate volatility specifications of Assumption 2.3.2, since the stochastic quantities $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$, $\mathcal{D}_{\beta_i}(t)$ and $\bar{f}(t)$ display Markovian dynamics.¹¹

¹¹As stated in Proposition 2.3.5, the Markovianisation obtained depends on the assumption that the market prices of jump risk are non-stochastic. If one in fact wished to allow these to be stochastic (say for empirical

We note that the drift term in (2.3.25) is a linear combination of $2n_w + n_p$ state variables, determined by (2.3.32), (2.3.33), (2.3.34) and the spot rate. In the following section, an exponentially affine term structure of interest rates in terms of these state variables is derived.

2.3.2.1. Affine Term Structure of Interest Rates. We obtain the multi-factor bond price formula in terms of the state variables $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$, and $\mathcal{D}_{\beta_i}(t)$, by using the Inui & Kijima (1998) approach. This consists of a direct substitution of the risk neutral forward rate dynamics (2.2.17) and the volatility specifications (2.3.21) and (2.3.2) into the fundamental relationship between bond prices and forward rates in equation (2.2.1) and manipulating the resulting integrals.

PROPOSITION 2.3.6. *Under Assumption 2.3.2 the bond price assumes the multi-factor exponential affine form given by*

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \bar{\mathcal{M}}(t, T) - \mathcal{N}_{\sigma_1}(t, T)r(t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}(t) \right. \quad (2.3.35) \\ \left. - \sum_{i=1}^{n_w} (\mathcal{N}_{\sigma_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} (\mathcal{N}_{\beta_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\beta_i}(t) \right\},$$

where $\bar{\mathcal{M}}(t, T)$ is defined as

$$\bar{\mathcal{M}}(t, T) \equiv \mathcal{N}_{\sigma_1}(t, T) f(0, t) - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s) \beta_i(s, y) [1 - e^{-\xi_i(s, y)}] dy ds \quad (2.3.36) \\ + \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds,$$

and $\mathcal{N}_x(t, T)$ is given by (2.3.20).

Proof. See Appendix 2.4 for details. □

The bond price formula (2.3.35) displays a finite dimensional affine structure in terms of a number of state variables ($2n_w + n_p$ in the state dependent volatility case). In particular, the state variables $\mathcal{E}_{\sigma_i}(t)$ are driven by jump-diffusion processes due to the dependency on the $\bar{f}(t)$, the state variables $\mathcal{D}_{\sigma_i}(t)$ are driven by pure diffusion processes, whereas the state variables $\mathcal{D}_{\beta_i}(t)$ are driven by pure jump processes. These stochastic factors (

studies) then one could still obtain a Markovian representation if the ψ_i were assumed to follow some Markovian system of stochastic differential equations.

namely $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$) have no easy economic interpretation, they are merely quantities that summarize the path dependency up to time t . It would be very convenient and more intuitive for applications if we could express these stochastic factors in terms of economic quantities observed in the market, like forward rates, whose dynamics would be driven by combined jump-diffusion processes. The next section shows that these stochastic factors can indeed be expressed in terms of benchmark forward rates with dynamics driven by jump-diffusion processes.

2.4. Finite Dimensional Affine Realisations in Terms of Forward Rates

Here we employ the basic ideas from Chiarella & Kwon (2003) and Björk & Svensson (2001), who show that, in a Markovian HJM framework with dynamics driven by diffusion processes, the state variables can be expressed as affine functions of a finite number of forward rates and yields. The jump component is introduced into their modelling framework and state dependent Wiener volatility functions and time deterministic Poisson volatility functions are assumed. It seems that the inclusion of jumps makes it very hard and likely impossible to derive Markovianisation results under more general volatility specifications that allow the jump volatility functions to be stochastic. However, Section 2.5 indicates how an approximate Markovianisation may be found in this case.

Under the different volatility specifications of Assumption 2.3.1 and Assumption 2.3.2, the number of the state variables of the Markovian affine term structures of interest rates is different. In the case of the deterministic volatility specifications of Assumption 2.3.1, the system includes $n_w + n_p$ state variables, while in the stochastic volatility specifications of Assumption 2.3.2, the number of state variables is $2n_w + n_p$. By using the corresponding exponential affine term structure of interest rates (2.3.18) and (2.3.35), the corresponding expressions for the forward rate are obtained and the instantaneous forward rate may be expressed in terms of these state variables. By taking the appropriate number of forward rates of fixed maturities, the state variables may be expressed in terms of these benchmark forward rates. This procedure is described next, for the stochastic volatility case of Assumption 2.3.2. The deterministic volatility case of Assumption 2.3.1 is treated similarly and the details are given in Appendix 2.5.

Consider the exponential affine term structure of interest rates (2.3.35), where the bond price is a function of the instantaneous spot rate $r(t)$, and the stochastic quantities $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\beta_i}(t)$, and $\mathcal{D}_{\sigma_i}(t)$. Then the instantaneous forward rate can be expressed as (from equation (2.2.1))

$$\begin{aligned} f(t, T) - f(0, T) + \frac{\partial \bar{\mathcal{M}}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} r(t) &= \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \mathcal{E}_{\sigma_i}(t) \\ &+ \sum_{i=2}^{n_w} \left(\frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\beta_i}(t), \end{aligned} \quad (2.4.1)$$

where the $\mathcal{N}_x(t, T)$ ($x \in \{\sigma_i, \beta_i\}$) are defined in equation (2.3.20).

Consider a number of fixed forward rate maturities equal to the number of state variables remaining after excluding the instantaneous spot rate $r(t)$. Then these state variables may be expressed in terms of forward rates with different fixed maturities. Thus, $\bar{n}_s (= 2n_w + n_p - 1)$ forward rates of different fixed maturities T_h are required.

PROPOSITION 2.4.1. *The forward rate of any maturity can be expressed in terms of the \bar{n}_s benchmark forward rates and the instantaneous spot rate $r(t)$ ¹² as*

$$f(t, T) = f(0, T) + \bar{\mathcal{Q}}(t, T) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h(t, T) f(t, T_h) + \bar{\mathcal{S}}(t, T) r(t), \quad (2.4.2)$$

where, for $l = n_w + q - 1$ and $k = 2n_w + i - 1$,

$$\begin{aligned} \bar{\mathcal{Q}}(t, T) &= \frac{\partial \bar{\mathcal{M}}(t, T)}{\partial T} - \sum_{h=1}^{n_s} \left(\frac{\partial \bar{\mathcal{M}}(t, T_h)}{\partial T_h} - f(0, T_h) \right) \left[\sum_{i=1}^{n_w} \bar{\omega}_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \right. \\ &+ \left. \sum_{q=2}^{n_w} \bar{\omega}_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \bar{\omega}_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \right], \end{aligned} \quad (2.4.3)$$

$$\begin{aligned} \bar{\mathcal{R}}_h(t, T) &= \sum_{i=1}^{n_w} \bar{\omega}_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) + \sum_{q=2}^{n_w} \bar{\omega}_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \\ &+ \sum_{i=1}^{n_p} \bar{\omega}_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right), \end{aligned} \quad (2.4.4)$$

¹²Only up to time $t = \min_h T_h$. By reparameterising in terms of fixed time-to-maturity forward rates $f(t, t + T_h)$, we may allow for any $t \in \mathbb{R}_+$, a representation which would actually be more amenable to empirical estimation.

and

$$\begin{aligned} \bar{S}(t, T) &= \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} - \sum_{h=1}^{n_s} \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \left(\sum_{i=1}^{n_w} \bar{\omega}_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \right) \\ &+ \sum_{q=2}^{n_w} \bar{\omega}_{1h} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \bar{\omega}_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right), \end{aligned} \quad (2.4.5)$$

and $\bar{\omega}_{\ell h}$ denotes the ℓh^{th} element of matrix $\bar{\mathbb{O}}^{-1}$, the inverse of the square matrix $\bar{\mathbb{O}}(t)$ such that

$$\bar{\mathbb{O}}(t) = \begin{bmatrix} \bar{\varphi}_1(t) & \bar{\varphi}_2(t) & \bar{\varphi}_3(t) \end{bmatrix}, \quad (2.4.6)$$

where, for $i = 1, 2, \dots, n_w$, $\bar{\varphi}_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma_i}(t, T_h)}{\partial T_h} \mathcal{N}_{\sigma_i}(t, T_h) \right]$ is a $\bar{n}_s \times n_w$ matrix,

for $q = 2, \dots, n_w$, $\bar{\varphi}_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma_q}(t, T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \right]$, is a $\bar{n}_s \times (n_w - 1)$ matrix, and

for $i = 1, 2, \dots, n_p$, $\bar{\varphi}_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_i}(t, T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \right]$, is a $\bar{n}_s \times n_p$ matrix.

Assume that $\bar{\mathbb{O}}(t)$ is invertible for all $t \in \{t; t = \min_h T_h\}$.

Proof. Considering equation (2.4.1) for the maturities $T_1, T_2, \dots, T_{\bar{n}_s}$ we obtain the system

$$\begin{bmatrix} f(t, T_1) - f(0, T_1) + \frac{\partial \bar{\mathcal{M}}(t, T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_1)}{\partial T_1} r(t) \\ f(t, T_2) - f(0, T_2) + \frac{\partial \bar{\mathcal{M}}(t, T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_2)}{\partial T_2} r(t) \\ \vdots \\ f(t, T_{\bar{n}_s}) - f(0, T_{\bar{n}_s}) + \frac{\partial \bar{\mathcal{M}}(t, T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} r(t) \end{bmatrix} = \bar{\mathbb{O}}(t) \times \begin{bmatrix} \mathcal{E}_{\sigma_1}(t) \\ \vdots \\ \mathcal{E}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\sigma_2}(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\beta_1}(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}(t) \end{bmatrix}.$$

By inverting the matrix $\bar{\mathbb{O}}(t)$, the state variables $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ are expressed in terms of forward rates of \bar{n}_s distinct maturities as

$$\begin{bmatrix} \mathcal{E}_{\sigma_1}(t) \\ \vdots \\ \mathcal{E}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\sigma_2}(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\beta_1}(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}(t) \end{bmatrix} = \bar{\mathbb{O}}^{-1}(t) \times \begin{bmatrix} f(t, T_1) - f(0, T_1) + \frac{\partial \bar{\mathcal{M}}(t, T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_1)}{\partial T_1} r(t) \\ f(t, T_2) - f(0, T_2) + \frac{\partial \bar{\mathcal{M}}(t, T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_2}(t, T_1)}{\partial T_2} r(t) \\ \vdots \\ f(t, T_{\bar{n}_s}) - f(0, T_{\bar{n}_s}) + \frac{\partial \bar{\mathcal{M}}(t, T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} r(t) \end{bmatrix}. \quad (2.4.7)$$

By substitution of expressions (2.4.7) for the state variables into the forward rate formula (2.4.1), one obtains (2.4.2) which expresses the forward rate of any maturity in terms of the \bar{n}_s forward rates and the instantaneous spot rate $r(t)$. \square

The following proposition displays the corresponding bond price formula.

PROPOSITION 2.4.2. *The zero-coupon bond prices in terms of the \bar{n}_s benchmark forward rates and the instantaneous spot rate $r(t)$ are given by the exponential affine form*

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \bar{\mathcal{Q}}^P(t, T) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^P(t, T) f(t, T_h) + \bar{\mathcal{S}}^P(t, T) r(t) \right\}, \quad (2.4.8)$$

where

$$\bar{\mathcal{Q}}^P(t, T) \equiv - \int_t^T \bar{\mathcal{Q}}(t, s) ds, \quad \bar{\mathcal{R}}_h^P(t, T) \equiv - \int_t^T \bar{\mathcal{R}}_h(t, s) ds, \quad \text{and} \quad \bar{\mathcal{S}}^P(t, T) \equiv - \int_t^T \bar{\mathcal{S}}(t, s) ds.$$

Proof. By substitution of (2.4.2) into the fundamental relationship (2.2.1). \square

The risk neutral dynamics for each benchmark forward rate $f(t, T_h)$ are given by (recall (2.2.17))

$$\begin{aligned} df(t, T_h) = & \left(\sum_{i=1}^{n_w} \sigma_i(t, T_h, \bar{f}(t)) \zeta_i(t, T_h, \bar{f}(t)) + \sum_{i=1}^{n_p} \psi_i(t) \beta_i(t, T_h) [1 - e^{-\xi_i(t, T_h)}] \right) dt \\ & + \sum_{i=1}^{n_w} \sigma_i(t, T_h, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_i(t, T_h) [dQ_i(t) - \psi_i(t) dt]. \end{aligned} \quad (2.4.9)$$

By using the system (2.4.7), the dynamics (2.3.25) for the spot rate $r(t)$, under the forward rate volatility specifications (2.3.21) of Assumption 2.3.2, can be expressed in terms of the state vector (set $k = \bar{n}_s$)

$$\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_{\bar{n}_s}))^\top,$$

as

$$\begin{aligned} dr(t) = & \left[D^f(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) f(t, T_h) - \bar{\mathcal{S}}^f(t) r(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (2.4.10)$$

where, for $l = n + q - 1$ and $k = 2n + i - 1$,

$$\bar{\mathcal{R}}_h^f(t) = \sum_{i=1}^{n_w} \varpi_{ih} - \sum_{q=2}^{n_w} \hat{\kappa}_{\sigma q}(t) \varpi_{lh} - \sum_{j=1}^{n_p} \hat{\kappa}_{\beta j}(t) \varpi_{kh}, \quad (2.4.11)$$

$$D^f(t) = \bar{D}(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) \left(-f(0, T_h) + \frac{\partial \bar{\mathcal{M}}(t, T_h)}{\partial T_h} \right), \quad (2.4.12)$$

and

$$\bar{\mathcal{S}}^f(t) = k_{\sigma 1}(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) \frac{\partial \mathcal{N}_{\sigma_h}(t, T_1)}{\partial T_h}. \quad (2.4.13)$$

Thus a closed Markovian system for all the elements of the state vector has been obtained.

The advantage in obtaining the bond pricing formula (2.4.8) and the forward rate formula (2.4.2) is that they allow us to transfer the market information of a certain set of distinct forward rate curves and the instantaneous spot rate (in addition to the initial forward curves included in the terms $\frac{\partial \bar{\mathcal{M}}(t, T_h)}{\partial T_h}$) into the bond price and the forward rate curve of any maturity respectively. Also, we have expressed the state variables in terms of the benchmark forward rates with dynamics driven by both Wiener and Poisson processes, rather than the situation that we have in Section 2.3.2.1, where some state variables are driven by pure diffusion processes, and some others are driven by only pure jump processes.

Moreover, the yield to maturity which is defined as $R(t, T) \equiv -\ln P(t, T)/(T - t)$, becomes in terms of the forward rate

$$R(t, T) = -\frac{\int_t^T f(t, u)du}{T - t}, \quad (2.4.14)$$

and using expression (2.4.2) we could express the yield to maturity in terms of the same set of forward rates mentioned above. Applying similar invertibility arguments¹³ we may express the forward curve in terms of a set of bonds or yields to maturity. Given that yields of different maturities are observed in the market, this model set-up proves to be very suitable for parameter estimation and model calibration.

REMARK 2.4.1. The reason to include or not the spot rate in the set of the state variables depends on the particular application. Essentially it allows us to relate the class of models developed here to the traditional models (e.g. Hull-White, Ritchken-Sankarasubramanian) that take the instantaneous spot rate as the underlying state variable, as we shall show in Chapter 3. However, the general framework developed does not tie us to such a choice. Any convenient set of interest rates including market observable rates such as forward LIBOR rate, may be used as the state variables. Appendix 2.6 sets out the results for the case when $r(t)$ is not one of the state variables.

2.5. State Dependent Poisson Volatility Structure

In the previous sections, the case in which only the Wiener volatility functions may depend on a number of state variables was considered. In the case where both Wiener and Poisson volatilities are state dependent, it does not seem possible to generalise the results of the previous section. We now indicate why, in the case that both Wiener and Poisson volatilities are state dependent, it seems impossible to obtain Markovian representations of the spot rate dynamics (2.2.15) and so we propose one way to obtain an approximate solution to the problem.

Assume that the Wiener volatilities follow the structure (2.3.21) for $i = 1, \dots, n_p$, and the Poisson volatilities are state dependent by the means of the functional form

$$\beta_i(s, t, \bar{f}(s)) = \beta_{0i}(s, \bar{f}(s))e^{-\int_s^t \kappa_{\beta i}(u)du}. \quad (2.5.1)$$

¹³See Corollary 2 of Chiarella & Kwon (2003).

The derivative of the volatility functions (2.5.1) with respect to the second argument (maturity) still satisfies (2.3.23), so the dynamics of the spot rate (2.3.25) still follow. Given the state dependent volatility specifications (2.3.21) and (2.5.1) (assume that the market prices of jump risk are non-stochastic), all the quantities $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{E}_{\beta_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ are now stochastic.

The difficulty in handling this case arises from the process $\mathcal{E}_{\beta_i}(t)$. Recall that

$$\mathcal{E}_{\beta_i}(t) = \int_0^t \psi_i(s) \beta_i^2(s, t) e^{-\xi_i(s, t)} ds, \quad (2.5.2)$$

and introduce for $n = 2, 3, \dots$, the quantities

$$\mathcal{E}_{\beta_i}^{(n)}(t, \bar{f}) = \int_0^t \psi_i(s) \beta_i^n(s, t, \bar{f}(s)) e^{-\xi_i(s, t, \bar{f}(s))} ds.$$

We seek to obtain the stochastic differential equation for $\mathcal{E}_{\beta_i}(t)$, which from (2.5.2) turns out to be

$$d\mathcal{E}_{\beta_i} = (\psi_i(t) \beta_{0i}(t, \bar{f}(t)) - \kappa_{\beta_i}(t) \mathcal{E}_{\beta_i}(t, \bar{f}(t)) - \mathcal{E}_{\beta_i}^{(2)}(t, \bar{f}(t))) dt.$$

The process $\mathcal{E}_{\beta_i}^{(2)}(t, \bar{f}(t))$ in turn satisfies the stochastic differential equation

$$d\mathcal{E}_{\beta_i}^{(2)}(t, \bar{f}(t)) = \left(\psi_i(t) \beta_{0i}^2(t, \bar{f}) - 2\kappa_{\beta_i}(t) \mathcal{E}_{\beta_i}^{(2)}(t, \bar{f}(t)) - \mathcal{E}_{\beta_i}^{(3)}(t, \bar{f}(t)) \right) dt.$$

Thus, we are dealing with an infinite sequence of processes $\mathcal{E}_{\beta_i}^{(n)}(t, \bar{f}(t))$, since for $n = 2, 3, \dots$ we find that

$$d\mathcal{E}_{\beta_i}^{(n)}(t, \bar{f}(t)) = \left(\psi_i(t) \beta_{0i}^n(t, \bar{f}(t)) - n\kappa_{\beta_i}(t) \mathcal{E}_{\beta_i}^{(n)}(t, \bar{f}(t)) - \mathcal{E}_{\beta_i}^{(n+1)}(t, \bar{f}(t)) \right) dt.$$

Therefore, when both Wiener and Poisson volatilities are state dependent, it seems that we cannot obtain a Markovian representation, at least not by an approach similar to the one that led to the spot rate dynamics equation (2.3.25). To “close” this sequence will require some approximation. In practice, it would be the case that $\beta^n(t) \simeq 0$, for sufficiently large n (see the magnitudes of the jump component obtained by Chiarella & Tô (2003))¹⁴ so in this way it is possible to achieve an approximate Markovianisation resulting in an approximate affine term structure.

¹⁴In practice the value of n would be fairly low, probably around 3 or 4.

2.6. Model Limitations

The Markovian term structure model developed here does not guarantee positivity of the interest rates, a feature that we must handle with caution given the state dependent nature of the model's volatility functions. In fact, there is a positive probability that this type of dynamics may drive interest rates to negative values. This is due to the functional properties of the jump adjusted drift coefficient. To understand this effect, the functional behavior of the drift coefficient of the forward rate is examined in detail. Recall the risk neutral forward rate dynamics (2.2.17), which under the volatility specifications of Assumption 2.3.2 are expressed as

$$\begin{aligned}
df(t, T) &= \sum_{i=1}^{n_w} \frac{1}{k_{\sigma i}} \sigma_{0i}^2(t, \bar{f}(t)) e^{-\kappa_{\sigma i}(T-t)} (1 - e^{-\kappa_{\sigma i}(T-t)}) dt \\
&\quad - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}(T-t)} \left[e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}(T-t)} - 1)} - 1 \right] dt - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}(T-t)} dt \\
&\quad + \sigma_0(t, \bar{f}(t)) e^{-\kappa_{\sigma}(T-t)} d\widetilde{W}(t) + \sum_{i=1}^2 \beta_{0i} e^{-k_{\beta i}(T-t)} dQ_i(t). \tag{2.6.1}
\end{aligned}$$

The drift function of the forward rate dynamics is bounded by the function $\mathbb{D}(\tau)$ (set $\tau = T - t$)

$$\begin{aligned}
\mathbb{D}(\tau) &= \sum_{i=1}^{n_w} \frac{\sigma_{0i}^2}{k_{\sigma i}} c_0 e^{-\kappa_{\sigma i}\tau} (1 - e^{-\kappa_{\sigma i}\tau}) - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}\tau} e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}\tau} - 1)}, \tag{2.6.2} \\
&= \mathbb{D}_G(\tau) + \mathbb{D}_J(\tau)
\end{aligned}$$

where $\mathbb{D}_G(\tau)$ is the Gaussian drift contribution and $\mathbb{D}_J(\tau)$ is the contribution of the jump component to the drift. The derivative of $\mathbb{D}(\tau)$ is

$$\begin{aligned}
\frac{d\mathbb{D}(\tau)}{d\tau} &= \sum_{i=1}^{n_w} \sigma_{0i}^2 c_0 e^{-\kappa_{\sigma i}\tau} (2e^{-\kappa_{\sigma i}\tau} - 1) \\
&\quad + \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}\tau} e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}\tau} - 1)} (k_{\beta i} + \beta_{0i} e^{-k_{\beta i}\tau}).
\end{aligned}$$

First by assuming only positive jump sizes, the drift function is originally negative for some time close to the maturity as the following arguments shows. The $\mathbb{D}_J(\tau)$ is an increasing function in τ with negative minimum $-\sum_{i=1}^{n_p} \psi_i \beta_{0i}$ at $\tau = 0$. As $\tau \rightarrow \infty$, $\mathbb{D}_J(\tau)$ converges to 0. The $\mathbb{D}_G(\tau)$ has a minimum of 0 at $\tau = 0$. As $\tau \rightarrow \infty$, $\mathbb{D}_G(\tau)$

converges to 0. Thus \mathbb{D} has always a negative minimum of $-\sum_{i=1}^{n_p} \psi_i \beta_{0i}$ at $\tau = 0$. Figure 2.6.1 shows how the forward rate drift changes over τ when positive jump sizes are allowed. The parameter values used are $\sigma_0 = 0.03$, $k_\sigma = 0.18$, $\beta_{01} = 0.01$, $\beta_{02} = 0.02$, $c_0 = 5$, $k_{\beta_1} = 0.3$, $k_{\beta_2} = 0.17$, $\psi_1 = 1$, $\psi_2 = 1.5$.

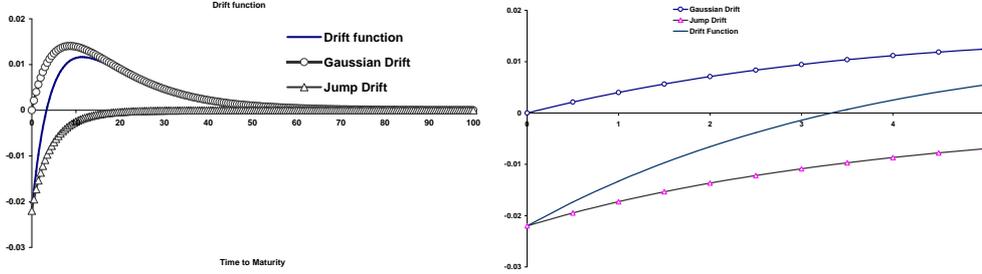


FIGURE 2.6.1. Forward Rate Drift. The parameter values used are $\sigma_0 = 0.03$, $k_\sigma = 0.18$, $\beta_{01} = 0.01$, $\beta_{02} = 0.02$, $c_0 = 5$, $k_{\beta_1} = 0.3$, $k_{\beta_2} = 0.17$, $\psi_1 = 1$, $\psi_2 = 1.5$.

By assuming only negative jump sizes, then the drift function is always positive. However, the negative jump noise terms may drive interest rates to negative values. Consequently, we cannot avoid the situation of interest rates becoming negative. Thus we need to ensure that the state dependent volatility specifications used are of the form that will either avoid negative interest rates or make the probability of their occurrence rather low as is the case, for example, with the Hull-White model.

For well defined volatility functions, special functional forms required that they are well defined for negative values. As an illustrative example, let assume that the state dependence is modelled as a linear combination of benchmark forward rates and the instantaneous spot rate of the state vector $\bar{f}(t)$, as

$$L_f(t) = c_0 r(t) + \sum_{h=1}^{n_d} c_h f(t, T_h).$$

When $L_f(s)$ becomes very small or negative then the model may behave as a deterministic volatility Hull-White type of model. Thus a suggested volatility function may be

$$\sigma_0(t, \bar{f}(t)) = \begin{cases} c_f \sigma_0, & L_f(t) < 0.005; \\ \sigma_0[(L_f(t) - 0.005)^\gamma + c_f], & L_f(t) \geq 0.005; \end{cases} \quad (2.6.3)$$

with $\gamma = \frac{1}{2}$ and $c_f > 0$.

2.7. Conclusions

This chapter develops a multi-factor jump-diffusion model of the HJM term structure of interest rates. In particular, the forward rate dynamics are driven by multi-dimensional Wiener and Poisson noise terms. Under specific volatility restrictions, we obtain Markovian representations of the spot rate dynamics and we derive exponential affine bond pricing formulas. The main contributions of this chapter can be summarised as follows:

- ★ Two particular forward rate volatility set-ups, one deterministic and the other stochastic, are considered. Under the deterministic volatility specifications, both Wiener and jump volatilities are time dependent, while under the stochastic volatility case, Wiener volatilities are state dependent and the jump volatilities are time deterministic functions. In these two volatility cases, the Markovian representations of the instantaneous spot rate and the bond prices are derived, as well as the corresponding exponential affine bond pricing formulas.
- ★ The state variables of these models are expressed in terms of a set of benchmark forward rates and yields, a fact which makes the model suitable for both calibration and parameter estimation. Thus for the case when the Wiener volatilities are state dependent but the Poisson ones are only time deterministic, we extend the results of Chiarella & Kwon (2003) to the jump-diffusion case.
- ★ Under state dependent Poisson volatility specifications, it becomes difficult to obtain Markovian representation of the system and so an “approximate” Markovian structure is proposed.

These two volatility set-ups will allow us in the next chapter to create the jump extended versions of a number of popular models such as Hull-White and Ritchken & Sankarasubramanian class of models and investigate their properties and distributional profiles.

Incorporation of the jump processes as well as stochastic volatility specifications into the HJM framework generates a model that should capture more effectively the statistical features of the market information. In addition, the achievement of Markovian structures at the expense of volatility restrictions, combined with the ability to express the state variables in terms of a finite number of benchmark forward rates and yields, provides a flexible model appropriate for parameter estimation and model calibration. However, full exploitation of these representations remains for future research.

Another direction of further research includes pricing instruments involving the possibility of early exercise in a jump-diffusion model with state dependent volatilities. Additionally, the framework developed here may be extended to deal with defaultable term structure models, a theme which is developed in Chapter 4.

Appendix 2.1. Bond Price Dynamics

By integrating the stochastic integral equation (2.2.5) for the instantaneous spot rate $r(t)$, from 0 to t we obtain

$$\begin{aligned} \int_0^t r(u)du &= \int_0^t f(0, u)du + \int_0^t \int_0^u \alpha(s, u)dsdu \\ &+ \sum_{i=1}^{n_w} \int_0^t \int_0^u \sigma_i(s, u, \bar{f}(s))dW_i(s)du + \sum_{i=1}^{n_p} \int_0^t \int_0^u \beta_i(s, u)[dQ_i(s) - \lambda_i ds]du. \end{aligned} \quad (\text{A 2.1.1})$$

Using the relationship $\ln P(t, T) = -\int_t^T f(t, u)du$, substituting the stochastic integral equation (2.2.4) for the forward rate and interchanging the order of the integration we obtain¹⁵

$$\begin{aligned} \ln P(t, T) &= -\int_t^T f(0, u)du - \int_0^t \int_t^T \alpha(s, u)duds \\ &- \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i(s, u, \bar{f}(s))dudW_i(s) - \sum_{i=1}^{n_p} \int_0^t \int_t^T \beta_i(s, u)du[dQ_i(s) - \lambda_i ds], \end{aligned}$$

which may be expressed as

$$\begin{aligned} \ln P(t, T) &= -\int_0^T f(0, u)du + \int_0^t f(0, u)du - \int_0^t \left(-\int_s^t \alpha(s, u)du + \int_s^T \alpha(s, u)du \right) ds \\ &- \sum_{i=1}^{n_w} \int_0^t \left(-\int_s^t \sigma_i(s, u, \bar{f}(s))du + \int_s^T \sigma_i(s, u, \bar{f}(s))du \right) dW_i(s) \\ &- \sum_{i=1}^{n_p} \int_0^t \left(-\int_s^t \beta_i(s, u)du + \int_s^T \beta_i(s, u)du \right) [dQ_i(s) - \lambda_i ds]. \end{aligned}$$

¹⁵We also make use of the results that for a sufficiently well-behaved function g

$$\int_0^t \int_s^t g(s, u)duds = \int_0^t \int_0^u g(s, u)dsdu \quad \text{and} \quad \int_0^t \int_s^t g(s, u)dudW(s) = \int_0^t \int_0^u g(s, u)dW(s)du.$$

Using equation (A 2.1.1) we may simplify the above dynamics to

$$\begin{aligned} \ln P(t, T) &= \ln P(0, T) + \int_0^t r(s) ds - \int_0^t \int_s^T \alpha(s, u) dud s \\ &\quad - \sum_{i=1}^{n_w} \int_0^t \int_s^T \sigma_i(s, u, \bar{f}(s)) dud W_i(s) - \sum_{i=1}^{n_p} \int_0^t \int_s^T \beta_i(s, u) du [dQ_i(s) - \lambda_i ds]. \end{aligned}$$

By taking the stochastic differential we find that the quantity $V(t, T) = \ln P(t, T)$ satisfies the stochastic differential equation

$$\begin{aligned} dV(t, T) &= \left(r(t) - \int_t^T \alpha(t, u) du \right) dt \\ &\quad - \sum_{i=1}^{n_w} \int_t^T \sigma_i(t, u, \bar{f}(t)) dud W_i(t) - \sum_{i=1}^{n_p} \int_t^T \beta_i(t, u) du [dQ_i(t) - \lambda_i dt]. \end{aligned}$$

Application of the jump-diffusion version of Ito's lemma to $P(t, T) = e^{V(t, T)}$ finally yields the stochastic differential equation for the bond price $P(t, T)$ namely

$$\frac{dP(t, T)}{P(t, T)} = [r(t) + H(t, T, \bar{f}(t))] dt - \sum_{i=1}^{n_w} \zeta_i(t, T, \bar{f}(t)) dW_i(t) - \sum_{i=1}^{n_p} (1 - e^{-\xi_i(t, T)}) dQ_i(t), \quad (\text{A 2.1.2})$$

where $\zeta_i(t, T, \bar{f}(t))$, $\xi_i(t, T)$ and $H(t, T, \bar{f}(t))$ are defined by (2.2.9), (2.2.10) and (2.2.11) respectively.

Appendix 2.2. The No-Arbitrage Condition in the Bond Market

Setting $n_H = n_w + n_p$, consider a hedging portfolio containing bonds of maturities $T_1, T_2, \dots, T_{n_H+1}$ in the proportions $w_1, w_2, \dots, w_{n_H+1}$ with $w_1 + w_2 + \dots + w_{n_H+1} = 1$. We denote by $P_h(t) = P(t, T_h)$ ($h = 1, 2, \dots, (n_H + 1)$) the value of these $n_H + 1$ zero-coupon bonds. For simplicity of notation we write the stochastic differential equation for P_h in the general form (see equation (2.2.8))

$$\frac{dP_h(t)}{P_h(t)} = \mu_{P_h}(t) dt + \sum_{i=1}^{n_w} \nu_{P_h, i}(t) dW_i(t) + \sum_{i=1}^{n_p} \chi_{P_h, i}(t) dQ_i(t),$$

where

$$\begin{aligned}\mu_{P_h}(t) &\equiv r(t) + H(t, T_h, \bar{f}(t)) \\ \nu_{P_{h,i}}(t) &\equiv -\zeta_i(t, T_h, \bar{f}(t)), \\ \chi_{P_{h,i}}(t) &\equiv e^{-\xi_i(t, T_h)} - 1.\end{aligned}$$

Let V be the value of the hedging portfolio then the return on the portfolio is given by

$$\begin{aligned}\frac{dV}{V} &= w_1 \frac{dP_1}{P_1} + w_2 \frac{dP_2}{P_2} + \cdots + w_{n_H+1} \frac{dP_{n_H+1}}{P_{n_H+1}} \\ &= \sum_{h=1}^{n_H+1} w_h \mu_{P_h} dt + \sum_{h=1}^{n_H+1} w_h \left(\sum_{i=1}^{n_w} \nu_{P_{h,i}} dW_i(t) + \sum_{i=1}^{n_p} \chi_{P_{h,i}} dQ_i(t) \right).\end{aligned}$$

In order to eliminate both Wiener and Poisson risks we need to choose $w_1, w_2, \dots, w_{n_H+1}$ so that

$$\sum_{h=1}^{n_H+1} w_h \nu_{P_{h,i}} = 0, \quad \text{for } i = 1, 2, \dots, n_w, \quad (\text{A 2.2.1})$$

and

$$\sum_{h=1}^{n_H+1} w_h \chi_{P_{h,i}} = 0, \quad \text{for } i = 1, 2, \dots, n_p. \quad (\text{A 2.2.2})$$

The hedging portfolio then becomes riskless, thus, it should earn the risk-free rate of interest $r(t)$, i.e.,

$$\frac{dV}{V} = \sum_{h=1}^{n_H+1} w_h \mu_{P_h} dt = r(t) dt.$$

From the last equality and the fact that $w_1 + w_2 + \cdots + w_{n_H+1} = 1$, we have

$$\sum_{h=1}^{n_H+1} w_h (\mu_{P_h} - r(t)) = 0. \quad (\text{A 2.2.3})$$

Equations (A 2.2.1), (A 2.2.2) and (A 2.2.3) form a system of $n_H + 1$ equations with $n_H + 1$ unknowns $w_1, w_2, \dots, w_{n_H+1}$. This system can only have a non-zero solution if

$$\begin{vmatrix} \nu_{P_{1,1}}(t) & \nu_{P_{2,1}}(t) & \cdots & \nu_{P_{n_H+1,1}}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \nu_{P_{1,n_w}}(t) & \nu_{P_{2,n_w}}(t) & \cdots & \nu_{P_{n_H+1,n_w}}(t) \\ \chi_{P_{1,1}}(t) & \chi_{P_{2,1}}(t) & \cdots & \chi_{P_{n_H+1,1}}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{P_{1,n_p}}(t) & \chi_{P_{2,n_p}}(t) & \cdots & \chi_{P_{n_H+1,n_p}}(t) \\ \mu_{P_1} - r(t) & \mu_{P_2} - r(t) & \cdots & \mu_{P_{n_H+1}} - r(t) \end{vmatrix} = 0.$$

This implies that for $h = 1, 2, \dots, (n_H + 1)$ there exists a vector $\Phi = (\phi_1(t), \phi_2(t), \dots, \phi_{n_w}(t))$ and a vector $\Psi = (\psi_1(t), \dots, \psi_{n_p}(t))$, such that

$$\mu_{P_h} - r(t) = - \sum_{i=1}^{n_w} \phi_i(t) \nu_{P_{h,i}}(t) - \sum_{i=1}^{n_p} \psi_i(t) \chi_{P_{h,i}}(t).$$

Since the bond maturities are arbitrary, it follows that for bonds of any maturity T we must have that

$$\mu_P - r(t) = - \sum_{i=1}^{n_w} \phi_i(t) \nu_{P_i}(t) - \sum_{i=1}^{n_p} \psi_i(t) \chi_{P_i}(t). \quad (\text{A 2.2.4})$$

The economic interpretation of condition (A 2.2.4) is that the excess return of each bond above the risk free rate is equal to the total risk premium required as compensation for bearing the risk associated with the Wiener processes and the Poisson processes. Consequently, we may interpret Ψ as the vector of the market prices of Poisson jump risk (one associated with each possible jump size) and Φ as the vector of the market prices of the Wiener diffusion risk. By recalling that $\mu_P(t) = r(t) + H(t, T, \bar{f}(t))$ and substituting the expressions for $\nu_{P_i}(t)$, with $i = 1, \dots, n_w$, and $\chi_{P_i}(t)$, with $i = 1, \dots, n_p$, we obtain

$$\begin{aligned} H(t, T, \bar{f}(t)) &\equiv - \int_t^T \alpha(t, u) du + \sum_{i=1}^{n_w} \frac{1}{2} \zeta_i^2(t, T, \bar{f}(t)) + \sum_{i=1}^{n_p} \lambda_i \xi_i(t, T) \\ &= \sum_{i=1}^{n_w} \phi_i(t) \zeta_i(t, T, \bar{f}(t)) - \sum_{i=1}^{n_p} \psi_i(t) (e^{-\xi_i(t, T)} - 1). \end{aligned} \quad (\text{A 2.2.5})$$

By taking the derivative of (A5.1.9) with respect to T and manipulating appropriately we derive the forward rate drift restriction that extends the HJM forward rate drift restriction

to now incorporate the jump feature, namely,

$$\alpha(t, T) = \sum_{i=1}^{n_w} \sigma_i(t, T, \bar{f}(t))(-\phi_i(t) + \zeta_i(t, T, \bar{f}(t))) - \sum_{i=1}^{n_p} \beta_i(t, T)(\psi_i(t)e^{-\xi_i(t, T)} - \lambda_i). \quad (\text{A 2.2.6})$$

Appendix 2.3. Simplification of Terms Occurring in (2.3.14) and (2.3.29).

In equation (2.3.14) we need to simplify the terms $\frac{\partial}{\partial t} \int_0^t \sigma_i(s, t) \zeta_i(s, t) ds$ and in equation (2.3.29) we need to simplify the terms $\frac{\partial}{\partial t} \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds$. The dependence of the integrands on the $\bar{f}(s)$ does not affect the following manipulations. Thus for more general results set¹⁶

$$S(s, t, \bar{f}(s)) = \sigma(s, t, \bar{f}(s)) \zeta(s, t, \bar{f}(s)) = \sigma(s, t, \bar{f}(s)) \int_s^t \sigma(s, u, \bar{f}(s)) du,$$

which given Assumption 2.3.2 can be written

$$S(s, t, \bar{f}(s)) = \sigma_0^2(s, \bar{f}(s)) e^{-\int_s^t \kappa_\sigma(v) dv} \int_s^t e^{-\int_s^u \kappa_\sigma(v) dv} du.$$

Then the derivative of $S(s, t, \bar{f}(s))$ with respect to the second argument is given by

$$\frac{\partial S(s, t, \bar{f}(s))}{\partial t} = -\kappa_\sigma(t) S(s, t, \bar{f}(s)) + \sigma_0^2(s, \bar{f}(s)) e^{-2 \int_s^t \kappa_\sigma(v) dv}.$$

Therefore,

$$\begin{aligned} & \frac{\partial}{\partial t} \int_0^t \left(\sigma(s, t, \bar{f}(s)) \int_s^t \sigma(s, u, \bar{f}(s)) du \right) ds \\ &= \int_0^t \frac{\partial}{\partial t} S(s, t, \bar{f}(s)) ds + S(t, t, \bar{f}(t)) \\ &= \int_0^t \left[-\kappa_\sigma(t) S(s, t, \bar{f}(s)) + \sigma_0^2(s, \bar{f}(s)) e^{-2 \int_s^t \kappa_\sigma(v) dv} \right] ds \\ &= \int_0^t \sigma^2(s, t, \bar{f}(s)) ds - \kappa_\sigma(t) \int_0^t S(s, t, \bar{f}(s)) ds. \end{aligned}$$

Now consider the corresponding term in equations (2.3.14) or (2.3.29), with the Poisson volatility functions, and let

$$F(s, t) = \beta(s, t) [1 - e^{-\int_s^t \beta(s, u) du}] = \beta_0(s) e^{-\int_s^t \kappa_\beta(v) dv} \left[1 - e^{-\int_s^t \beta_0(s) e^{-\int_s^u \kappa_\beta(v) dv} du} \right].$$

¹⁶Similar results are obtained when the dependency on $\bar{f}(s)$ is dropped out.

Then

$$\frac{\partial F(s, t)}{\partial t} = -\kappa_\beta(t)F(s, t) + \beta^2(s, t)e^{-\int_s^t \beta(s, u)du},$$

and so

$$\frac{\partial}{\partial t} \int_0^t \psi(s)F(s, t)ds = \int_0^t \psi(s)\beta^2(s, t)e^{-\int_s^t \beta(s, u)du}ds - \int_0^t \psi(s)\kappa_\beta(t)F(s, t)ds.$$

Appendix 2.4. Derivation of the Bond Price Formula in Proposition 2.3.18 and Proposition 2.3.35.

We derive the bond price formula using the Inui & Kijima (1998) approach. In Proposition 2.3.18 the Wiener volatilities are assumed to be time deterministic whereas in Proposition 2.3.35 the Wiener volatilities are state dependent. The following manipulations apply similarly for both cases thus the more general set-up is derived here. The forward rate dynamics under the risk neutral measure are (recall equation (2.2.17))

$$f(t, T) = f(0, T) + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, T, \bar{f}(s))\zeta_i(s, T, \bar{f}(s))ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, T, \bar{f}(s))d\widetilde{W}_i(s) \quad (\text{A 2.4.1})$$

$$+ \sum_{i=1}^{n_p} \int_0^t \psi_i(s)\beta_i(s, T)[1 - e^{-\xi_i(s, T)}]ds + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, T)[dQ_i(s) - \psi_i(s)ds].$$

Using the fundamental relationship $P(t, T) = \exp\left(-\int_t^T f(t, y)dy\right)$, we may write after interchanging the order of the integration¹⁷

$$\begin{aligned} P(t, T) &= \\ &= \exp\left(-\int_t^T f(0, y)dy - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i(s, y, \bar{f}(s))\zeta_i(s, y, \bar{f}(s))dyds - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i(s, y, \bar{f}(s))dyd\widetilde{W}_i(s) \right. \\ &\quad \left. - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s)\beta_i(s, y)[1 - e^{-\xi_i(s, y)}]dyds - \sum_{i=1}^{n_p} \int_0^t \int_t^T \beta_i(s, y)dy[dQ_i(s) - \psi_i(s)ds]\right). \end{aligned} \quad (\text{A 2.4.2})$$

¹⁷We assume that the conditions for application of stochastic Fubini theorem are satisfied.

Next we incorporate the volatility specifications (2.3.21) and (2.3.2) and functions (2.3.20), to derive¹⁸

$$\int_t^T \sigma_i(s, y, \bar{f}(s)) dy = \sigma_i(s, t, \bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy \equiv \sigma_i(s, t, \bar{f}(s)) \mathcal{N}_{\sigma_i}(t, T), \quad (\text{A 2.4.3})$$

so that for $i = 1, \dots, n_w$

$$\int_0^t \int_t^T \sigma_i(s, y, \bar{f}(s)) dy d\widetilde{W}_i(s) = \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i(s, t, \bar{f}(s)) d\widetilde{W}_i(s). \quad (\text{A 2.4.4})$$

Similarly

$$\int_t^T \beta_i(s, y) dy = \beta_i(s, t) \int_t^T e^{-\int_t^y \kappa_{\beta_i}(u) du} dy = \beta_i(s, t) \mathcal{N}_{\beta_i}(t, T). \quad (\text{A 2.4.5})$$

and hence for $i = 1, \dots, n_p$

$$\int_0^t \int_t^T \beta_i(s, y) dy [dQ_i(s) - \psi_i(s) ds] = \mathcal{N}_{\beta_i}(t, T) \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds]. \quad (\text{A 2.4.6})$$

Similarly, for $i = 1, \dots, n_w$, we manipulate the term

$$\begin{aligned} \int_t^T \sigma_i(s, y, \bar{f}(s)) \zeta_i(s, y, \bar{f}(s)) dy &= \int_t^T \sigma_i(s, y, \bar{f}(s)) \int_s^y \sigma_i(s, v, \bar{f}(s)) dv dy \\ &= \sigma_i(s, t, \bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy \int_s^t \sigma_i(s, v, \bar{f}(s)) dv + \sigma_i^2(s, t, \bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} \int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv dy \\ &= \sigma_i(s, t, \bar{f}(s)) \mathcal{N}_{\sigma_i}(t, T) \zeta_i(s, t, \bar{f}(s)) + \frac{1}{2} \sigma_i^2(s, t, \bar{f}(s)) \mathcal{N}_{\sigma_i}^2(t, T), \end{aligned} \quad (\text{A 2.4.7})$$

since

$$\begin{aligned} \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} \int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv dy &= \int_t^T d \left(\frac{1}{2} \left[\int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv \right]^2 \right) \\ &= \frac{1}{2} \left(\int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy \right)^2 = \frac{1}{2} \mathcal{N}_{\sigma_i}^2(t, T). \end{aligned} \quad (\text{A 2.4.8})$$

¹⁸Note that

$\sigma_i(s, y, \bar{f}(s)) = \sigma_{0i}(s, \bar{f}(s)) e^{-\int_s^y \kappa_{\sigma_i}(u) du} = \sigma_{0i}(s, \bar{f}(s)) e^{-\int_s^t \kappa_{\sigma_i}(u) du} e^{-\int_t^y \kappa_{\sigma_i}(u) du} = \sigma_i(s, t, \bar{f}(s)) e^{-\int_t^y \kappa_{\sigma_i}(u) du}.$

Therefore integrating equation (A 2.4.7) from 0 to t we obtain (for $i = 1, \dots, n_w$)

$$\begin{aligned} \int_0^t \int_t^T \sigma_i(s, y, \bar{f}(s)) \zeta_i(s, y, \bar{f}(s)) dy ds = \\ \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds + \frac{1}{2} \mathcal{N}_{\sigma_i}^2(t, T) \int_0^t \sigma_i^2(s, t, \bar{f}(s)) ds. \end{aligned} \quad (\text{A 2.4.9})$$

Substituting the results¹⁹ (A 2.4.4), (A 2.4.6) and (A 2.4.9) into equation (A 2.4.2), and collecting like terms, the bond price formula simplifies to

$$\begin{aligned} P(t, T) = \exp \left(- \int_t^T f(0, y) dy - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i(s, t, \bar{f}(s)) \zeta_i(s, t, \bar{f}(s)) ds \right. \\ \left. - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \int_0^t \sigma_i^2(s, t, \bar{f}(s)) ds - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i(s, t, \bar{f}(s)) d\widetilde{W}_i(s) \right. \\ \left. - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds] \right. \\ \left. - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s) \beta_i(s, y) [1 - e^{-\xi_i(s, y)}] dy ds \right). \end{aligned} \quad (\text{A 2.4.10})$$

By using the definitions (2.3.27), (2.3.28) and (2.3.13), equation (A 2.4.10) simplifies further to

$$\begin{aligned} P(t, T) = \exp \left(- \int_t^T f(0, y) dy - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}(t) \right. \\ \left. - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \mathcal{D}_{\beta_i}(t) - \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \right\} \right. \\ \left. - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s) \beta_i(s, y) [1 - e^{-\xi_i(s, y)}] dy ds \right). \end{aligned} \quad (\text{A 2.4.11})$$

Thus the bond price formula, where the bond price is a function of the state variables $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\beta_i}(t)$ and $\mathcal{D}_{\sigma_i}(t)$, may be expressed as,

$$\begin{aligned} P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \tilde{\mathcal{M}}(t, T) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}(t) \right. \\ \left. - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \mathcal{D}_{\beta_i}(t) \right\}, \end{aligned} \quad (\text{A 2.4.12})$$

¹⁹The results (A 2.4.4) and (A 2.4.9) have been already proven in Inui & Kijima (1998).

where,

$$\begin{aligned} \tilde{\mathcal{M}}(t, T) = & - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s) \beta_i(s, y) [1 - e^{-\xi_i(s, y)}] dy ds \\ & + \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \right\}. \end{aligned} \quad (\text{A 2.4.13})$$

To include the spot rate in the set of the state variables, substitute the expression (2.3.31) for the $\mathcal{D}_{\sigma_1}(t)$ into the bond price formula (A 2.4.12) to obtain in Proposition 2.3.18 the multi-factor affine term structure of interest rates in the form

$$\begin{aligned} P(r(t), t, T) = & \frac{P(0, T)}{P(0, t)} \exp \left\{ \mathcal{M}(t, T) - \mathcal{N}_{\sigma_1}(t, T) r(t) \right. \\ & \left. - \sum_{i=2}^{n_w} (\mathcal{N}_{\sigma_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} (\mathcal{N}_{\beta_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\beta_i}(t) \right\}, \end{aligned} \quad (\text{A 2.4.14})$$

where (recall the $\mathcal{E}_{\sigma_i}(t)$ are time deterministic in this case)

$$\mathcal{M}(t, T) = \mathcal{N}_{\sigma_1}(t, T) f(0, t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}(t) + \tilde{\mathcal{M}}(t, T).$$

In Proposition 2.3.35 the multi-factor affine term structure of interest rates is expressed in the form

$$\begin{aligned} P(r(t), t, T) = & \frac{P(0, T)}{P(0, t)} \exp \left\{ \bar{\mathcal{M}}(t, T) - \mathcal{N}_{\sigma_1}(t, T) r(t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}(t) \right. \\ & \left. - \sum_{i=2}^{n_w} (\mathcal{N}_{\sigma_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} (\mathcal{N}_{\beta_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T)) \mathcal{D}_{\beta_i}(t) \right\}, \end{aligned} \quad (\text{A 2.4.15})$$

where

$$\bar{\mathcal{M}}(t, T) = \mathcal{N}_{\sigma_1}(t, T) f(0, t) + \tilde{\mathcal{M}}(t, T),$$

with $\mathcal{N}_x(t, T)$ ($x \in \{\sigma_i, \beta_i\}$) defined as in equation (2.3.20).

Appendix 2.5. Finite Dimensional Affine Realisations in Terms of Forward Rates under Deterministic Volatility Specifications

In Section 2.4, the state variables were expressed as affine functions of a finite number of forward rates. Similarly here, by using the affine term structure of interest rates (2.3.18), which includes $r(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ as state variables, we may express the instantaneous forward rate and the bond price in terms of forward rates of $n_s (= n_w + n_p - 1)$ different maturities T_h as the following propositions show. The instantaneous forward rate can be expressed here as (from equation (2.3.18))

$$\begin{aligned} f(t, T) - f(0, T) + \frac{\partial \mathcal{M}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} r(t) = & \quad (\text{A 2.5.1}) \\ + \sum_{i=2}^{n_w} \left(\frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\beta_i}(t), \end{aligned}$$

PROPOSITION 2.5.1. *The forward rate of any maturity can be expressed in terms of the instantaneous spot rate $r(t)$ and the n_s benchmark forward rates $f(t, T_h)$ for $h = 1, \dots, n_s$, as*

$$f(t, T) = \mathcal{Q}(t, T) + \sum_{h=1}^{n_s} \mathcal{R}_h(t, T) f(t, T_h) + \mathcal{S}(t, T) r(t), \quad (\text{A 2.5.2})$$

where, for $l = q - 1$ and $k = n_w + i - 1$,

$$\begin{aligned} \mathcal{Q}(t, T) = \frac{\partial \mathcal{M}(t, T)}{\partial T} - \sum_{h=1}^{n_s} \frac{\partial \mathcal{M}(t, T_h)}{\partial T_h} \left[\sum_{q=2}^{n_w} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \right. & \quad (\text{A 2.5.3}) \\ \left. + \sum_{i=1}^{n_p} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \right], \end{aligned}$$

$$\mathcal{R}_h(t, T) = \sum_{q=2}^{n_w} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right),$$

and

$$\begin{aligned} \mathcal{S}(t, T) = & \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} - \sum_{h=1}^{n_s} \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \left(\sum_{q=2}^{n_w} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \right. \\ & \left. + \sum_{i=1}^{n_p} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \right). \end{aligned} \quad (\text{A 2.5.4})$$

Denote as ϖ_{wh} the wh^{th} element of matrix $\mathbb{O}^{-1}(t)$, the inverse of the square matrix $\mathbb{O}(t)$, such as

$$\mathbb{O}(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) \end{bmatrix},$$

where, for $2 \leq q \leq n_w$, $\varphi_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma_q}(t, T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \right]$ is an $n_s \times (n_w - 1)$ matrix and for $1 \leq i \leq n_p$, $\varphi_2(t) = \left[\frac{\partial \mathcal{N}_{\beta_i}(t, T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_h)}{\partial T_h} \right]$ is an $n_s \times n_p$ matrix.

Assume that $\mathbb{O}(t)$ is invertible for all $t \in \{t; t = \min_h T_h\}$.

Proof. From equation (A 2.5.1) and for the maturities T_1, T_2, \dots, T_{n_s} , we obtain the system

$$\begin{bmatrix} f(t, T_1) - f(0, T_1) + \frac{\partial \mathcal{M}(t, T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_1)}{\partial T_1} r(t) \\ f(t, T_2) - f(0, T_2) + \frac{\partial \mathcal{M}(t, T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_2)}{\partial T_2} r(t) \\ \vdots \\ f(t, T_{n_s}) - f(0, T_{n_s}) + \frac{\partial \mathcal{M}(t, T_{n_s})}{\partial T_{n_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_{n_s})}{\partial T_{n_s}} r(t) \end{bmatrix} = \mathbb{O}(t) \times \begin{bmatrix} \mathcal{D}_{\sigma_2}(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\beta_1}(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}(t) \end{bmatrix}.$$

By inverting the matrix $\mathbb{O}(t)$, the state variables $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ are expressed in terms of forward rates of n_s distinct maturities as

$$\begin{bmatrix} \mathcal{D}_{\sigma_2}(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}(t) \\ \mathcal{D}_{\beta_1}(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}(t) \end{bmatrix} = \mathbb{O}^{-1}(t) \times \begin{bmatrix} f(t, T_1) - f(0, T_1) + \frac{\partial \mathcal{M}(t, T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_1)}{\partial T_1} r(t) \\ f(t, T_2) - f(0, T_2) + \frac{\partial \mathcal{M}(t, T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_2}(t, T_1)}{\partial T_2} r(t) \\ \vdots \\ f(t, T_{n_s}) - f(0, T_{n_s}) + \frac{\partial \mathcal{M}(t, T_{n_s})}{\partial T_{n_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T_{n_s})}{\partial T_{n_s}} r(t) \end{bmatrix}, \quad (\text{A 2.5.5})$$

Substitution of expressions (A 2.5.5) for the state variables into the forward rate formula (A 2.5.1) leads to (A 2.5.2), an expression of the forward rate of any maturity in

terms of the instantaneous spot rate $r(t)$ and the forward rates of n_s fixed maturities.

□

PROPOSITION 2.5.2. *The zero-coupon bond price in terms of the benchmark forward rates $f(t, T_h)$ is given by*

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \mathcal{Q}^P(t, T) - \sum_{h=1}^{n_s} \mathcal{R}_h^P(t, T) f(t, T_h) - \mathcal{S}^P(t, T) r(t) \right\}, \quad (\text{A 2.5.6})$$

where

$$\mathcal{Q}^P(t, T) \equiv \int_t^T \mathcal{Q}(t, s) ds, \quad \mathcal{R}_h^P(t, T) \equiv \int_t^T \mathcal{R}_h(t, s) ds, \quad \text{and} \quad \mathcal{S}^P(t, T) \equiv \int_t^T \mathcal{S}(t, s) ds. \quad (\text{A 2.5.7})$$

Proof. Substitute (A 2.5.2) into the relationship (2.2.1). □

Note that the risk neutral dynamics (2.4.9) drive the instantaneous forward rates $f(t, T_h)$, although the Wiener volatility functions are now only time dependent.

Appendix 2.6. Benchmark Forward Rates as Sole State Variables

As pointed out in Remark 2.4.1, the set of the state variables may or may not include the spot rate, depending on the application. In Section 2.3 and Section 2.4, results are derived in the case that the spot rate is included in the set of the state variables. In this appendix we summarise these results when the spot rate is not a state variable, thus we derive a model with benchmark forward rates as the sole state variables.

By substituting (2.3.27), (2.3.11), (2.3.28) and (2.3.13) into the stochastic differential equation (2.3.30), we obtain the dynamics for the spot rate in terms of the stochastic factors $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$, as

$$\begin{aligned} dr(t) = & \left[\tilde{D}(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma_i}(t) - \sum_{i=1}^{n_w} k_{\sigma_i}(t) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} k_{\beta_i}(t) \mathcal{D}_{\beta_i}(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\tilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (\text{A 2.6.1})$$

where

$$\tilde{D}(t) = \frac{\partial}{\partial t} f(0, t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta_i}(t). \quad (\text{A 2.6.2})$$

The corresponding affine term structure of interest rates (see Appendix 2.4 for details) is given by (A 2.4.12) which is a function of the state variables $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$. Then using equation (2.2.1), we can obtain the relation between the instantaneous forward rate curve and these state variables as

$$\begin{aligned} f(t, T) - f(0, T) + \frac{\partial \tilde{\mathcal{M}}(t, T)}{\partial T} &= \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \mathcal{E}_{\sigma_i}(t) \\ &+ \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} \mathcal{D}_{\beta_i}(t), \end{aligned} \quad (\text{A 2.6.3})$$

where $\mathcal{N}_x(t, T)$ ($x \in \{\sigma_i, \beta_i\}$) are defined as in equation (2.3.20).

Taking a number of fixed maturity forward rates equal to the number of the state variables, it becomes possible to express the state variables in terms of forward rates with different fixed maturities. Thus, we consider forward rates of $\tilde{n}_s (= 2n_w + n_p)$ different fixed maturities T_h , as shown in the following proposition.

PROPOSITION 2.6.1. *The forward rate of any maturity can be expressed in terms of the \tilde{n}_s benchmark forward rates $f(t, T_h)$, ($h = 1, \dots, \tilde{n}_s$) as*

$$f(t, T) = f(0, T) - f(0, t) + \tilde{\mathcal{Q}}(t, T) + \sum_{h=1}^{\tilde{n}_s} \tilde{\mathcal{R}}_h(t, T) f(t, T_h), \quad (\text{A 2.6.4})$$

where, for $l = n_w + i$ and $k = 2n_w + i$,

$$\tilde{\mathcal{R}}_h(t, T) = \sum_{i=1}^{n_w} \left(\tilde{\omega}_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) + \tilde{\omega}_{lh} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \tilde{\omega}_{kh} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T}, \quad (\text{A 2.6.5})$$

and

$$\begin{aligned} \tilde{\mathcal{Q}}(t, T) &= \frac{\partial \tilde{\mathcal{M}}(t, T)}{\partial T} - \sum_{h=1}^{\tilde{n}_s} \left(\frac{\partial \tilde{\mathcal{M}}(t, T_h)}{\partial T_h} - f(0, T_h) + f(0, t) \right) \left[\sum_{i=1}^{n_w} \left(\tilde{\omega}_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \right. \right. \\ &\quad \left. \left. + \tilde{\omega}_{lh} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \tilde{\omega}_{kh} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} \right]. \end{aligned} \quad (\text{A 2.6.6})$$

Denote as $\tilde{\omega}_{\ell h}$ the ℓh^{th} element of matrix $\tilde{\mathbb{O}}^{-1}(t)$, the inverse of the square matrix $\tilde{\mathbb{O}}(t)$, such that

$$\tilde{\mathbb{O}}(t) = \begin{bmatrix} \tilde{\varphi}_1(t) & \tilde{\varphi}_2(t) & \tilde{\varphi}_3(t) \end{bmatrix},$$

where, for $i = 1, 2, \dots, n_w$, $\tilde{\varphi}_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma_i}(t, T_h)}{\partial T_h} \mathcal{N}_{\sigma_i}(t, T_h) \right]$ is an $\tilde{n}_s \times n_w$ matrix,

$\tilde{\varphi}_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma_i}(t, T_h)}{\partial T_h} \right]$ is an $\tilde{n}_s \times n_w$ matrix, and

for $i = 1, 2, \dots, n_p$, $\tilde{\varphi}_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_i}(t, T_h)}{\partial T_h} \right]$ is an $\tilde{n}_s \times n_p$ matrix.

Assume that $\tilde{\mathbb{O}}(t)$ is invertible for all $t \in \{t; t = \min_h T_h\}$.

Proof. Similar manipulations as Proposition 2.4.1. □

PROPOSITION 2.6.2. *The zero-coupon bond prices in terms of the benchmark forward rates $f(t, T_h)$ is given by*

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left(\tilde{\mathcal{Q}}^P(t, T) + \sum_{h=1}^{\tilde{n}_s} \tilde{\mathcal{R}}_h^P(t, T) f(t, T_h) \right), \quad (\text{A 2.6.7})$$

where

$$\tilde{\mathcal{Q}}^P(t, T) \equiv - \int_t^T \tilde{\mathcal{Q}}(t, s) ds, \quad \text{and} \quad \tilde{\mathcal{R}}_h^P(t, T) \equiv - \int_t^T \tilde{\mathcal{R}}_h(t, s) ds. \quad (\text{A 2.6.8})$$

Proof. Substitution of (A 2.6.4) into (2.2.1). □

Thus by taking the set of the state dependent variables $\bar{f}(t)$ of the forward rate volatility functions considered in Assumption 2.3.2 the vector where elements are the set of the benchmark forward rates,

$$\bar{f}(t) = (f(t, T_1), f(t, T_2), \dots, f(t, T_{\tilde{n}_s}))^\top,$$

and so we have a closed Markovian system.

Extending Some Popular Term Structure Models to the Jump-Diffusion Setting.

In this chapter we relate the structure developed in Chapter 2 to the existing class of jump-diffusion term structure models whose starting point is a jump-diffusion process for the spot rate. In particular we obtain natural jump-diffusion versions of the Hull & White (1990, 1994) models and the Ritchken & Sankarasubramanian (1995) model within the HJM framework. We also give some numerical simulations to gauge the effect of the jump-component on yield curves and the implications of various volatility specifications for the spot rate distribution.

3.1. Introduction

Within the framework developed in Chapter 2, we develop some particular classes of jump-diffusion spot rate models. The general model developed is related to known models and provides an extension of these models to incorporate jumps components. In particular, we develop what we believe is the natural extension of the Hull & White (1990), (1994) (hereafter HW) class of models and the Ritchken & Sankarasubramanian (1995) (hereafter RS) class of models to the jump-diffusion case. The HW model is a deterministic volatility model and it is extended to deal with jump-diffusion driving dynamics. Similarly, the RS model, which is a stochastic volatility model, is extended to incorporate jump noise driving the spot rate dynamics. The volatility structure, which now is state dependent, is expressed in terms of the spot rate itself and a number of benchmark forward rates.

The contributions of this chapter are as follows. In Section 3.2 we develop a class of multi-factor HW jump-diffusion models. By using the results from Appendix 2.5, an expression for the forward rate in terms of benchmark forward rates is obtained. Under the specific example that considers one Wiener and two jump noise terms driving the dynamics of instantaneous forward rate and consequently the dynamics of the instantaneous spot

rate, we examine the effect of the different benchmark forward rate levels on the forward curve. Ritchken & Sankarasubramanian (1995) jump-diffusion models are considered in Section 3.3, where, using the results of Section 2.4, finite dimensional affine realisations in terms of forward rates are investigated when noise is generated by one Wiener and two jump terms. In Section 3.4, we carry out a number of numerical simulations to gauge the implications of the various volatility specifications that generate these models. Section 3.5 concludes.

3.2. Hull & White Type models

One of the characteristic features of HW type models is that the underlying dynamics involve a mean reverting process for the instantaneous spot rate of interest, which is the underlying state variable¹ in this class of models. Furthermore, the volatility function is only time deterministic. So, to obtain HW type models under jump-diffusions, we assume the deterministic volatility specifications of Assumption 2.3.1. The corresponding Markovian dynamics of the instantaneous spot rate (2.3.6) are derived in Section 2.3.1 and generalise the structure of the Hull & White (1994) two-factor model where the spot rate is driven by only one Wiener process. We recall that the basic idea of Hull & White (1994) was to add to the drift term a stochastic factor driven by another Wiener process. In our set-up, the instantaneous spot rate dynamics (2.3.6), namely,

$$dr(t) = \left[D(t) - \sum_{i=2}^{n_w} \hat{k}_{\sigma_i}(t) \mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} \hat{k}_{\beta_i}(t) \mathcal{D}_{\beta_i}(t) - k_{\sigma_1}(t)r(t) \right] dt \\ + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\tilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t)dt],$$

have a drift containing $n_w + n_p$ stochastic factors, including the mean reverting term for the instantaneous spot rate $r(t)$ and with some of the stochastic factors driven by Wiener processes and some by jump processes. Using the finite dimensional affine realisations in terms of benchmark forward rates, discussed in Section 2.4, the instantaneous spot rate

¹Of course, there is no reason why one could not define a class of HW or RS models where, say, some other rate, e.g. the 6-month LIBOR rate serves as the underlying state variable. This might even be more reasonable as such rates are observed in the market whereas the instantaneous spot rate is not. However it has become traditional for a wide class of models to use the instantaneous spot rate as the underlying state variable.

dynamics can be expressed in terms of the state vector

$$\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_{\bar{n}_s}))^\top,$$

of the benchmark forward rates and the spot rate, as (recall expression (2.4.10))

$$\begin{aligned} dr(t) = & \left[D^f(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) f(t, T_h) - \bar{\mathcal{S}}^f(t) r(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t) [dQ_i(t) - \psi_i(t) dt]. \end{aligned}$$

It is also worth pointing out that as it is defined within the HJM framework, the spot rate process (2.3.6) is automatically calibrated to the currently observed yield curve through the $D(t)$ term. For these reasons we suggest that this representation is the natural extension of the HW model to the multi-factor jump-diffusion situation.

In the following examples, the initial forward rate curve considered has the functional form $f(0, t) = (a_0 + a_1 t + a_2 t^2) e^{-vt}$ with parameters being estimated as $a_0 = 0.033287$, $a_1 = 0.014488$, $a_2 = -0.000117$, and $v = 0.0925$, which result in an upward sloping forward curve. The data used for interpolation are the US zero yields on July 20, 2001, up to 10 years maturity including the overnight rate.

We will now consider the case of the one Wiener-two Poisson HW type of model. Thus for $n_w = 1$ and $n_p = 2$, consider the volatility functions

$$\sigma(t, T) = \sigma_0 e^{-\kappa_\sigma(T-t)}, \quad (3.2.1)$$

and,

$$\beta_i(t, T) = \beta_{0i} e^{-\kappa_{\beta_i}(T-t)}, \quad \text{with } i = 1, 2. \quad (3.2.2)$$

The number of the stochastic variables, in this case, is 3(= $n_w + n_p$), and using the results from Proposition 2.5.1 and Proposition 2.5.2 of Appendix 2.5 we may express these stochastic variables in terms of 2 benchmark forward rates and the spot rate. In turn, the forward rate $f(t, T)$ and the bond prices $P(t, T)$ can be expressed in terms of the spot rate $r(t)$ and these benchmark forward rates. The state variables used here are the spot rate, the 5-year forward rate $f(t, 5)$ and the 10-year forward rate $f(t, 10)$.

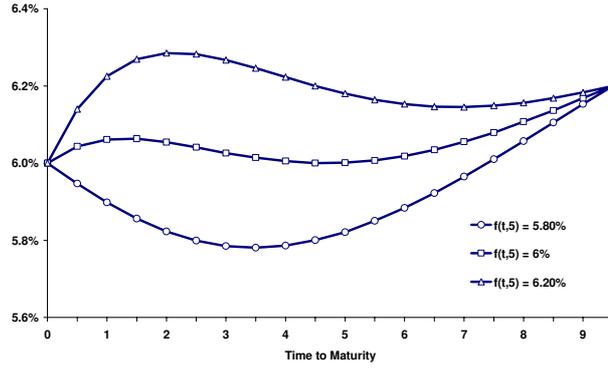


FIGURE 3.2.1. Forward rate curves at $t = 6$ months, for the one Wiener and two Poisson HW type model when $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\psi_1 = 1$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$ and $\psi_2 = 1.5$. The corresponding curves represent $f(t, T)$ when $f(t, 10) = 6.2\%$ and $f(t, 5)$ takes the values of 5.8% , 6% and 6.2% .

The parameter values of the Wiener volatility are $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$. The parameter specifications for the jump volatility terms are $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$. The jump intensities are $\psi_1 = 1$ and $\psi_2 = 1.5$ respectively. The forward rate curves (by using equation (A 2.5.2)) shown in Figure 3.2.1 are at 6 months time, when it is assumed that $r = 6\%$, the 10-year forward rate $f(t, 10)$ is 6.2% and the 5-year forward rate takes the values of 5.8% , 6% and 6.2% . For these volatility specifications, the spot rate volatility is 3.5% and the 10-year forward rate volatility is 16.5% of the spot rate volatility.

3.3. Ritchken & Sankarasubramanian Type Models

The RS class of models considered in this application is characterised by state dependent Wiener volatility functions, so that

$$\sigma(t, T) = \sigma_1(T - t)\sigma_2(\bar{f}(t))e^{-\int_t^T \kappa_\sigma(u)du}, \quad (3.3.1)$$

where σ_1 is a time deterministic function and σ_2 is a well behaved function of the vector $\bar{f}(t)$. In the original Ritchken & Sankarasubramanian (1995) paper, the forward rate volatility functions considered are of the form $\sigma(r)e^{-\int_t^T \kappa_\sigma(u)du}$. Subsequently, Ritchken & Chuang (1999) consider the forward rate volatility functions $(a_0 + a_1(T - t))e^{-k(T-t)}$. The form (3.3.1) generalises this type of volatility structure.

We consider again the case that $n_w = 1$ and $n_p = 2$. The number of the state variables, in this case, is $4 (= 2n_w + n_p)$. Using the results from Proposition 2.4.1 and Proposition 2.4.2 we may express these state variables in terms of 3 benchmark forward rates and the spot rate. Thus we may set $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), f(t, T_3))^T$. In turn, the forward rate $f(t, T)$ and the bond prices $P(t, T)$ can be expressed in terms of the spot rate $r(t)$ and these benchmark forward rates. The state variables used here are the spot rate, the 2.5-year forward rate $f(t, 2.5)$, the 5-year forward rate $f(t, 5)$ and the 10-year forward rate $f(t, 10)$.

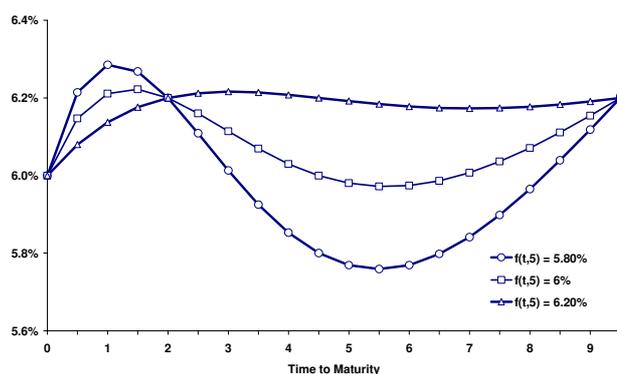


FIGURE 3.3.1. Forward rate curves at $t = 6$ months, for the One Wiener and Two Poisson RS type models when $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\psi_1 = 1$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$ and $\psi_2 = 1.5$. The corresponding curves represent $f(t, T)$, when $f(t, 2.5) = f(t, 10) = 6.2\%$ and $f(t, 5)$ takes the values of 5.8% , 6% and 6.2% .

The initial forward rate curve and the volatility specifications considered here are the same as in Section 3.2. Equation (2.4.2) implies the forward rate curves shown in Figure 3.3.1 in 6 months time, when $r = 6\%$, the 2.5-year forward rate and the 10-year forward rate is 6.2% and the 5-year forward rate takes the values of 5.8% , 6% and 6.2% .

In order to compare the different class of models examined, we select the model parameters so as to maintain, for all models, the spot rate volatility at 3.5% and the 10-year forward rate volatility at 16.5% of the spot rate volatility. To obtain these volatility levels, the set of the Wiener and Poisson volatility parameter values is the one used in each of the above examples.

Comparing Figure 3.2.1 and Figure 3.3.1 we see that the state dependent volatility models display forward rate curves with sharper curvature changes than the equivalent HW

type models of Section 3.2. This is expected since the state dependent volatility models incorporate a larger number of state variables, which makes the model more flexible and able to capture more realistic forward rate behavior such as sharper movements.

REMARK 3.3.1. The results obtained in this section imply that, under the framework developed in Chapter 2, what matters for the “richness” of different shapes of the term structure is more the number of state variables, rather than the number of driving sources of uncertainty.

3.4. Simulated Distributions

In this section we perform simulations of the stochastic differential equation system that results from the Markovianisation procedure. We examine and compare the simulated normalised distributions of the HW class of models and the RS class of models and in particular when one Wiener and two Poisson noise terms drive the forward rate dynamics. For all the simulation examples performed in this section, an Euler-Maruyama approximation is employed and we discretize the time interval $[0, 1]$ into $N = 400$ equal subintervals of length $\Delta t = 1/N$, and generate 100,000 paths for $r(t)$. Furthermore, in order to compare the leptokurtosis levels of the two classes of models, we use normalised distributions which means that the volatility parameters (Wiener and Poisson) have been selected as to provide the same variance of the simulated distributions, with variance here being 0.0017 in all cases.

3.4.1. Hull White Models. For the One Wiener and Two Poisson HW type of models, the volatility specifications considered are (3.2.1) for the Wiener term and (3.2.2) for the jump terms and, the ψ_i are constant.

We consider the discretised system of the instantaneous spot rate dynamics (2.3.6) with the two state variables $\mathcal{D}_{\beta_i}(t)$ expressed in terms of the two benchmark forward rates $f(t, 5)$ and $f(t, 10)$, by making use of the system (A 2.5.5).

Figure 3.4.1 shows the simulated normalised distributions of $r(t)$ for the HW type of models at $t = 1$. The volatility parameter values used are $\kappa_\sigma = 0.18$, $\kappa_{\beta_1} = 0.31$, $\kappa_{\beta_2} = 0.17$, $\psi_1 = 1$ and $\psi_2 = 1.5$. We consider three sets of volatility magnitude parameters, one with high jump volatility, one with low jump volatility and one with no jump volatility

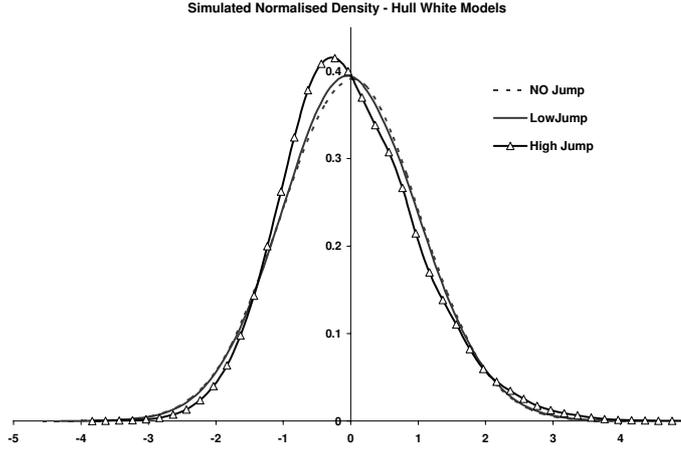


FIGURE 3.4.1. Simulated Normalised Density of the Instantaneous Spot Rate for the HW type of models at $t = 1$. The volatility magnitudes are; for the high jump volatility case $\sigma_0 = 0.9\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$; for the low jump volatility case $\sigma_0 = 3.8\%$, $\beta_{01} = 2\%$, $\beta_{02} = -1.2\%$; and for the no jump volatility case $\sigma_0 = 4.5\%$.

which are respectively; a) $\sigma_0 = 0.9\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$, b) $\sigma_0 = 3.8\%$, $\beta_{01} = 2\%$, $\beta_{02} = -1.2\%$ and c) $\sigma_0 = 4.5\%$. We consider the no-jump volatility case, which has a Gaussian distribution, in order to compare with the distributional outcomes in the situation with jumps. In fact, in the absence of jumps, the model reduces to the Gaussian case. Figure 3.4.1 shows that, compared to the normal distribution, with increasing jump magnitude, the distribution becomes asymmetric. In addition, long tail to the right is obtained, as the positive jump size has been chosen to be larger than the negative jump size. However the jump magnitude needs to be of a reasonable size for this effect to become pronounced.

3.4.2. Ritchken & Sankarasubramanian Models. For the RS type models, the Wiener volatilities are state dependent having the functional form (3.3.1). In particular, for the One Wiener and Two Poisson RS type models, we need four state variables to Markovianise the system, thus $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), f(t, T_3))^T$. We further assume that $\sigma_1(T - t) = \sigma_0$ constant, and

$$\sigma_2(t, \bar{f}(t)) = \begin{cases} 0.05, & L_f(t) < 0.005; \\ (L_f(t) - 0.005)^\gamma + 0.05, & L_f(t) \geq 0.005; \end{cases}$$

with $L_f(t) = c_0 r(t) + \sum_{h=1}^3 c_h f(t, T_h)$, and $\gamma = \frac{1}{2}$. Taking into consideration the discussion of Section 2.6, we consider the above square root process for the Wiener volatilities

of the type (2.6.3), thus the state dependent volatility will be a well defined function. Figure 3.4.2 shows the volatility function. For the Poisson volatility specifications, we consider $\beta_i(s, t) = \beta_{0i}e^{-k_{\beta_i}(t-s)}$ and constant ψ_i .

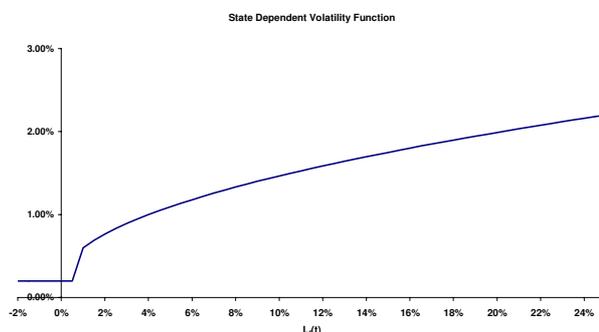


FIGURE 3.4.2. State Dependent Volatility Function.

We now consider the discretised system of the spot rate dynamics (2.3.25) with the state variables $\mathcal{E}_\sigma(t)$ and $\mathcal{D}_{\beta_i}(t)$ expressed in terms of the three benchmark forward rates $f(t, 2.5)$, $f(t, 5)$, $f(t, 10)$ and the spot rate by using the system (2.4.7).

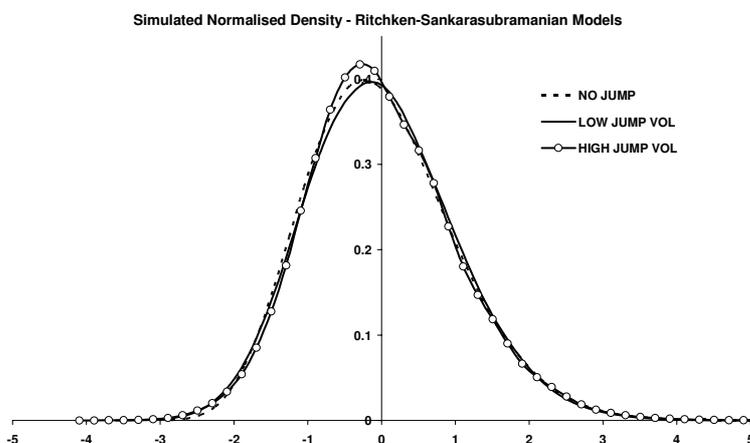


FIGURE 3.4.3. Simulated Normalised Density of the Instantaneous Spot Rate for the RS type models at $t = 1$. The volatility magnitudes are $\sigma_0 = 1.2\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$ for the high jump volatility case; $\sigma_0 = 5.2\%$, $\beta_{01} = 2.4\%$, $\beta_{02} = -1.5\%$ for the low jump volatility case; and $\sigma_0 = 6.8\%$ for the no-jump volatility case.

The simulated normalised distributions of $r(t)$ at $t = 1$ for the RS type of models are shown in Figure 3.4.3. The volatility parameter values used are $\kappa_\sigma = 0.18$, $\kappa_{\beta_1} = 0.31$,

$\kappa\beta_2 = 0.17$, $\psi_1 = 1$ and $\psi_2 = 1.5$. We also set $c_0 = 1$, $c_1 = 2$, $c_2 = 1$, $c_3 = 2$. we consider three sets of volatility magnitudes, $\sigma_0 = 1.2\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$ for the high jump volatility case; $\sigma_0 = 5.2\%$, $\beta_{01} = 2.4\%$, $\beta_{02} = -1.5\%$ for the low jump volatility case; and $\sigma_0 = 6.8\%$ for the no-jump volatility case. In the no-jump volatility case, in other words relying on state dependent volatilities only, the skewness obtained is relatively large. Adding jumps does not change the order of the magnitude of the skewness. However, in the HW models the jump magnitude significantly changes the order of magnitude of the skewness (see Table 3.4.1 and Table 3.4.2).

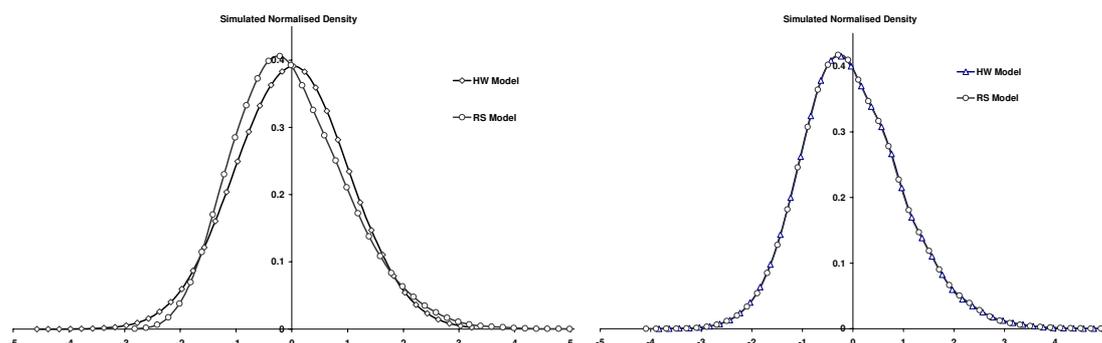


FIGURE 3.4.4. Comparison of Simulated Normalised Density of the Instantaneous Spot Rate for the HW and RS type of models at $t = 1$ when a) there is no jump and b) large jump volatility is considered.

Figure 3.4.4 compares the simulated normalised distribution of the $r(t)$ for the HW and RS type of models at $t = 1$ for the cases considered earlier. The no jump cases are to the left and the large jump volatility cases to the right. In the no jump case, there is a visible difference in the distributions and indeed in the amount of skewness (see Table 3.4.1). In the large jump volatility cases similar distributions are obtained indicating that the jump feature dominates the stochastic volatility feature, at least for large jump sizes. However the two models show differences when we compare the statistical properties of the spot rate changes as Table 3.4.2 illustrates. Furthermore, in order to gauge the effect of the jump parameters and the state dependent volatility on the simulated normalised distributions, we compare in Table 3.4.1 and Table 3.4.2 the statistical properties of the simulated distributions of the spot rate and the spot rate changes (recall that variance of the spot rate is 0.17% in all cases and expressed in percentage terms).

We observe that when the jump volatilities are low the HW model is very close to a Gaussian one, although the RS model exhibits a variation from the Gaussian model with

Statistical Information on $r(t)$						
	no-jump		low jump		high jump	
	HW	RS	HW	RS	HW	RS
Mean	6.59	6.60	6.62	6.61	6.61	6.60
Variance	0.17	0.17	0.17	0.17	0.17	0.17
Skewness	0.0045	0.5004	0.0396	0.3463	0.4313	0.4494
Kurtosis	3.0081	3.3555	3.0438	3.224	3.4875	3.5451

TABLE 3.4.1. The statistical measures of the spot rate from simulated distributions for different jump magnitudes under the HW and RS models.

Statistical Information on $dr(t)$						
	no-jump		low jump		high jump	
	HW	RS	HW	RS	HW	RS
Mean	0.0009	0.0019	0.0006	0.0012	-0.0011	-0.0015
Variance	0.0005	0.0005	0.0004	0.0003	0.00001	0.00002
Skewness	-0.0019	-0.0025	0.0016	-0.0139	0.0018	-0.0634
Kurtosis	2.9820	3.3040	2.9899	3.3037	3.0033	3.2354

TABLE 3.4.2. The statistical measures of the spot rate changes from simulated distributions for different jump magnitudes under the HW and RS models.

high skewness and kurtosis. Also, the state dependent models (RS) with or without jumps certainly display higher kurtosis and higher skewness of the spot rate compared to the equivalent (with respect to the jump size) deterministic volatility models (HW). This indicates that state dependent volatilities capture more efficiently the asymmetric feature of the empirical spot rate distribution.

By increasing the jump volatilities, both models exhibit asymmetric normalised distributions with a long tail to the right as we made the choice of the positive jump size to dominate the negative jump size. However, the state dependent RS model without jump, has high kurtosis for the spot rate changes but not particularly high negative skewness. When the state dependent model is combined with the jump-diffusion model, such as RS with jumps, then both higher kurtosis and skewness of the spot rate changes are obtained. Thus, jumps on one hand and state dependent volatility on the other hand, yield models that capture better the stylised empirical facts of interest rate movements. However, the combination of both state dependent volatilities and jumps succeeds in accommodating most of the empirical distributional behavior of the spot rate and the spot rate changes.

Note that in Chapter 4, (see Section 4.7) more emphasis is given on measuring the ability of these class of models to capture the empirical facts of interest rates.

The feature that has made it tractable and possible to quantify these characteristics is the ability to obtain Markovian structures for the interest rate dynamics. This Markovian class of models that incorporates the more realistic jump-diffusion processes combined with stochastic volatility may be employed for more accurate derivative pricing and hedging and also in empirical studies of interest rate markets.

3.5. Conclusions

This chapter deals with the extension of some popular term structure models to the jump-diffusion setting; in particular the well known HW and RS type of models. The deterministic volatility modelling platform of Chapter 2 served as the extension of the multi-factor HW framework to incorporate jumps, and the state dependent volatility set-up of Chapter 2 served as the jump-diffusion version of the multi-factor RS type of models. Some numerical examples illustrate the nature of the HW and RS class of models when they are extended to incorporate jumps. More specifically, making use of the affine finite dimensional realisations in terms of benchmark forward rates derived in Chapter 2, we investigate the patterns of the forwards rate curves under these two classes of models. Additionally, Monte Carlo simulations of the Markovian instantaneous spot rate dynamics are performed and the impact of the volatility specifications on the spot rate and spot rate changes distribution is discussed for both types of models.

The main results of the investigations performed in this chapter are:

- ★ The state dependent volatility models (RS) have a tendency to generate forward rate curves with sharper curvature changes than the equivalent deterministic volatility models (HW). This is probably due to the fact that the state dependent volatility models incorporate a larger number of state variables, which makes the model more flexible and able to capture more realistic forward rate behavior.
- ★ State dependent volatility models capture more effectively the asymmetric feature of the empirical spot rate distribution compared to deterministic volatility models.

- ★ Both jump-diffusion models, HW and RS, exhibit asymmetric distributions, a feature that becomes more pronounced, as jump volatility levels increase.
- ★ The combination of both state dependent volatilities and jumps succeeds in accommodating most of the empirical distributional behavior of the spot rate and the spot rate changes which will in turn provide more accurate derivative security pricing models and better models for econometric estimation.

The tractability of the Markovian structures obtained provides an efficient and more accurate basis for Monte Carlo simulations, that may be employed for derivative pricing as is presented in Chapter 5. Furthermore the framework developed here may be extended to credit risk models, as will be discussed in the next chapter. It may also be used as a framework for calibration as well as for econometric estimation of term structure models; see Chiarella & Tô (2004) with regard to econometric estimation . Further development of these themes is left for further research.

CHAPTER 4

Markovianisation of Defaultable HJM Term Structure Models

This chapter discusses the Markovianisation of defaultable term structure models, within the HJM framework. Multi-dimensional Poisson jump processes are employed to model default events and at the same time the volatility structure is assumed to be state dependent. With regard to the default intensities, both cases of a deterministic default intensity as well as the more realistic set-up of a stochastic jump intensity are examined. Under appropriate volatility specifications, a finite dimensional Markovian defaultable spot rate structure and the corresponding affine term structure of interest rates are obtained. Additionally, these models admit finite dimensional affine realisations in terms of benchmark defaultable forward rates. When the dynamics are Markovian in the entire yield curve, an approximate Markovian scheme is proposed.

4.1. Introduction

The past decade has witnessed a rapidly increasing interest in research on pricing and hedging financial instruments subject to default risk, which has inevitably changed the way that financial institutions and security traders deal with investment and risk management. The main two credit risk approaches include the structural approach where default is triggered (at maturity or any time during the lifetime of the contract) when the value of the firm falls below a barrier value and the reduced form approach¹ where the time of default is modelled directly using jump (e.g. Cox, marked point) processes. In the intensity class of models, default is triggered by exogenous sources in an unpredictable manner, providing a more realistic modelling set-up, however the empirical implementation of such models is still quite limited.

¹Examples of reduced form models or the so-called intensity models include those of Artzner & Delbaen (1992), Lando (1994), Artzner & Delbaen (1995), Jarrow & Turnbull (1995), Das & Tufano (1996), Duffie & Huang (1996), Jarrow, Lando & Turnbull (1997), Duffie & Singleton (1997), Madan & Unal (1998), Schönbucher (1998), Lando (1998), Duffie & Singleton (1999), Schönbucher (2000), Schönbucher (2003).

The extension of the jump-diffusion versions of the HJM framework to the defaultable case may be regarded as an excellent modelling platform within the reduced form class of models from which to generate a tractable class of defaultable models appropriate for numerical applications. In addition, there is a large literature on modelling default-free instruments by using jump-diffusion processes, some of the results of which map directly to the defaultable case. We refer in particular to the relevant literature in Chapter 2. Thus while the Markovianisation of default-free term structure models has been extensively studied, there is a relatively limited literature on the development of Markovian representations of defaultable interest rate models.

This chapter considers a multi-factor jump-diffusion model of the defaultable term structure of interest rates under a specific volatility structure. The defaultable forward rate dynamics are driven by multi-dimensional Wiener and Poisson processes and the volatility structure is such that the Wiener volatility functions are state dependent and the Poisson volatility functions are time and maturity dependent. A parameterisation of the Schönbucher (2000), (2003) general HJM framework where jumps in the defaultable term structure cause jumps and defaults to the defaultable bond prices is studied. Thus working within the HJM framework, bond prices in an arbitrage free environment are obtained, even though the spot rate dynamics are non-Markovian. Imposing restrictions on the volatility structure, a Markovian multi-factor model is achieved. It turns out that the state variables of this model can be expressed as functions of a finite number of benchmark defaultable forward rates. The model that we thereby develop provides a fairly broad tractable class of defaultable term structure models that would be suitable for both calibration and econometric estimation. However, the resulting class of defaultable term structure models imposes a deterministic structure on the credit spreads. Therefore the model is extended to allow for stochastic intensities. It then becomes difficult to obtain Markovian representations of the system and so we propose one way of obtaining an “approximate” Markovian structure. Alternatively, another way to restore the path independence is to consider constant Poisson volatilities.

Schönbucher (2000), (2003) extends the HJM framework and conditions for the absence of arbitrage to include the term structure of defaultable bond prices. In this case, jumps and defaults are linked in that at times of default, there is a jump in defaultable forward

rates. The setting considered here is slightly more general in that not every jump in interest rates is necessarily linked to a default event. For general volatility specifications the defaultable rate dynamics are non-Markovian, making numerical implementation difficult and computationally intensive. A Markovian representation of the stochastic dynamic system driving defaultable bond prices is derived by considering certain specifications of the volatility functions of the instantaneous defaultable forward rate. Essentially, an extension to the defaultable jump-diffusion case of the approach of the Markovianisation of HJM models developed by the techniques discussed in Chapter 2 is examined.

This chapter is organized as follows. In Section 4.2 the Schönbucher (2000), (2003) defaultable term structure framework with discrete jumps is reviewed focusing on the economic intuition of the underlying hedging argument. In Section 4.3, for a specific volatility structure, the corresponding Markovian representation of the spot rate in terms of a finite number of state variables that are driven by Markovian diffusion and jump processes is obtained. To make the model more easily interpretable, in Section 4.4, these state variables are expressed as finite dimensional affine realisations in terms of economic quantities observed in the market, such as forward rates and yields. Section 4.5 extends the model by incorporating stochastic spreads and discusses the reason why a Markovian representation is not possible in this case. However, a way in which an approximate Markovian representation may be developed is suggested. Section 4.6 discusses model limitations. In Section 4.7, we consider the Markovian defaultable term structure model under the deterministic and stochastic default intensity set-up and we perform numerical simulations for a range of jump sizes and magnitudes as well as time horizons, in order to gauge the effect of the volatility and default intensity specifications on the distribution of the defaultable spot rate. Section 4.8 concludes.

4.2. Modelling Defaultable Term Structure within the HJM Framework

We adapt the Schönbucher (2000), (2003) general HJM framework where jumps in the defaultable term structure $f^d(t, T)$ are equivalent to defaults and jumps in the defaultable bond prices $P^d(t, T)$. In the model we consider here, we start from a general HJM-type jump-diffusion model for the defaultable rates. This allows one to link some of the jumps

to default events while others remain jumps in interest rates only, and may be interpreted as being caused by economic influences other than defaults.

The defaultable forward rate dynamics are driven by jump-diffusions. Thus, the stochastic differential equation for the instantaneous defaultable forward rate $f^d(t, T)$ driven by both Gaussian and Poisson risks, is given by,

$$df^d(t, T) = \alpha^d(t, T)dt + \sum_{i=1}^{n_w} \sigma_i^d(t, T)dW_i(t) + \sum_{i=1}^{n_p} \beta_i^d(t, T)[dQ_i(t) - \lambda_i dt], \quad (4.2.1)$$

where $\alpha^d : [0, T] \rightarrow R_+$ is the drift function, $W_i(t)$ are standard Wiener processes ($i = 1, 2, \dots, n$), $Q_i(t)$ is a Poisson process with constant intensity λ_i ($i = 1, 2, \dots, n_p$).² The Poisson process Q_i is employed to model the arrival time of the jump events. At the Poisson jump times, the jump size is equal to $\beta_i^d(t, T)$. Under these assumptions, the jump feature is modelled by a multivariate point process, allowing for a finite number of jumps.

The volatility specifications allow for $\sigma_i^d : [0, T] \rightarrow R_+$, the volatility functions associated with the Wiener noise processes, to be state dependent according to

$$\sigma_i^d(t, T) = \sigma_i^d(t, T, \bar{f}^d(t)), \quad \text{for } i = 1, \dots, n_w, \quad (4.2.2)$$

where the σ_i^d are well-defined functions and there are n_f state variables, e.g., forward rates of n_f different maturities, so that $\bar{f}^d(t) = (f^d(t, T_1), f^d(t, T_2), \dots, f^d(t, T_{n_f}))^\top$. The volatility functions $\beta_i^d : [0, T] \rightarrow R_+$, ($i = 1, 2, \dots, n_p$) associated with the Poisson noise processes are assumed to be only time and maturity dependent.³

The price at time t of a defaultable zero-coupon bond with maturity T , a so called ‘pseudo’ bond, is given by⁴

$$\hat{P}(t, T) = \exp\left(-\int_t^T f^d(t, s)ds\right). \quad (4.2.3)$$

²The Wiener processes $W_i(t)$ and the Poisson process $Q_i(t)$ with intensity λ_i generate the \mathbb{P} -augmentation of the filtration \mathcal{F}_t .

³See Chapter 2 for the intuition of using only time and maturity dependent Poisson volatility functions. Basically, state dependent Poisson volatility functions result in infinite dimensional Markovian structures.

⁴This is the price of the defaultable zero-coupon bond given that it has not defaulted before time t .

The dynamics of $\hat{P}(t, T)$ follow from those for $f^d(t, T)$ in equation (4.2.1) and are given by (see equation (2.2.8))

$$\frac{d\hat{P}(t, T)}{\hat{P}(t^-, T)} = [r^d(t) + H^d(t, T, \bar{f}^d(t))]dt - \sum_{i=1}^{n_w} \zeta_i^d(t, T, \bar{f}^d(t))dW_i(t) - \sum_{i=1}^{n_p} [1 - e^{-\xi_i^d(t, T)}]dQ_i(t), \quad (4.2.4)$$

where

$$\zeta_i^d(t, T, \bar{f}^d(t)) = \int_t^T \sigma_i^d(t, u, \bar{f}^d(t))du, \quad (4.2.5)$$

$$\xi_i^d(t, T) = \int_t^T \beta_i^d(t, u)du, \quad (4.2.6)$$

$$H^d(t, T, \bar{f}^d(t)) = - \int_t^T \alpha^d(t, u)du + \sum_{i=1}^{n_w} \frac{1}{2} \zeta_i^{d2}(t, T, \bar{f}^d(t)) + \sum_{i=1}^{n_p} \lambda_i \xi_i^d(t, T). \quad (4.2.7)$$

A key feature of the ‘pseudo’ bond price is that the influence of previous defaults has been removed.

Following the Schönbucher (2000) set-up we assume fractional recovery (e.g. restructure and reduction of the notional in case of default), thus allowing for multiple defaults. The actual value of every defaultable bond $P^d(t, T)$ is given by

$$P^d(t, T) = \hat{P}(t, T) \bar{Q}(t), \quad (4.2.8)$$

where $\bar{Q}(t)$ is the reduction on the bond’s face value due to the number $\eta(t)$ of defaults up to time t . Note that $\eta(t) = \sum_{i=1}^{n_p} \eta_i(t)$, where $\eta_i(t)$ is the number of defaults up to time t due to the source of jump events $dQ_i(t)$. Let τ_{ik} denote the time of the k^{th} jump in Q_i , and q_{ik} the loss fraction due to the default triggered by the k^{th} jump in Q_i .⁵ At the bond’s maturity T it pays out

$$\bar{Q}(T) := \prod_{i=1}^{n_p} \left(\prod_{\tau_{ik} \leq T} (1 - q_{ik}) \right), \quad (4.2.9)$$

the remainder after all fractional losses.

⁵Setting $q_{ik} = 0$ means that this particular jump does not trigger a default.

Let us assume that the fractional losses due to the defaults related to the $dQ_i(t)$ term are $q_i(t)$ at each and every jump time, so that

$$\bar{Q}(t) := \prod_{i=1}^{n_p} \prod_{k=1}^{\eta_i(t)} (1 - q_i(\tau_{ik})), \quad (4.2.10)$$

which is the solution (Doléans-Dade exponential formula⁶) of the stochastic differential equation

$$\frac{d\bar{Q}(t)}{\bar{Q}(t-)} = - \sum_{i=1}^{n_p} q_i(t) dQ_i(t), \quad (4.2.11)$$

subject to the initial condition $\bar{Q}(0) = 1$. Note that when q_i are assumed to be constant, then the solution (4.2.10) reduces to the expression

$$\bar{Q}(t) := \prod_{i=1}^{n_p} (1 - q_i)^{\eta_i(t)}. \quad (4.2.12)$$

By an application of Ito's lemma, the dynamics of the “real” defaultable zero-coupon bond $P^d(t, T)$, defined by (4.2.8), with $\bar{Q}(t)$ driven by (4.2.11), are given by

$$\begin{aligned} \frac{dP^d(t, T)}{P^d(t-, T)} &= [r^d(t) + H^d(t, T, \bar{f}^d(t))]dt - \sum_{i=1}^{n_w} \zeta_i^d(t, T, \bar{f}^d(t)) dW_i(t) \\ &\quad - \sum_{i=1}^{n_p} [(1 - q_i(t))[1 - e^{-\xi_i^d(t, T)}] + q_i(t)] dQ_i(t). \end{aligned} \quad (4.2.13)$$

4.2.1. The Hedging Argument in the Defaultable Bond Market. Since we are using the HJM framework, the bond prices that we derive will match the observed yield curve, by construction. We are able to obtain bond-pricing formulae in the case of a finite number of jumps, each of which is associated with a jump volatility, by extending the Shirakawa (1991) approach, that assumes only a finite number of possible jump sizes and the existence of a sufficient number of traded bonds to hedge away all of the jump risks and to guarantee market completeness.

Following the same structure as Chapter 2 we focus on the economic intuition of the hedging argument based on the classical approach of Vasicek (1977) that adapts the original Black-Scholes hedging argument to interest rate term structure models.⁷ We have $n_w + n_p$

⁶Appendix 4.1 derives these results, by using the Doléans-Dade exponential formula that has been proved in Jacod & Shiryaev (2003).

⁷See Appendix 4.2 for full details on the hedging argument in the defaultable bond market.

sources of risk, n_w due to the Wiener processes $W_i(t)$ ($i = 1, \dots, n_w$), and n_p due to the Poisson processes Q_i ($i = 1, \dots, n_p$), thus we need to place bonds of $n_w + n_p + 1$ maturities in the hedging portfolio to hedge all of these risks.

By taking an appropriate position in the $n_w + n_p + 1$ defaultable bonds⁸ it is possible to eliminate both Wiener and Poisson risks. The condition that the riskless hedged portfolio earns the risk-free rate of interest $r(t)$, implies that there must exist vectors $\Phi^d = (\phi_1^d, \dots, \phi_{n_w}^d)^\top$ and $\Psi^d = (\psi_1^d, \dots, \psi_{n_p}^d)^\top$ such that for each maturity $h = 1, 2, \dots, (n_w + n_p + 1)$ the expected excess bond return satisfies

$$[r^d(t) + H^d(t, T_h, \bar{f}^d(t))] - r(t) = \sum_{i=1}^{n_w} \phi_i^d(t) \zeta_i^d(t, T_h, \bar{f}^d(t)) + \sum_{i=1}^{n_p} \psi_i^d(t) [(1 - q_i(t))(1 - e^{-\xi_i^d(t, T)}) + q_i(t)]. \quad (4.2.14)$$

Since the maturities of the hedging bonds may be chosen arbitrarily, it must be the case that, for bonds of any maturity T , the expected excess bond return satisfies

$$[r^d(t) + H^d(t, T, \bar{f}^d(t))] - r(t) = \sum_{i=1}^{n_w} \phi_i^d(t) \zeta_i^d(t, T, \bar{f}^d(t)) + \sum_{i=1}^{n_p} \psi_i^d(t) [(1 - q_i(t))(1 - e^{-\xi_i^d(t, T)}) + q_i(t)]. \quad (4.2.15)$$

The economic interpretation of condition (4.2.15) is that the expected excess return of each bond above the risk free rate is equal to the total risk premium required as compensation for bearing the risk associated with the Wiener processes and the Poisson processes. Consequently, we may interpret Ψ as the vector of the market prices of Poisson jump risks (one associated with each possible jump size) and Φ as the vector of the market prices of the Wiener risks. By taking the derivative of (4.2.15) with respect to T and manipulating appropriately, we derive the extension of the HJM forward rate drift restriction to the defaultable bond market, namely,

$$\begin{aligned} \alpha^d(t, T) &= \sum_{i=1}^{n_w} \sigma_i^d(t, T, \bar{f}^d(t)) (-\phi_i^d(t) + \zeta_i^d(t, T, \bar{f}^d(t))) \\ &\quad + \sum_{i=1}^{n_p} \beta_i^d(t, T) (\lambda_i - \psi_i^d(t) [1 - q_i(t)] e^{-\xi_i^d(t, T)}). \end{aligned} \quad (4.2.16)$$

By substituting expression (4.2.7) for $H^d(t, T, \bar{f}^d(t))$ into equation (4.2.15) it follows that the short rate spread is the sum of the products between the intensity of a default and the

⁸Note that the defaultable bonds considered are bonds of a single defaultable issuer.

corresponding expected loss quota, that is

$$r^d(t) - r(t) = \sum_{i=1}^{n_p} \psi_i^d(t) q_i(t). \quad (4.2.17)$$

Thus the spot rate spread (as measured by the difference of defaultable and default-free instantaneous spot rate) is driven solely by the $\psi_i^d(t)$ and $q_i(t)$.

4.2.2. The Risk Neutral Dynamics under a General Volatility Specification. By an application of Girsanov's theorem (Bremaud (1981)), for every fixed finite time horizon T , we can obtain a new risk neutral measure $\widetilde{\mathbb{P}}^9$, under which $\widetilde{W}_i(t) = -\int_0^t \phi_i^d(s) ds + W_i(t)$ is a standard Wiener process for $i = 1, \dots, n_w$, and Q_i is a Poisson process associated with intensity $\psi_i^d(t)$ for $i = 1, \dots, n_p$. Furthermore the \widetilde{W}_i and Q_i are mutually independent.

Using equation (4.2.15), the stochastic differential equation for the bond price under the risk neutral measure reduces to

$$\begin{aligned} \frac{dP^d(t, T)}{P^d(t^-, T)} &= r(t)dt - \sum_{i=1}^{n_w} \zeta_i^d(t, T, \bar{f}^d(t)) d\widetilde{W}_i(t) \\ &\quad - \sum_{i=1}^{n_p} [(1 - q_i(t))(1 - e^{-\xi_i^d(t, T)}) + q_i(t)] (dQ_i(t) - \psi_i^d(t)dt). \end{aligned} \quad (4.2.18)$$

Furthermore, by defining the relative defaultable bond price as

$$Z^d(t, T) = \frac{P^d(t, T)}{B(t)}, \quad (0 \leq t \leq T),$$

where $B(t)$ is the accumulated money account

$$B(t) = \exp\left(\int_0^t r(s)ds\right),$$

and using equation (4.2.18) as well as Ito's lemma, the stochastic differential equation for $Z^d(t, T)$ is

$$\frac{dZ^d(t, T)}{Z^d(t^-, T)} = -\sum_{i=1}^{n_w} \zeta_i^d(t, T) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} [(1 - q_i(t))(1 - e^{-\xi_i^d(t, T)}) + q_i(t)] [dQ_i(t) - \psi_i^d(t)dt]. \quad (4.2.19)$$

⁹The Wiener processes $\widetilde{W}_i(t)$ ($i = 1, \dots, n_w$) and the Poisson processes $Q_i(t)$ ($i = 1, \dots, n_p$) with intensity Ψ generate the $\widetilde{\mathbb{P}}_t$ -augmentation of the filtration \mathcal{F}_t .

The relative bond price process $Z^d(t, T)$ thus becomes a martingale under $\tilde{\mathbb{P}}$, so that

$$\tilde{\mathbb{E}}[dZ^d(t, T) \mid \mathcal{F}_t] = 0,$$

where $\tilde{\mathbb{E}}$ is the expectation (given information at time t) with respect to the equivalent probability measure $\tilde{\mathbb{P}}$. The above equation implies that

$$Z^d(t, T) = \tilde{\mathbb{E}}[Z^d(T, T) \mid \mathcal{F}_t],$$

and as a result, the bond price can be expressed as

$$\begin{aligned} P^d(t, T) &= \tilde{\mathbb{E}} \left[\frac{B(t)}{B(T)} P^d(T, T) \mid \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(- \int_t^T r(s) ds \right) \bar{Q}(T) \mid \mathcal{F}_t \right]. \end{aligned} \quad (4.2.20)$$

Using equation (4.2.1) and setting $T = t$, the stochastic integral equation for the defaultable spot rate $r^d(t)$ under the historical measure is given by

$$\begin{aligned} r^d(t) &\equiv f^d(t, t) \\ &= f^d(0, t) + \int_0^t \alpha^d(s, t) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i^d(s, t) [dQ_i(s) - \lambda_i ds]. \end{aligned} \quad (4.2.21)$$

By substitution of the drift restriction (4.2.16) for $\alpha^d(s, t)$ into the equation (4.2.21), we obtain the dynamics of the instantaneous defaultable spot interest rate $r^d(t)$ under the risk neutral measure $\tilde{\mathbb{P}}$, in the form

$$\begin{aligned} r^d(t) &= f^d(0, t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\tilde{W}_i(s) \\ &+ \sum_{i=1}^{n_p} \int_0^t \beta_i^d(s, t) \psi_i^d(s) [1 - (1 - q_i(s)) e^{-\xi_i^d(s, t)}] ds + \sum_{i=1}^{n_p} \int_0^t \beta_i^d(s, t) [dQ_i(s) - \psi_i^d(s) ds]. \end{aligned} \quad (4.2.22)$$

The dynamics for $r^d(t)$ implied by (4.2.22) are non-Markovian, under a general volatility setting. A specific volatility structure is required, as already discussed in Chapter 2 for the default-free case, in order to obtain Markovian representations of the defaultable spot rate dynamics (4.2.22), a theme that is developed in the following section.

4.3. A Specific Volatility Structure

We consider the volatility specifications of

ASSUMPTION 4.3.1. *For $i = 1, \dots, n_w$, the state dependent Wiener volatility structure (4.2.2) is of the form*

$$\sigma_i^d(s, t, \bar{f}^d(s)) = \sigma_{0i}^d(s, \bar{f}^d(s)) e^{-\int_s^t \kappa_{\sigma_i}(u) du}, \quad (4.3.1)$$

and for $i = 1, \dots, n_p$, the time dependent Poisson volatility functions are of the form

$$\beta_i^d(s, t) = \beta_{0i}^d(s) e^{-\int_s^t \kappa_{\beta_i}(u) du}, \quad (4.3.2)$$

where $\kappa_{\sigma_i}(t)$, $\kappa_{\beta_i}(t)$ and $\beta_{0i}^d(t)$ are deterministic functions and $\sigma_{0i}^d(t, \bar{f}^d(t))$ are state dependent functions.

As with the volatility functions in Chapter 2, the crucial property of the volatility functions (4.3.1) and (4.3.2) is that their derivatives with respect to the second argument (maturity) are given by

$$\frac{\partial}{\partial t} \sigma_i^d(s, t, \bar{f}^d(s)) = -\kappa_{\sigma_i}(t) \sigma_i^d(s, t, \bar{f}^d(s)), \quad (4.3.3)$$

for $i = 1, \dots, n_w$, and

$$\frac{\partial}{\partial t} \beta_i^d(s, t) = -\kappa_{\beta_i}(t) \beta_i^d(s, t), \quad (4.3.4)$$

for $i = 1, \dots, n_p$. This is a natural consequence of the functional forms (4.3.1) and (4.3.2), that allows the separation of the time dependent component from the maturity dependent component.

PROPOSITION 4.3.1. *Given that Assumption 4.3.1 is satisfied, the dynamics under the risk-neutral measure for the instantaneous defaultable spot rate $r^d(t)$ can be expressed as*

$$r^d(t) = f^d(0, t) + \sum_{i=1}^{n_w} \mathcal{D}_{\sigma_i}^d(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}^d(t), \quad (4.3.5)$$

in stochastic integral form, or,

$$\begin{aligned} dr^d(t) = & \left[D(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} k_{\sigma_i}(t) \mathcal{D}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} k_{\beta_i}(t) \mathcal{D}_{\beta_i}^d(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}^d(t) (dQ_i(t) - \psi_i^d(t) dt), \end{aligned} \quad (4.3.6)$$

in stochastic differential form, where

$$D(t) = \frac{\partial}{\partial t} f^d(0, t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta_i}^d(t), \quad (4.3.7)$$

and

$$\mathcal{E}_{\sigma_i}^d(t) \equiv \int_0^t \sigma_i^d(s, t, \bar{f}^d(s))^2 ds, \quad (4.3.8)$$

$$\mathcal{E}_{\beta_i}^d(t) \equiv \int_0^t \psi_i^d(s) \beta_i^{d^2}(s, t) (1 - q_i(s)) e^{-\xi_i^d(s, t)} ds, \quad (4.3.9)$$

$$\mathcal{D}_{\sigma_i}^d(t) \equiv \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds + \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s), \quad (4.3.10)$$

$$\mathcal{D}_{\beta_i}^d(t) \equiv \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - (1 - q_i(s)) e^{-\xi_i^d(s, t)}] ds + \int_0^t \beta_i^d(s, t) (dQ_i(s) - \psi_i^d(s) ds). \quad (4.3.11)$$

Proof. Taking the differential of (4.2.22) and making use of (4.3.3) and (4.3.4), the stochastic differential equation for the instantaneous spot rate under the risk neutral measure becomes

$$\begin{aligned} dr^d(t) = & \left[\frac{\partial f^d(0, t)}{\partial t} + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds \right. \\ & + \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - (1 - q_i(s)) e^{-\xi_i^d(s, t)}] ds \right) \\ & \left. - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) - \sum_{i=1}^{n_p} \kappa_{\beta_i}(t) \int_0^t \beta_i^d(s, t) (dQ_i(s) - \psi_i^d(s) ds) \right] dt \end{aligned} \quad (4.3.12)$$

$$+ \sum_{i=1}^{n_w} \sigma_{0_i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0_i}^d(t) [dQ_i(t) - \psi_i^d(t) dt],$$

which, by using the results obtained in Appendix 2.3, adjusted for the defaultable forward rate volatilities, may be expressed as

$$\begin{aligned}
dr^d(t) = & \left[\frac{\partial f^d(0, t)}{\partial t} + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s))^2 ds - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds \right. \\
& + \sum_{i=1}^{n_p} \int_0^t \psi_i^d(s) \beta_i^d(s, t)^2 (1 - q_i(s)) e^{-\xi_i^d(s, t)} ds \\
& - \sum_{i=1}^{n_p} \kappa_{\beta_i}(t) \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - (1 - q_i(s)) e^{-\xi_i^d(s, t)}] ds \\
& \left. - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) - \sum_{i=1}^{n_p} \kappa_{\beta_i}(t) \int_0^t \beta_i^d(s, t) [dQ_i(s) - \psi_i^d(s) ds] \right] dt \\
& + \sum_{i=1}^{n_w} \sigma_{0i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}^d(t) [dQ_i(t) - \psi_i^d(t) dt].
\end{aligned} \tag{4.3.13}$$

Substituting expressions (4.3.8), (4.3.9), (4.3.10) and (4.3.11) into equation (4.3.13), we obtain the dynamics (4.3.6). \square

Note that the $\mathcal{E}_{\beta_i}^d(t)$ are deterministic functions of time, whereas the $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ are stochastic quantities that satisfy Markovian stochastic differential equations (drifts and diffusion terms depend on these processes) as the next Proposition shows.

PROPOSITION 4.3.2. *Given the forward rate volatility specifications of Assumption 4.3.1 and assuming that the market prices of jump risk are time deterministic, the stochastic quantities $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ for $i = 1, \dots, n_w$, and $\mathcal{D}_{\beta_i}^d(t)$, for $i = 1, 2, \dots, n_p$, satisfy the stochastic differential equations,*

$$d\mathcal{E}_{\sigma_i}^d(t) = [\sigma_{0i}^d(t, \bar{f}^d(t))^2 - 2\kappa_{\sigma_i}(t)\mathcal{E}_{\sigma_i}^d(t)]dt, \tag{4.3.14}$$

$$d\mathcal{D}_{\sigma_i}^d(t) = [\mathcal{E}_{\sigma_i}^d(t) - \kappa_{\sigma_i}(t)\mathcal{D}_{\sigma_i}^d(t)]dt + \sigma_{0i}^d(t, \bar{f}^d(t))d\widetilde{W}_i(t), \tag{4.3.15}$$

and

$$d\mathcal{D}_{\beta_i}^d(t) = [\beta_{0i}^d(t)\psi_i^d(t)q_i(t) + \mathcal{E}_{\beta_i}^d(t) - \kappa_{\beta_i}(t)\mathcal{D}_{\beta_i}^d(t)]dt + \beta_{0i}^d(t)(dQ_i(t) - \psi_i^d(t)dt). \tag{4.3.16}$$

Proof. Follows immediately from the functional form of the quantities defined in (4.3.8), (4.3.9) and (4.3.10). \square

Section 4.4 explains how the stochastic quantities $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ may be expressed in terms of the set of the benchmark defaultable forward rates $\bar{f}^d(t)$, and vice versa, the set of the benchmark defaultable forward rates, $\bar{f}^d(t)$, in terms of these stochastic terms. Thus, the instantaneous defaultable spot rate dynamics (4.3.6) are Markovian under the forward rate volatility specifications (4.3.1) and (4.3.2), since the stochastic quantities $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ display Markovian dynamics. In the following section, we derive an exponentially affine term structure of interest rates in terms of the same stochastic quantities.

4.3.1. Affine Term Structure of Interest Rates. Using the Markovian structure (4.3.6) and applying the Inui & Kijima (1998) approach, we obtain the multi-factor bond price formula in terms of the state variables $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$.

PROPOSITION 4.3.3. *The multi-factor affine term structure of interest rates is*

$$P^d(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp \left\{ \mathcal{M}^d(t, T) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \mathcal{D}_{\beta_i}^d(t) \right\}, \quad (4.3.17)$$

where,

$$\begin{aligned} \mathcal{M}^d(t, T) &= \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})) \\ &\quad - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [1 - [1 - q_i(s)] e^{-\xi_i^d(s, y)}] dy ds \\ &\quad + \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - e^{-\xi_i^d(s, t)}] ds \right\}. \end{aligned} \quad (4.3.18)$$

with $\mathcal{N}_x(t, T)$ defined by (2.3.20).

Proof. See Appendix 4.3. □

The defaultable bond price formula (4.3.17) displays a finite dimensional affine structure in terms of a number of state variables ($2n_w + n_p$ in our case), however the resulting formula is no longer Markovian (in terms of the set of state variables that “Markovianise” the instantaneous defaultable spot rate dynamics) due to the dependence on the counting

functions $\eta_i(t)$, that count the number of defaults up to time t originating from the jump sources $Q_i(t)$, as well as, the dependency of the accumulated fractional loss at time t on the history of the jump times.¹⁰

REMARK 4.3.1. The affine term structure of interest rates (4.3.17) may become Markovian by restricting further the volatility specifications by assuming constant Poisson volatilities and also by restricting the fractional losses to be constant. In this case, the $\eta_i(t)$ may be retrieved from the processes $\mathcal{D}_{\beta_i}^d(t)$ as follows. The state variable $\mathcal{D}_{\beta_i}^d(t)$ for constant β_i^d becomes

$$\begin{aligned}\mathcal{D}_{\beta_i}^d(t) &= \beta_i^d \int_0^t \psi_i^d(s) [1 - (1 - q_i)e^{-\beta_i^d(t-s)}] ds + \int_0^t \beta_i^d (dQ_i(s) - \psi_i^d(s) ds) \\ &= -\beta_i^d \int_0^t \psi_i^d(s) (1 - q_i) e^{-\beta_i^d(t-s)} ds + \sum_{m=1}^{\eta_i(t)} \beta_{im}^d \\ &= -\beta_i^d \int_0^t \psi_i^d(s) (1 - q_i) e^{-\beta_i^d(t-s)} ds + \beta_i^d \eta_i(t).\end{aligned}\tag{4.3.19}$$

Thus the $\eta_i(t)$ can be expressed in terms of the $\mathcal{D}_{\beta_i}^d(t)$. In addition, for constant fractional losses, the leading term in (4.3.18) reduces to $\sum_{i=1}^{n_p} \eta_i(t) \ln(1 - q_i)$. Therefore, in this case, the affine term structure may be expressed in terms of the same state variables used for the Markovian structure of the defaultable spot rate.

In the next section, we will show that the state variables $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ can be expressed in terms of benchmark defaultable forward rates with dynamics driven by both Wiener and Poisson processes. Thus as in the default-free case in Chapter 2, we are able to obtain a set of state variables that are economically interpretable.

4.4. Finite Dimensional Affine Realisations in Terms of Defaultable Forward Rates

Working along the lines of Chiarella & Kwon (2003) and Björk & Svensson (2001), who show that, in a Markovian HJM framework with dynamics driven by diffusion processes, the state variables can be expressed as affine functions of a finite number of forward rates and yields, Section 2.4 shows that similar representations may be obtained when incorporating jumps with state dependent Wiener volatility functions and time deterministic

¹⁰Obviously (4.3.17) becomes Markovian if one is able to express the accumulated fractional loss at time t , $\sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik}))$, as additional state variables that satisfy Markovian dynamics.

Poisson volatility functions. Moving from the default-free to the defaultable case, we are also able to express the defaultable forward rate structure and the defaultable bond prices in terms of benchmark forward rates, since the “pseudo” defaultable bond prices exhibit Markovian dynamics.

We show in Appendix 4.3 that the dynamics of the “pseudo” defaultable bond $\hat{P}(t, T)$, may be rewritten (as a function of the state variables) in the form¹¹

$$\hat{P}(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp \left\{ \hat{\mathcal{M}}^d(t, T) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \mathcal{D}_{\beta_i}^d(t) \right\}, \quad (4.4.1)$$

where,

$$\hat{\mathcal{M}}^d(t, T) = \mathcal{M}^d(t, T) - \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})). \quad (4.4.2)$$

Using the definition (4.2.3), the instantaneous defaultable forward interest rate may be expressed in terms of these state variables as

$$f^d(t, T) + \frac{\partial \hat{\mathcal{M}}^d(t, T)}{\partial T} = \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \mathcal{E}_{\sigma_i}^d(t) + \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{D}_{\sigma_i}^d(t) + \sum_{i=1}^{n_p} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} \mathcal{D}_{\beta_i}^d(t). \quad (4.4.3)$$

Thus we have an affine relationship between state variables and forward rates and we can re-express the state of the Markovian system in terms of $n_d (= 2n_w + n_p)$ forward rates of fixed maturities.

PROPOSITION 4.4.1. *For T_h different fixed maturities, $h = 1, 2, \dots, n_d$, consider the square matrix*

$$\mathbb{O}^d(t) = \begin{bmatrix} \chi_1(t) & \chi_2(t) & \chi_3(t) \end{bmatrix},$$

where, $\chi_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma_i}(t, T_h)}{\partial T_h} \mathcal{N}_{\sigma_i}(t, T_h) \right]$ is an $n_d \times n_w$ matrix, for $i = 1, 2, \dots, n_w$,

$\chi_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma_i}(t, T_h)}{\partial T_h} \right]$ is an $n_d \times n_w$ matrix, for $i = 1, 2, \dots, n_w$, and

$\chi_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_i}(t, T_h)}{\partial T_h} \right]$ is an $n_d \times n_p$ matrix, for $i = 1, 2, \dots, n_p$. Assume that $\mathbb{O}^d(t)$ is

¹¹See equation (A4.3.12).

invertible for all $t \in \{t; t = \min_h T_h\}$. Then the defaultable forward rate of any maturity can be expressed in terms of the n_d benchmark forward rates $f^d(t, T_h)$, ($h = 1, \dots, n_d$) as

$$f^d(t, T) = -\mathcal{Q}^d(t, T) + \sum_{h=1}^{n_d} \mathcal{R}_h^d(t, T) f^d(t, T_h), \quad (4.4.4)$$

where, for $l = n + i$ and $k = 2n + i$,

$$\mathcal{R}_h^d(t, T) = \sum_{i=1}^{n_w} \left(\varpi_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) + \varpi_{lh} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \varpi_{kh} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T}, \quad (4.4.5)$$

and

$$\begin{aligned} \mathcal{Q}^d(t, T) = & -\frac{\partial \hat{\mathcal{M}}^d(t, T)}{\partial T} + \sum_{h=1}^{n_d} \frac{\partial \hat{\mathcal{M}}^d(t, T_h)}{\partial T_h} \left[\sum_{i=1}^{n_w} \left(\varpi_{ih} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \right. \right. \\ & \left. \left. + \varpi_{lh} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \right) + \sum_{i=1}^{n_p} \varpi_{kh} \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} \right], \end{aligned} \quad (4.4.6)$$

and ϖ_{lh} denotes the lh^{th} element of matrix $(\mathbb{O}^d)^{-1}(t)$, the inverse of the matrix $\mathbb{O}^d(t)$.

Proof. See Appendix 4.4. □

The defaultable bond price cannot be expressed in terms of the benchmark forward rates only but also depends on the counting functions $\eta_i(t)$, as the following proposition states.

PROPOSITION 4.4.2. *The defaultable bond prices in terms of the benchmark defaultable forward rates $f^d(t, T_h)$ is given by*

$$P^d(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \prod_{i=1}^{n_p} \prod_{k=1}^{\eta_i(t)} (1 - q_i(\tau_{ik})) \exp \left\{ \bar{\mathcal{Q}}^d(t, T) - \sum_{h=1}^{n_d} \bar{\mathcal{R}}_h^d(t, T) f^d(t, T_h) \right\}, \quad (4.4.7)$$

where

$$\bar{\mathcal{R}}_h^d(t, T) = \int_t^T \mathcal{R}_h^d(t, s) ds, \quad \text{and} \quad \bar{\mathcal{Q}}^d(t, T) = \int_t^T \mathcal{Q}^d(t, s) ds. \quad (4.4.8)$$

Proof. By substitution of (4.4.4) into the relationship (4.2.8) in conjunction with (4.2.10).

□

The advantage of the representation (4.4.7) is that it expresses bond prices in terms of economically interpretable quantities. The risk neutral dynamics for each defaultable forward rate $f^d(t, T_h)$ are given by

$$df^d(t, T_h) = \left(\sum_{i=1}^{n_w} \sigma_i^d(t, T_h, \bar{f}^d(t)) \zeta_i^d(t, T_h, \bar{f}^d(t)) - \sum_{i=1}^{n_p} \psi_i^d(t) \beta_i^d(t, T_h) [[1 - q_i(t)] e^{-\xi_i^d(t, T)} - 1] \right) dt + \sum_{i=1}^{n_w} \sigma_i^d(t, T_h, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_i^d(t, T_h) [dQ_i(t) - \psi_i^d(t) dt], \quad (4.4.9)$$

which are driven by both Wiener and Poisson processes. Thus by setting

$$\bar{f}^d(t) = (f^d(t, T_1), f^d(t, T_2), \dots, f^d(t, T_{n_d}))^\top$$

we have a closed system.

4.5. Modelling Defaultable Term Structure with Stochastic Intensity

In the analysis above we have assumed only deterministic structures for the jump intensities $\psi_i^d(t)$. Recalling the result (4.2.17), namely that

$$r^d(t) - r(t) = \sum_{i=1}^{n_p} \psi_i^d(t) q_i(t),$$

we see that such an assumption imposes a deterministic structure on the credit spreads (short rate spread), something that it is not consistent with empirical observations. The empirical study on credit spreads by Sarig & Warga (1989) shows that credit spreads are in general non-zero, whilst Prigent, Renault & Scaillet (2000) provide evidence of mean reversion. To extend the model to incorporate stochastic spreads that are consistent with empirical observations we will need to use Cox processes (Poisson process with stochastic intensity) instead of simple Poisson processes to model the jump (some of which may be defaults) arrivals.

ASSUMPTION 4.5.1. *The process for each of the intensities $\psi_i^d(t)$ associated with the Cox processes $Q_i(t)$, for $i = 1, 2, \dots, n_p$, under the risk-neutral measure satisfies the stochastic differential equation*

$$d\psi_i^d(t) = \theta_i (\bar{\psi}_i^d - \psi_i^d(t)) dt + \sqrt{\psi_i^d(t)} \sum_{k=1}^{n_w} \sigma_{\psi_{ki}^d} d\widetilde{W}_k(t), \quad (4.5.1)$$

where θ_i , $\bar{\psi}_i^d$ and $\sigma_{\psi_i^d}$ are constant.

The square root process of Cox et al. (1985) has been selected to model the intensities $\psi^d(t)$ in order to provide positive credit spreads. Given the stochastic dynamics of the credit spreads, the quantities $\mathcal{E}_{\beta_i}^d(t)$ defined by (4.3.9) are no longer deterministic.

Under Assumption 4.5.1 the quantities $\mathcal{E}_{\beta_i}^d(t)$ display non-Markovian dynamics. To see this, we calculate

$$d\mathcal{E}_{\beta_i}^d(t) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{oi}^{d,2}(t) - 2\kappa_{\beta_i}(t)\mathcal{E}_{\beta_i}^d(t) - \int_0^t \psi_i^d(s)(1 - q_i(s))\beta_i^{d,3}(s, t)e^{-\xi_i(s, t)} ds \right) dt. \quad (4.5.2)$$

and by introducing for $n = 2, 3, \dots$, the notation

$$\mathcal{E}_{\beta_i}^{(n)}(t) = \int_0^t \psi_i^d(s)(1 - q_i(s))\beta_i^{d,n}(s, t)e^{-\xi_i(s, t)} ds,$$

the process $\mathcal{E}_{\beta_i}^d(t) = \mathcal{E}_{\beta_i}^{(2)}(t)$ satisfies the stochastic differential equation

$$d\mathcal{E}_{\beta_i}^{(2)}(t) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{oi}^{d,2}(t) - 2\kappa_{\beta_i}(t)\mathcal{E}_{\beta_i}^{(2)}(t) - \mathcal{E}_{\beta_i}^{(3)}(t) \right) dt. \quad (4.5.3)$$

In turn, the quantity $\mathcal{E}_{\beta_i}^{(3)}(t)$ satisfies the stochastic differential equation

$$d\mathcal{E}_{\beta_i}^{(3)}(t, \bar{f}) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{oi}^{d,3}(t) - 3\kappa_{\beta_i}(t)\mathcal{E}_{\beta_i}^{(3)}(t) - \mathcal{E}_{\beta_i}^{(4)}(t) \right) dt.$$

Thus, an infinite sequence of processes $\mathcal{E}_{\beta_i}^{(n)}(t)$ is generated, so that for $n = 2, 3, \dots$

$$d\mathcal{E}_{\beta_i}^{(n)}(t) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{oi}^n(t) - n\kappa_{\beta_i}(t)\mathcal{E}_{\beta_i}^{(n)}(t) - \mathcal{E}_{\beta_i}^{(n+1)}(t) \right) dt.$$

Therefore, it does not seem possible to obtain a Markovian representation of the spot rate dynamics under stochastic credit spreads within the current framework.

4.5.1. Approximate Markovianisation. We have just seen that under the stochastic dynamics of the credit spreads, the quantities $\mathcal{E}_{\beta_i}^d(t)$ defined by (4.3.9) are stochastic quantities with non-Markovian dynamics since evaluating the stochastic differential equation for $\mathcal{E}_{\beta_i}^d(t)$ requires the dynamics of the infinite sequence of processes $\mathcal{E}_{\beta_i}^{(n)}(t)$.

However we may “close” this sequence, in other words to obtain a finite sequence by some approximation procedure. In practice, given the magnitude of the jump volatility, it would be the case that $\beta_i^{d,n}(s, t) \simeq 0$, for sufficiently large n .

Another approach that we may employ to achieve Markovianisation is to restrict further the Poisson volatility functions. In the next subsection, we derive a Markovian structure for the defaultable forward rate and the defaultable bond prices, using constant jump volatilities under Assumption 4.5.1.

4.5.2. Constant Jump Volatility. In the current set-up, to obtain a Markovian representation of the spot rate dynamics (4.3.6), we shall impose more specific jump volatility restrictions on the existing volatility specifications of Assumption 4.3.1.

ASSUMPTION 4.5.2. *For $i = 1, \dots, n_p$, the time dependent Poisson volatility functions are of the form*

$$\beta_i^d(s, t) = \beta_{0i}^d, \quad (4.5.4)$$

where β_{0i}^d are constant.

Under the jump volatility specification of Assumption 4.5.2, the quantities $\mathcal{E}_{\beta_i}^d(t)$ defined by (4.3.9) are stochastic and satisfy the Markovian stochastic differential equations,

$$d\mathcal{E}_{\beta_i}^d(t) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{0i}^{d^2} - \beta_{0i}^d \mathcal{E}_{\beta_i}^d(t) \right) dt.$$

Thus the instantaneous defaultable spot rate $r^d(t)$ evolves according to the following proposition

PROPOSITION 4.5.1. *Under Assumption 4.3.1 and Assumption 4.5.2, the dynamics for the instantaneous defaultable spot rate $r^d(t)$ can be expressed as*

$$r^d(t) = f^d(0, t) + \sum_{i=1}^{n_w} \mathcal{D}_{\sigma_i}^d(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}^d(t), \quad (4.5.5)$$

in stochastic integral form, or

$$\begin{aligned} dr^d(t) = & \left[\frac{\partial}{\partial t} f^d(0, t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma_i}^d(t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta_i}^d(t) - \sum_{i=1}^{n_w} k_{\sigma_i}(t) \mathcal{D}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} \beta_{0i}^d(t) \psi_i^d(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}^d(t) dQ_i(t), \end{aligned} \quad (4.5.6)$$

in stochastic differential form, where the stochastic quantities $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{E}_{\beta_i}^d(t)$ and $\mathcal{D}_{\sigma_i}^d(t)$ are defined as in Proposition 4.3.1 and satisfy the Markovian system

$$d\mathcal{E}_{\sigma_i}^d(t) = [\sigma_{0i}^{d,2}(t, \bar{f}^d(t)) - 2\kappa_{\sigma_i}(t)\mathcal{E}_{\sigma_i}^d(t)]dt, \quad (4.5.7)$$

$$d\mathcal{E}_{\beta_i}^d(t) = \left(\psi_i^d(t)(1 - q_i(t))\beta_{0i}^{d,2} - \beta_{0i}^d\mathcal{E}_{\beta_i}^d(t) \right) dt, \quad (4.5.8)$$

and

$$d\mathcal{D}_{\sigma_i}^d(t) = [\mathcal{E}_{\sigma_i}^d(t) - \kappa_{\sigma_i}(t)\mathcal{D}_{\sigma_i}^d(t)]dt + \sigma_{0i}^d(t, \bar{f}^d(t))d\widetilde{W}_i(t). \quad (4.5.9)$$

with $\psi_i^d(t)$ having dynamics driven by (4.5.1).

Proof. Using Assumption 4.5.2 in combination with Assumption 4.3.1 the dynamics simplify as above. See Appendix 4.5 for details. \square

Thus, we have obtained a Markovian representation of the spot rate dynamics, namely equation (4.5.6), using stochastic intensities, level dependent Wiener volatilities and constant Poisson volatilities.

Following the same course as in Section 4.3.1, we obtain the multi-factor defaultable bond price formula.

PROPOSITION 4.5.2. *The defaultable bond price is given by the exponential affine form*

$$P^d(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp \left\{ \check{\mathcal{M}}^d(t, T) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} \mathcal{C}_{\beta_i}^d(t, T) \mathcal{E}_{\beta_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) - (T - t) \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}^d(t) \right\}, \quad (4.5.10)$$

where,

$$\check{\mathcal{M}}^d(t, T) = \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})), \quad (4.5.11)$$

and

$$\mathcal{C}_{\beta_i}^d(t, T) = \frac{(T - t)\beta_{0i}^d + e^{-\beta_{0i}^d(T-t)} - 1}{\beta_{0i}^{d,2}}. \quad (4.5.12)$$

Proof. See Appendix 4.6. □

Using the affine term structure of interest rates (4.5.10) and the definition (4.2.3), the instantaneous defaultable forward rate of any maturity is expressed in terms of the $2n_w + 2n_p$ state variables as

$$\begin{aligned} f^d(t, T) = & f^d(0, T) + \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_p} \frac{\partial \mathcal{C}_{\beta_i}(t, T)}{\partial T} \mathcal{E}_{\beta_i}^d(t) \\ & + \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{D}_{\sigma_i}^d(t) + \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}^d(t). \end{aligned} \quad (4.5.13)$$

In the following sections we deal with some model implementation issues, we discuss model limitations and we perform numerical simulations of the Markovian defaultable term structures.

4.6. Model Limitations

Just as in the situation in Section 2.6, it is also the case that there is a positive probability that the Markovian term structure model developed here may drive interest rates to negative values. To understand this effect, the functional behavior of the jump adjusted drift coefficient of the defaultable forward rate is examined in detail. Recall the risk neutral defaultable forward rate dynamics (4.4.9), which under the volatility specifications of Assumption 4.3.1 are expressed as

$$\begin{aligned} df^d(t, T) = & \sum_{i=1}^{n_w} \frac{1}{k_{\sigma_i}} \sigma_{0i}^d{}^2(t, \bar{f}^d(t)) e^{-\kappa_{\sigma_i}(T-t)} \left(1 - e^{-\kappa_{\sigma_i}(T-t)}\right) dt \\ & - \sum_{i=1}^{n_p} \psi_i \beta_{0i}^d e^{-k_{\beta_i}(T-t)} \left[(1 - q_i) e^{\frac{\beta_{0i}^d}{k_{\beta_i}} (e^{-k_{\beta_i}(T-t)} - 1)} - 1 \right] dt - \sum_{i=1}^{n_p} \psi_i \beta_{0i}^d e^{-k_{\beta_i}(T-t)} dt \\ & + \sigma_0^d(t, \bar{f}^d(t)) e^{-\kappa_{\sigma}(T-t)} d\widetilde{W}(t) + \sum_{i=1}^2 \beta_{0i}^d e^{-k_{\beta_i}(T-t)} dQ_i(t). \end{aligned} \quad (4.6.1)$$

The drift function of the defaultable forward rate dynamics is bounded by the function $\mathbb{D}^d(\tau)$ (set $\tau = T - t$)

$$\begin{aligned} \mathbb{D}^d(\tau) = & \sum_{i=1}^{n_w} \frac{\sigma_{0i}^d{}^2}{k_{\sigma_i}} c_0 e^{-\kappa_{\sigma_i}\tau} (1 - e^{-\kappa_{\sigma_i}\tau}) - \sum_{i=1}^{n_p} \psi_i \beta_{0i}^d e^{-k_{\beta_i}\tau} [1 - q_i] e^{\frac{\beta_{0i}^d}{k_{\beta_i}} (e^{-k_{\beta_i}\tau} - 1)}, \\ & = \mathbb{D}_G^d(\tau) + \mathbb{D}_J^d(\tau), \end{aligned} \quad (4.6.2)$$

where $\mathbb{D}_G^d(\tau)$ is the Gaussian drift contribution and $\mathbb{D}_J^d(\tau)$ is the contribution of the jump component to the drift. The derivative of $\mathbb{D}^d(\tau)$ is

$$\begin{aligned} \frac{d\mathbb{D}^d(\tau)}{d\tau} &= \sum_{i=1}^{n_w} \sigma_{0i}^d{}^2 c_0 e^{-\kappa_{\sigma i}\tau} (2e^{-\kappa_{\sigma i}\tau} - 1) \\ &\quad + \sum_{i=1}^{n_p} \psi_i \beta_{0i}^d [1 - q_i] e^{-k_{\beta i}\tau} e^{\frac{\beta_{0i}^d}{k_{\beta i}}(e^{-k_{\beta i}\tau} - 1)} (k_{\beta i} + \beta_{0i}^d e^{-k_{\beta i}\tau}). \end{aligned}$$

Positive jump sizes cause the drift function to be originally negative for some time close to the maturity and positive elsewhere. Negative jump sizes, on the other hand, cause the drift function to be always positive however the negative jump noise terms may drive the process to negative interest rates. Thus, this model allows interest rates to become negative with a positive probability. To overcome this problem, as we already proposed for the default-free model in Section 2.6, we use functional forms that are well defined for negative values. As an illustrative example, assume that the state dependency is modelled as a linear combination of the defaultable forward rates $\bar{f}^d(t)$, as

$$L_f^d(t) = c_0 + \sum_{h=1}^{n_d} c_h f^d(t, T_h).$$

When $L_f(s)$ becomes very small or negative then the model may behave as a deterministic volatility Hull-White type of model. Thus, a suggested volatility function may be

$$\sigma_0^d(t, \bar{f}^d(t)) = \begin{cases} c_f \sigma_0^d, & L_f(t) < 0.005; \\ \sigma_0^d [(L_f^d(t) - 0.005)^\gamma + c_f], & L_f(t) \geq 0.005; \end{cases} \quad (4.6.3)$$

with $\gamma = \frac{1}{2}$, for example.

4.7. Simulated Distributions

In this chapter, the Markovianisation of a multi-factor jump-diffusion defaultable HJM term structure model under stochastic volatility specifications is discussed. The stochastic volatility feature is handled by the means of state dependent Wiener volatility functions. This general model can be related to known term structure models and in particular to provide extensions of these models to the defaultable environment by incorporating jump components and relating jumps to default events. More specifically, given that the Wiener volatilities are state dependent, the model considered here may be seen as a RS

type model extended to a defaultable set-up. In particular, we consider a state dependent Wiener volatility structure which is expressed in terms of a number of benchmark defaultable forward rates while the jump volatilities are assumed to be time deterministic. Under these volatility specifications, Markovian representations of the defaultable term structure have been obtained, under the assumptions of deterministic and stochastic default intensities. Taking advantage of these Markovian representations, we perform simulations of the Markovian defaultable spot rate dynamics and we compare the simulated distributions of the model with deterministic jump intensities to those from the model incorporating stochastic jump intensities. The noise is generated by one Wiener and two Poisson jump terms, thus we consider the case that $n_w = 1$ and $n_p = 2$. The initial defaultable forward rate curve is assumed to have the functional form $f^d(0, t) = (a_0 + a_1 t + a_2 t^2) e^{-vt}$ with parameters set to $a_0 = 6.2382$, $a_1 = 0.4086$, $a_2 = -0.0113$, and $v = 0.0170$, resulting in an upward sloping forward curve. This curve is typical of observed defaultable forward curves. To interpolate default information to obtain initial defaultable forward rate curves is an ongoing research topic and is beyond the scope of this thesis. The recent work of Bystrom & Kwon (2003) gives one interesting approach to this empirically difficult issue. For the simulation examples, an Euler-Maruyama approximation is employed, the time interval $[0, 1]$ is discretised into $N = 400$ equal subintervals of length $\Delta t = 1/N$, and 100,000 paths for $r^d(t)$ are generated.

4.7.1. Deterministic Jump Intensities. The jump intensities now are deterministic and in particular in our example, they are considered constant. The number of state variables of the term structure is four. Thus, we shall require four state variables to complete the system. The Wiener volatility specifications have the functional form (4.3.1), namely

$$\sigma^d(s, t, \bar{f}^d(s)) = \sigma_0^d(s, \bar{f}^d(s)) e^{-\int_s^t \kappa_\sigma(u) du}, \quad (4.7.1)$$

where $\bar{f}^d(s) = (f^d(t, T_1), f^d(t, T_2), f^d(t, T_3), f^d(t, T_4))^\top$. We assume a similar functional form as (2.6.3), which ensures that the state dependent volatility is a well defined function, as Section 4.6 explains,

$$\sigma_0^d(t, \bar{f}^d(t)) = \begin{cases} 0.05 \sigma_0^d, & L_f(t) < 0.005; \\ \sigma_0^d [(L_f(t) - 0.005)^\gamma + 0.05], & L_f(t) \geq 0.005; \end{cases} \quad (4.7.2)$$

where $L_f(t) = c_0 + \sum_{h=1}^4 c_h f(s, T_h)$ and $\gamma = \frac{1}{2}$. For the Poisson volatility specifications, we consider $\beta_i^d(s, t) = \beta_{0i}^d e^{-k_{\beta_i}(t-s)}$ and constant ψ_i . For $\ell = 1, 2, \dots, N$, we set $t_\ell = \ell \Delta t$, and we consider the discretised system of the instantaneous defaultable spot rate dynamics (4.3.5) with the four state variables $\mathcal{D}_\sigma(t)$, $\mathcal{D}_{\beta_1}(t)$ and $\mathcal{D}_{\beta_2}(t)$ expressed in terms of the four benchmark defaultable forward rates $f^d(t, 2.5)$, $f^d(t, 5)$, $f^d(t, 7.5)$ and $f^d(t, 10)$, by using the system (A4.4.2). Appendix 4.7 provides the details of the simulated system.

Parameter Values							
	no-jump	negative jumps		positive jumps		+ve & -ve jumps	
		small	large	small	large	small	large
σ_0	3.5 %	3.2 %	1.5 %	3.2 %	1.5 %	3.2 %	1.5 %
β_{01}	0 %	- 0.8 %	-2 %	0.8 %	2 %	0.8 %	2 %
β_{02}	0 %	-1.5 %	-3 %	1.5 %	3 %	- 1.5 %	- 3 %

TABLE 4.7.1. The volatility parameter values used in the simulations of the deterministic default intensity models.

The volatility parameter values used are $k_\sigma = 0.18$, $c_0 = 1$, $c_1 = 2$, $c_2 = 1$, $c_3 = 2$, $c_4 = 1$, $k_{\beta_1} = 0.31$ and $k_{\beta_2} = 0.17$. Also we set $\psi_1 = 1$, $\psi_2 = 1.5$, $q_1 = 0$ and $q_2 = 60\%$. We consider three cases for the volatility sizes combined with three different jump magnitudes. Thus the cases considered are the case of zero jump sizes, the case of small jump sizes and the case of large jump sizes where for the non zero jump sizes we allow for only positive jumps, only negative jumps and the more realistic case of positive and negative jumps. The volatility parameter values used are summarised in Table 4.7.1.

In addition, the simulations have been performed over two different time horizons, one year and 6 months. In order to compare the simulated normalised distribution of the defaultable spot rate, the volatility specifications have been selected so as to provide the same variance of the simulated distributions, which is 0.17% for the 1-year time horizon and 0.09% for the 6 months time horizon.

Figure 4.7.1 shows the effect of the jump component on the simulated normalised distribution of the defaultable spot rate at $t = 1$.

Figure 4.7.2 shows the effect of the jump component on the simulated distribution of the defaultable spot rate at $t = 1$. It is obvious that there is a positive probability of negative

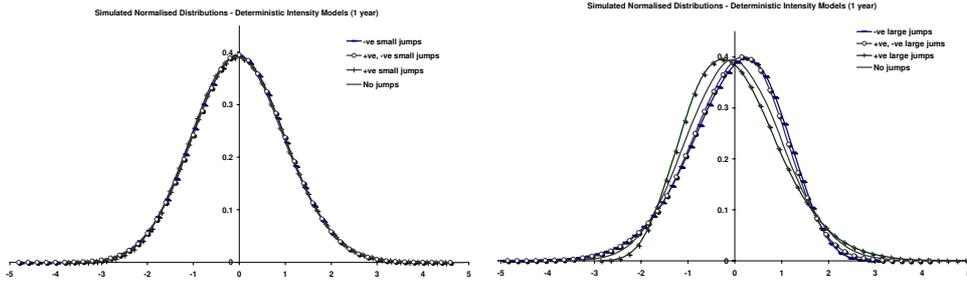


FIGURE 4.7.1. The skewness of the simulated normalised density of defaultable spot rate increases as the jump size increases - Deterministic jump intensities and 1 year time horizon.

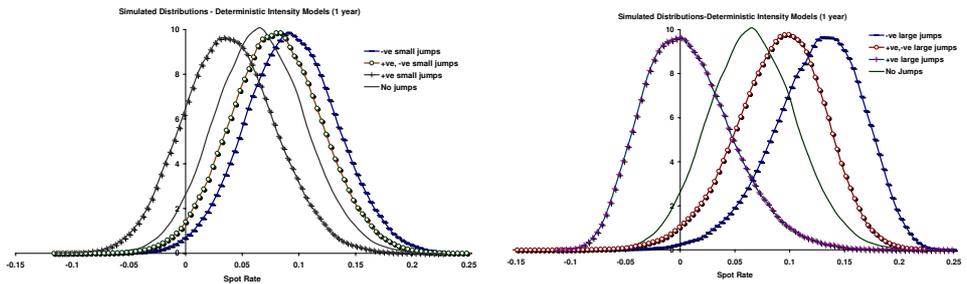


FIGURE 4.7.2. The simulated density of the defaultable spot rate - Deterministic jump intensities and 1 year time horizon.

interest rates that can be reduced by decreasing the time horizon. When the time horizon reduces to 6 months (for $t = 0.5$), as Figure 4.7.4 shows, the probability of negative interest rates declines compared to the ones at the 1 year horizon displayed in Figure 4.7.2 and this is so for a range of jump magnitudes and sizes. In addition, by reducing the time horizon, excess kurtosis and skewness may be achieved using more realistic jump sizes. Figure 4.7.3 shows the simulated normalised distribution of the defaultable spot rate for a range of jump sizes at $t = 0.5$. Comparing with the simulated normalised distribution of the defaultable spot rate at $t = 1$ (see Figure 4.7.1), excess kurtosis and skewness is obtained under the 6 months time horizon and for the large jump sizes. These results are also illustrated in Table 4.7.2 and Table 4.7.3 which display the statistical measures of the defaultable spot rate under 1 year and 6 months time horizons respectively.

The zero jump size case captures the effect of the state dependent volatility specifications solely on the distribution. Figure 4.7.1 shows that state dependent volatilities skew slightly the normalised distribution (see Table 4.7.2 Table 4.7.3) however the effect is

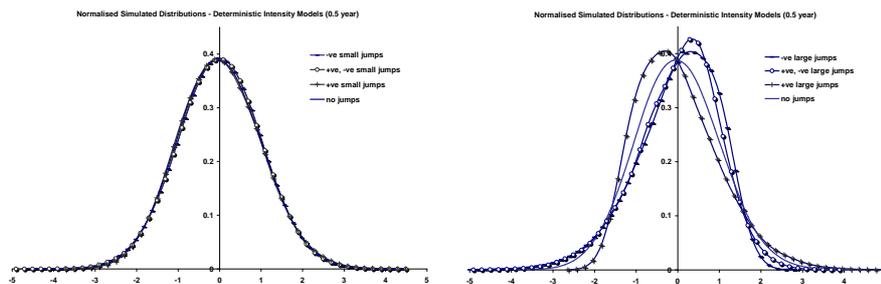


FIGURE 4.7.3. The skewness of the simulated normalised density of defaultable spot rate increases as the jump size increases - Deterministic jump intensities and 0.5 year time horizon.

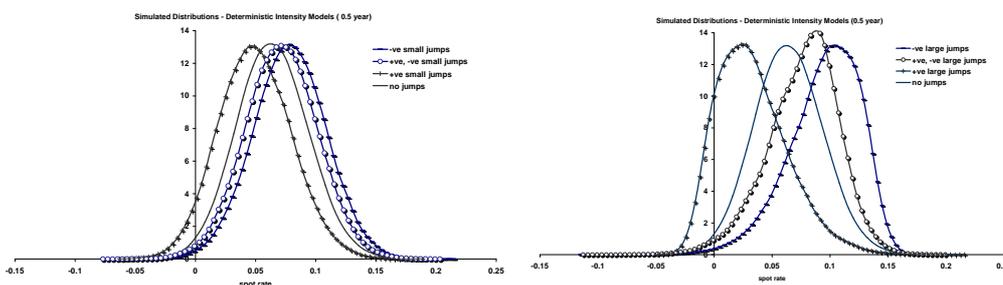


FIGURE 4.7.4. The simulated density of the defaultable spot rate - Deterministic jump intensities and 0.5 year time horizon.

stronger with increasing jump sizes. By increasing the jump component, the normalised distribution is asymmetric (a characteristic of empirical spot rate distributions), a feature that becomes more pronounced when the time horizon decreases. The long tail appears to right when only positive jumps are allowed and to the left when only negative jumps are allowed. An illustrative numerical comparison of the effect of the jump magnitudes

Statistical Information on $r^d(t)$ - 1 year							
	no-jump	negative jumps		positive jumps		+ve & -ve jumps	
		small	large	small	large	small	large
Mean	6.61	9.36	12.46	3.87	0.73	7.98	9.02
Variance	0.17	0.17	0.17	0.17	0.17	0.17	0.17
Skewness	0.0982	0.0421	-0.4833	0.1478	0.5243	0.0463	-0.3478
Kurtosis	3.0407	3.0242	3.3017	3.0440	3.3308	3.0522	3.3254

TABLE 4.7.2. The statistical measures (in percentage terms) of the defaultable spot rate simulated distributions for different jump magnitudes with the deterministic default intensity model - Time horizon 1 year.

on the simulated normalised distribution is provided in Table 4.7.2 for the 1 year time horizon and in Table 4.7.3 for the 6 months time horizon. We consider these two different

Statistical Information on $r^d(t)$ - 6 months							
	no-jump	negative jumps		positive jumps		+ve & -ve jumps	
		small	large	small	large	small	large
Mean	6.40	7.83	9.46	4.95	3.32	7.12	7.64
Variance	0.09	0.09	0.09	0.09	0.09	0.09	0.09
Skewness	0.0799	-0.0236	-0.7211	0.1344	0.7290	-0.0133	-0.4792
Kurtosis	2.9940	3.0293	3.7180	3.0195	3.6627	3.0708	3.5895

TABLE 4.7.3. The statistical measures (in percentage terms) of the defaultable spot rate simulated distributions for different jump magnitudes with the deterministic default intensity model - Time horizon 0.5 year.

time horizons to show that with more realistic jump sizes (smaller than 2%) we can obtain reasonable skewness and kurtosis.

Statistical Information on $dr^d(t)$ - 1 year							
	no-jump	negative jumps		positive jumps		+ve & -ve jumps	
		small	large	small	large	small	large
Mean	0.0013	0.0078	0.0154	-0.0053	-0.0121	0.0043	0.0067
Variance	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005
Skewness	-0.0040	-1.2351	-10.3845	1.1863	10.0087	-0.9868	-7.2874
Kurtosis	3.0438	10.8871	135.2202	10.4195	127.9270	10.7025	134.8426

TABLE 4.7.4. The statistical measures (in percentage terms) of the changes of the defaultable spot rate for different jump magnitudes with the deterministic default intensity model - Time horizon 1 year.

In addition, by focusing on the effect of a decreasing time horizon on the leptokurtosis, we make the comparison more extreme and we consider in Table 4.7.4, the statistical information of the changes of the defaultable spot rate. Given that the time horizon is 1 year and the discretisation level is 400, the information in the table can be viewed as the statistical measures of $r^d(t)$ over approximately one day time horizon. It is obvious that the model captures the empirical fact of increasing leptokurtosis as the time horizon decreases. Concluding, the models developed provide flexibility on the shape of the spot rate distributions and also succeed in accommodating the stylized facts of such distributions.

4.7.2. Stochastic Jump Intensities. The jump intensities are now assumed to be stochastic, evolving as in (4.5.1) so that the Markovian dynamics of the defaultable spot rate are determined by Proposition 4.5.1. The number of state variables has increased to 8, therefore we introduce six defaultable forward rates of differing maturities in order to

complete the system. Recall that the other two state variables are the two stochastic default intensities. Thus the Wiener volatility specifications are given by

$$\sigma^d(s, t, \bar{f}^d(s)) = \sigma_0^d(s, \bar{f}^d(s)) e^{-\int_s^t \kappa_\sigma(u) du}, \quad (4.7.3)$$

where $\bar{f}^d(s) = (f^d(t, T_1), f^d(t, T_2), f^d(t, T_3), f^d(t, T_4), f^d(t, T_5), f^d(t, T_6))^\top$. We assume a similar functional form as (2.6.3), that ensures that the state dependent volatility is a well defined function,

$$\sigma_0^d(t, \bar{f}^d(t)) = \begin{cases} 0.05 \sigma_0^d, & L_f^d(t) < 0.005; \\ \sigma_0^d((L_f^d(t))^\gamma + 0.05), & L_f^d(t) \geq 0.005; \end{cases} \quad (4.7.4)$$

where $L_f^d(t) = c_0 + \sum_{h=1}^6 c_h f^d(s, T_h)$ and $\gamma = \frac{1}{2}$. The Poisson volatilities are now constant.

We set the Wiener volatility specifications as $\kappa_\sigma = 0.08$, $c_0 = 1$, $c_1 = 2$, $c_2 = 1$, $c_3 = 2$, $c_4 = 1$, $c_5 = 2$, $c_6 = 1$, and the jump volatility specifications as $\psi_1 = 1$, $\psi_2 = 1.5$, $q_1 = 0$ and $q_2 = 60\%$. We also set $T_h = \frac{10}{6}h$ years, for $h = 1, 2, \dots, 6$. See Appendix 4.7 for the details of the simulated system. The volatility parameter values considered are the one indicated in Table 4.7.5.

Parameter Values						
	negative jumps		positive jumps		+ve & -ve jumps	
	small	large	small	large	small	large
σ_0	2.9 %	0.9 %	2.9 %	0.9 %	2.9 %	0.9 %
β_{01}	-0.8 %	-2 %	0.8 %	2 %	0.8 %	2 %
β_{02}	-1.5 %	-3 %	1.5 %	3 %	-1.5 %	-3 %

TABLE 4.7.5. The volatility parameter values used in the simulations of the stochastic default intensity models.

In order to compare the simulated normalised distribution of the defaultable spot rate, the volatility specifications have been selected so as to provide the same variance ($= 0.17\%$) for the simulated distributions. The simulated normalised distribution of the defaultable spot rate at $t = 1$ (and under the stochastic default intensity case) is displayed in Figure 4.7.5 for the range of volatility sizes given in Table 4.7.5.

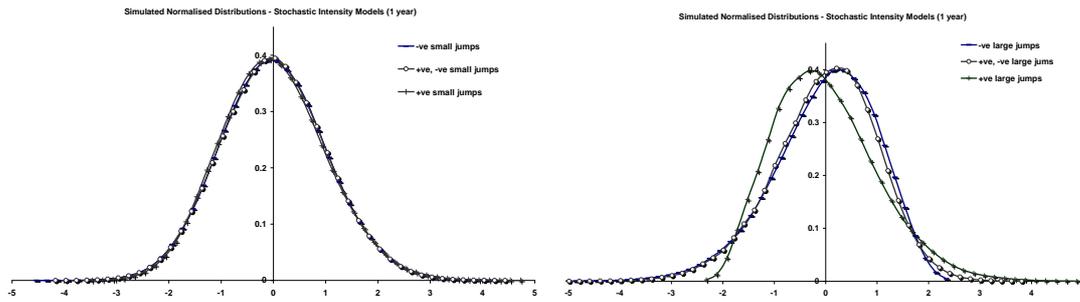


FIGURE 4.7.5. The Skewness of the Simulated Normalised Density of Defaultable Spot Rate Increases as the Jump Size Increases - Stochastic Default Intensities and 1 year Time Horizon.

The normalised distribution becomes leptokurtic, as the jump size increases. The long tail appears to the right when only positive jumps are allowed and to the left when only negative jumps are allowed.

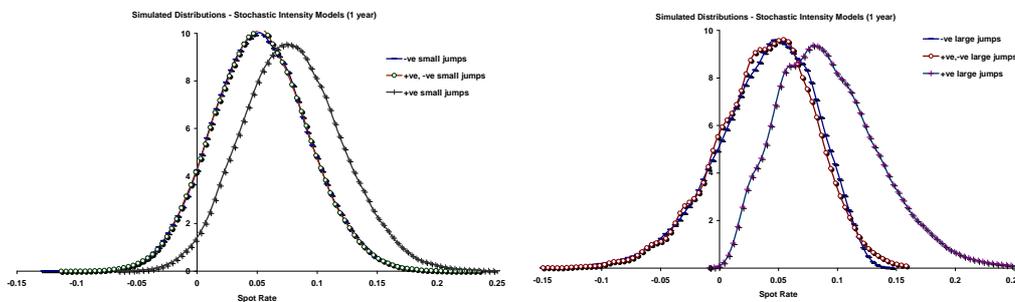


FIGURE 4.7.6. Simulated Density of Defaultable Spot Rate - Stochastic Default Intensities and 1 year Time Horizon.

Figure 4.7.6 shows the simulated distribution of the defaultable spot rate at $t = 1$, and for the volatility sizes on Table 4.7.5. The skewness and kurtosis of the simulated density of the defaultable spot rate increases as the jump size increases. Also note that, similarly as in the case of deterministic intensities, there is a positive probability of negative interest rates that can be reduced by decreasing the time horizon. Also, at smaller time horizons, excess kurtosis and skewness can be obtained by using reasonable small jump sizes.

However the model with the stochastic intensity displays, in general, higher skewness and kurtosis compared to the equivalent model with deterministic intensities as Table 4.7.6 shows. By adding a range of realistic features in this defaultable term structure model such as allowing jumps in the defaultable forward and spot rate, state dependent volatility specifications and stochastic default intensities, we obtain classes of defaultable term structure models with leptokurtosis.

Statistical Information of $r^d(t)$ - 1 year						
	negative jumps		positive jumps		+ve & -ve jumps	
	small	large	small	large	small	large
Mean	5.25	3.90	7.96	9.26	5.25	3.87
Variance	0.17	0.17	0.17	0.17	0.17	0.17
Skewness	0.0904	-0.5879	0.2309	0.6234	0.0956	-0.3793
Kurtosis	3.0153	3.3863	3.0840	3.4439	3.0313	3.4231

TABLE 4.7.6. The statistical measures (in percentage terms) of the defaultable spot rate simulated distributions for different jump magnitudes - Stochastic Default Intensity and 1 year Time Horizon .

Therefore, these type of models, which accommodate the tractability of Markovian representations as well as the complexity of stochastic volatility jump-diffusion models, capture the stylised empirical facts of defaultable interest rates and are convenient for numerical applications. As a consequence, these classes of models would provide a good modelling platform for credit derivative pricing and hedging.

4.8. Conclusions

In this chapter, a multi-factor jump-diffusion model of the defaultable term structure of interest rates within the HJM framework is considered. By an appropriate choice of a state dependent and time dependent forward rate volatility functions and deterministic credit spreads, Markovian representations of the defaultable spot rate dynamics are obtained and exponential affine formulas for the defaultable bond price are derived. Furthermore, the state variables of the model are expressed in terms of a set of benchmark defaultable forward rates, a fact which makes the model suitable for both calibration and parameter estimation. Making the model more realistic, the case of a stochastic credit spread is investigated, in which case it becomes difficult to obtain Markovian representation of the system. Then an “approximate” Markovian structure or constant Poisson volatilities are proposed.

Finally, by assuming initially deterministic default intensities, and further, the more realistic case of stochastic default intensities, some Monte Carlo simulations are performed to measure the effect of the volatility specifications on the distributions of the defaultable spot rate and show that the stochastic intensity models display more pronounced leptokurtic effects compared to the deterministic intensity models. In addition, we explain

how this model extends the RS default-free stochastic volatility model to a defaultable stochastic volatility term structure model under jump-diffusions.

In summary, this chapter makes the following main contributions:

- ★ Under the generalised Schönbucher (2000), (2003) framework and a specific formulation of state and time dependent volatility specifications, Markovian defaultable spot rate and defaultable bond price dynamics are obtained.
- ★ Finite dimensional affine realisations of the defaultable term structure in terms of forward rates and yields are obtained.
- ★ By assuming stochastic intensities, model limitations are discussed and an approximate Markovianisation of the system is developed.
- ★ The numerical implementations provide us with the following findings: State dependent volatilities skew slightly the distribution. However, the effect is stronger with increasing jump volatilities. Stochastic jump intensity adds skewness to the defaultable spot rate distribution compared to the case of deterministic jump intensity. Increasing the jump size, the normalised distribution becomes asymmetric (long tail to the right for positive jump sizes or long tail to the left for negative jump sizes).

The proposed defaultable term structure developed in this chapter combines the tractability of Markovian representations and the complexity of stochastic volatility jump-diffusion models and as the numerical simulations show succeeds in capturing the stylised empirical features of the distributions of defaultable interest rates. Therefore, this Markovian class of defaultable models that incorporates the realistic characteristics of stochastic volatility jump-diffusion defaultable forward rate dynamics combined with stochastic default intensities, may be employed for more accurate credit derivative pricing and hedging as well as model calibration and econometric estimation techniques.

Appendix 4.1. Doléans-Dade Exponential Formula

Assume that the fractional losses $q_i(t)$ are deterministic functions of time. The solution to the stochastic differential equation (4.2.11), may be derived by using results from Jacod

& Shiryaev (2003). To this end, it is convenient to define the process X_t by

$$dX_t = - \sum_i^{n_p} q_i(t) dQ_i(t).$$

Note that X_t has finite variation. Then appealing to equation (4.63) of Jacod & Shiryaev (2003), the solution of the equation (4.2.11) is of the form

$$\begin{aligned} \bar{Q}(t) &= e^{X_t - X_0} \prod_{s \leq t} (1 - \sum_i q_i(s) \Delta Q_i(s)) e^{-\Delta X_s} \\ &= \prod_{s \leq t} (1 - \sum_{i=1}^{n_p} q_i(s) \Delta Q_i(s)). \end{aligned} \quad (\text{A4.1.1})$$

Assuming that we have only single jumps¹² then equation (A4.1.1) becomes

$$\begin{aligned} \bar{Q}(t) &= \prod_{i=1}^{n_p} \prod_{s \leq t} (1 - q_i(s) \Delta Q_i(s)) \\ &= \prod_{i=1}^{n_p} \prod_{k=1}^{\eta_i(t)} (1 - q_i(\tau_{ik})). \end{aligned} \quad (\text{A4.1.2})$$

For constant fractional losses q_i , expression (A4.1.2) reduces to

$$\bar{Q}(t) = \prod_{i=1}^{n_p} (1 - q_i)^{\eta_i(t)}. \quad (\text{A4.1.3})$$

Appendix 4.2. The No-Arbitrage Condition in the Defaultable Bond Market

We set $n_H = n_w + n_p$. We consider a hedging portfolio containing defaultable bonds of maturities $T_1, T_2, \dots, T_{n_H+1}$ in proportions $w_1, w_2, \dots, w_{n_H+1}$ with $w_1 + w_2 + \dots + w_{n_H+1} = 1$. Denote by $P_h(t) = P^d(t, T_h)$ ($h = 1, 2, \dots, (n_H + 1)$) the value of these $n_H + 1$ defaultable zero-coupon bonds. For simplicity of notation we write the stochastic differential equation for P_h in the general form

$$\frac{dP_h(t)}{P_h(t)} = \mu_{P_h}(t) dt + \sum_{i=1}^{n_w} \nu_{P_h,i}(t) dW_i(t) + \sum_{i=1}^{n_p} \chi_{P_h,i}(t) dQ_i(t),$$

¹²This is the case here given that the jumps are modelled by Poisson processes, so the probability of more than two jumps over Δs is $o(\Delta s)$.

where

$$\begin{aligned}\mu_{P_h}(t) &= r^d(t) + H(t, T_h, \bar{f}^d(t)) \\ \nu_{P_{h,i}}(t) &= -\zeta_i^d(t, T_h, \bar{f}^d(t)), \\ \chi_{P_{h,i}}(t) &= -\left[(1 - q_i(t)) \left(1 - e^{-\xi_i^d(t, T_h)}\right) + q_i(t)\right].\end{aligned}$$

Let V be the value of the hedging portfolio. Then the return on the portfolio is given by

$$\begin{aligned}\frac{dV}{V} &= w_1 \frac{dP_1}{P_1} + w_2 \frac{dP_2}{P_2} + \cdots + w_{n_w+n_p+1} \frac{dP_{n_w+n_p+1}}{P_{n_w+n_p+1}} \\ &= \sum_{h=1}^{n_w+n_p+1} w_h \mu_{P_h} dt + \sum_{h=1}^{n_w+n_p+1} w_h \sum_{i=1}^{n_w} \nu_{P_{h,i}} dW_i(t) + \sum_{h=1}^{n_w+n_p+1} w_h \sum_{i=1}^{n_p} \chi_{P_{h,i}} dQ_i(t).\end{aligned}$$

In order to eliminate both Gaussian and Poisson risks we need to choose $w_1, w_2, \dots, w_{n_H+1}$ so that

$$\sum_{h=1}^{n_H+1} w_h \nu_{P_{h,i}} = 0, \quad \text{for } i = 1, 2, \dots, n_w \quad (\text{A4.2.1})$$

$$\sum_{h=1}^{n_H+1} w_h \chi_{P_{h,i}} = 0, \quad \text{for } i = 1, 2, \dots, n_p. \quad (\text{A4.2.2})$$

The hedging portfolio then becomes riskless. Thus, it should earn the risk-free (and default free) rate of interest $r(t)$, of the Gaussian bond market, i.e.,

$$\frac{dV}{V} = \sum_{h=1}^{n_H+1} w_h \mu_{P_h} dt = r(t) dt,$$

which can be simplified to

$$\sum_{h=1}^{n_H+1} w_h (\mu_{P_h} - r(t)) = 0, \quad (\text{A4.2.3})$$

using also the fact that $w_1 + w_2 + \cdots + w_{n_H+1} = 1$. Equations (A4.2.1), (A4.2.2) and (A4.2.3) form a system of $n_H + 1$ equations with $n_H + 1$ unknowns $w_1, w_2, \dots, w_{n_H+1}$.

This system can only have a non-zero solution if the determinant

$$\begin{vmatrix} \nu_{P_{1,1}}(t) & \nu_{P_{2,1}}(t) & \cdots & \nu_{P_{n_H+1,1}}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \nu_{P_{1,n_w}}(t) & \nu_{P_{2,n_w}}(t) & \cdots & \nu_{P_{n_H+1,n_w}}(t) \\ \chi_{P_{1,1}}(t) & \chi_{P_{2,1}}(t) & \cdots & \chi_{P_{n_H+1,1}}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{P_{1,n_p}}(t) & \chi_{P_{2,n_p}}(t) & \cdots & \chi_{P_{n_H+1,n_p}}(t) \\ \mu_{P_1} - r(t) & \mu_{P_2} - r(t) & \cdots & \mu_{P_{n_H+1}} - r(t) \end{vmatrix}$$

is equal to zero. This implies that for $h = 1, 2, \dots, (n_H + 1)$, there exist maturity independent functions $\phi_1^d(t), \phi_2^d(t), \dots, \phi_{n_w}^d(t)$ and $\psi_1^d(t), \psi_2^d(t), \dots, \psi_{n_p}^d(t)$, such that

$$\mu_{P_h} - r(t) = - \sum_{i=1}^{n_w} \phi_i^d(t) \nu_{P_{h,i}}(t) - \sum_{i=1}^{n_p} \psi_i^d(t) \chi_{P_{h,i}}(t).$$

Thus, for bonds of any maturity T , we must have that

$$\mu_P - r(t) = - \sum_{i=1}^{n_w} \phi_i^d(t) \nu_{P_i}(t) - \sum_{i=1}^{n_p} \psi_i^d(t) \chi_{P_i}(t). \quad (\text{A4.2.4})$$

By recalling that $\mu_P(t) = r^d(t) + H(t, T, \bar{f}^d(t))$ and substituting the expressions for $\nu_{P_i}(t)$, with $i = 1, \dots, n_w$, and $\chi_{P_i}(t)$, with $i = 1, \dots, n_p$, we obtain

$$r^d(t) + H^d(t, T, \bar{f}^d(t)) - r(t) = \sum_{i=1}^{n_w} \phi_i^d(t) \zeta_i^d(t, T) + \sum_{i=1}^{n_p} \psi_i^d(t) \left[(1 - q_i(t)) \left(1 - e^{-\xi_i^d(t, T_h)} \right) + q_i(t) \right]. \quad (\text{A4.2.5})$$

Substituting expression (4.2.7) for $H^d(t, T, \bar{f}^d(t))$ into (A4.2.5), the short interest rate spread is given by

$$\begin{aligned} r^d(t) - r(t) &= \int_t^T \alpha^d(t, u) du - \sum_{i=1}^{n_w} \frac{1}{2} (\zeta_i^d)^2(t, T, \bar{f}^d(t)) - \sum_{i=1}^{n_p} \lambda_i \xi_i^d(t, T) \\ &+ \sum_{i=1}^{n_w} \phi_i^d(t) \zeta_i^d(t, T, \bar{f}^d(t)) + \sum_{i=1}^{n_p} \psi_i^d(t) \left[(1 - q_i(t)) \left(1 - e^{-\xi_i^d(t, T_h)} \right) + q_i(t) \right]. \end{aligned} \quad (\text{A4.2.6})$$

By substitution of the forward rate drift restriction (4.2.16) into equation (A4.2.6),

$$\begin{aligned}
r^d(t) - r(t) = & - \sum_{i=1}^{n_w} \phi_i^d(t) \int_t^T \sigma_i^d(t, u, \bar{f}^d(t)) du + \sum_{i=1}^{n_w} \int_t^T \sigma_i^d(t, u, \bar{f}^d(t)) \zeta_i^d(t, u, \bar{f}^d(t)) du \\
& + \sum_{i=1}^{n_p} \lambda_i \int_t^T \beta_i^d(t, u) du - \sum_{i=1}^{n_p} \psi_i^d(t) [1 - q_i(t)] \int_t^T \beta_i^d(t, u) e^{-\xi_i^d(t, u)} du \\
& - \sum_{i=1}^{n_w} \frac{1}{2} \zeta_i^d(t, T, \bar{f}^d(t))^2 - \sum_{i=1}^{n_p} \lambda_i \xi_i^d(t, T) \\
& + \sum_{i=1}^n \phi_i^d(t) \zeta_i^d(t, T, \bar{f}^d(t)) + \sum_{i=1}^{n_p} \psi_i^d(t) \left[(1 - q_i(t)) \left(1 - e^{-\xi_i^d(t, T)} \right) + q_i(t) \right],
\end{aligned} \tag{A4.2.7}$$

or, after some manipulations, the short rate spread is the sum of the products between the intensity of a default and the corresponding expect loss quota, i.e.

$$r^d(t) - r(t) = \sum_{i=1}^{n_p} \psi_i^d(t) q_i(t). \tag{A4.2.8}$$

Appendix 4.3. Derivation of the Defaultable Bond Price Formula

We derive the bond price formula using the Inui & Kijima (1998) approach, which consists of a direct substitution of the risk neutral forward rate dynamics and the volatility specifications (4.3.1) and (4.3.2) into the fundamental relationship between bond prices and forward rates.

By substituting the forward rate drift restriction under the risk neutral measure,

$$\begin{aligned}
\alpha^d(t, T) = & \sum_{i=1}^{n_w} \sigma_i^d(t, T, \bar{f}^d(t)) (-\phi_i^d(t) + \int_t^T \sigma_i^d(t, u, \bar{f}^d(t)) du) \\
& - \sum_{i=1}^{n_p} \beta_i^d(t, T) \left(\psi_i^d(t) [1 - q_i(t)] \exp \left[- \int_t^T \beta_i^d(t, u) du \right] - \lambda_i \right),
\end{aligned} \tag{A4.3.1}$$

into the stochastic differential equation (4.2.1), the forward rate dynamics under the risk neutral measure become

$$\begin{aligned} f^d(t, T) = & f^d(0, T) + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, T, \bar{f}^d(s)) \zeta_i^d(s, T, \bar{f}^d(s)) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, T, \bar{f}^d(s)) d\widetilde{W}_i(s) \\ & - \sum_{i=1}^{n_p} \int_0^t \psi_i^d(s) \beta_i^d(s, T) [1 - q_i(t)] e^{-\xi_i^d(s, T)} - 1 ds + \sum_{i=1}^{n_p} \int_0^t \beta_i^d(s, T) [dQ_i(s) - \psi_i^d(s) ds]. \end{aligned} \quad (\text{A4.3.2})$$

Using the fundamental relationship $\hat{P}(t, T) = \exp\left(-\int_t^T f^d(t, y) dy\right)$, we may write the price of the ‘pseudo’ bond as¹³

$$\begin{aligned} \hat{P}(t, T) = & \exp\left(-\int_t^T f^d(0, y) dy - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) \zeta_i^d(s, y, \bar{f}^d(s)) dy ds - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) dy d\widetilde{W}_i(s) \right. \\ & \left. - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [1 - [1 - q_i(t)] e^{-\xi_i^d(s, y)}] dy ds - \sum_{i=1}^{n_p} \int_0^t \int_t^T \beta_i^d(s, y) dy [dQ_i(s) - \psi_i^d(s) ds]\right). \end{aligned} \quad (\text{A4.3.3})$$

Consider further the volatility specifications of Assumption 4.3.1, then¹⁴

$$\int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) dy = \sigma_i(s, t, \bar{f}^d(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy = \sigma_i(s, t, \bar{f}^d(s)) \mathcal{N}_{\sigma_i}(t, T), \quad (\text{A4.3.4})$$

and similarly

$$\int_t^T \beta_i^d(s, y) dy = \beta_i(s, t) \int_t^T e^{-\int_t^y \kappa_{\beta_i}(u) du} dy = \beta_i(s, t) \mathcal{N}_{\beta_i}(t, T). \quad (\text{A4.3.5})$$

Therefore, by integrating from 0 to t and for $i = 1, \dots, n_w$

$$\int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) dy d\widetilde{W}_i(s) = \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s), \quad (\text{A4.3.6})$$

¹³We assume that the conditions for application of stochastic Fubini theorem are satisfied.

¹⁴Note that

$\sigma_i(s, y, \bar{f}^d(s)) = \sigma_{0i}(s, \bar{f}^d(s)) e^{-\int_s^y \kappa_{\sigma_i}(u) du} = \sigma_{0i}(s, \bar{f}^d(s)) e^{-\int_s^t \kappa_{\sigma_i}(u) du} e^{-\int_t^y \kappa_{\sigma_i}(u) du} = \sigma_i(s, t, \bar{f}^d(s)) e^{-\int_t^y \kappa_{\sigma_i}(u) du}$.

and for $i = 1, \dots, n_p$

$$\int_0^t \int_t^T \beta_i^d(s, y) dy [dQ_i(s) - \psi_i^d(s) ds] = \mathcal{N}_{\beta_i}(t, T) \int_0^t \beta_i^d(s, t) [dQ_i(s) - \psi_i^d(s) ds]. \quad (\text{A4.3.7})$$

Similarly, for $i = 1, \dots, n_w$, we manipulate the term

$$\begin{aligned} & \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) \zeta_i^d(s, y, \bar{f}^d(s)) dy = \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) \int_s^y \sigma_i^d(s, v, \bar{f}^d(s)) dv dy \\ & = \sigma_i^d(s, t, \bar{f}^d(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} \left[\int_s^t \sigma_i^d(s, v, \bar{f}^d(s)) dv + \int_t^y \sigma_i^d(s, v, \bar{f}^d(s)) dv \right] dy \\ & = \sigma_i^d(s, t, \bar{f}^d(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy \int_s^t \sigma_i^d(s, v, \bar{f}^d(s)) dv \\ & + \sigma_i^{d2}(s, t, \bar{f}^d(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} \int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv dy \\ & = \sigma_i^d(s, t) \mathcal{N}_{\sigma_i}(t, T) \zeta_i^d(s, t, \bar{f}^d(s)) + \frac{1}{2} \sigma_i^{d2}(s, t, \bar{f}^d(s)) \mathcal{N}_{\sigma_i}^2(t, T), \end{aligned} \quad (\text{A4.3.8})$$

since

$$\begin{aligned} & \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} \int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv dy = \int_t^T \frac{d}{dy} \left(\int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv \right) \int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv dy \\ & = \int_t^T d \left(\frac{1}{2} \left[\int_t^y e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv \right]^2 \right) \\ & = \frac{1}{2} \left[\int_t^T e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv \right]^2 - \frac{1}{2} \left[\int_t^t e^{-\int_t^v \kappa_{\sigma_i}(u) du} dv \right]^2 \\ & = \frac{1}{2} \left(\int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy \right)^2 = \frac{1}{2} \mathcal{N}_{\sigma_i}^2(t, T). \end{aligned} \quad (\text{A4.3.9})$$

Therefore integrating equation (A4.3.8) from 0 to t and for $i = 1, \dots, n_w$, we obtain

$$\begin{aligned} & \int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) \zeta_i^d(s, y, \bar{f}^d(s)) dy ds = \\ & \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds + \frac{1}{2} \mathcal{N}_{\sigma_i}^2(t, T) \int_0^t \sigma_i^{d2}(s, t, \bar{f}^d(s)) ds. \end{aligned} \quad (\text{A4.3.10})$$

Substitute the results (A4.3.6), (A4.6.4) and (A4.3.10) into equation (A4.3.3), collect like terms, and the bond price formula will simplify to

$$\begin{aligned}
\hat{P}(t, T) = \exp & \left(- \int_t^T f^d(0, y) dy - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(t)) \zeta_i^d(s, t, \bar{f}^d(t)) ds \right. \\
& - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \int_0^t \sigma_i^{d2}(s, t, \bar{f}^d(s)) ds - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) \\
& - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \int_0^t \beta_i^d(s, t) [dQ_i(s) - \psi_i^d(s) ds] \\
& \left. + \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [[1 - q_i(s)] e^{-\xi_i^d(s, y)} - 1] dy ds \right).
\end{aligned} \tag{A4.3.11}$$

By using the definitions (4.3.8), (4.3.10) and (4.3.11), equation (A4.3.11) simplifies further to

$$\begin{aligned}
\hat{P}(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp & \left(- \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) \right. \\
& - \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \mathcal{D}_{\beta_i}^d(t) - \int_0^t \psi_i^d(s) \beta_i^d(s, t) [[1 - q_i(s)] e^{-\xi_i^d(s, t)} - 1] ds \right\} \\
& \left. + \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [[1 - q_i(s)] e^{-\xi_i^d(s, y)} - 1] dy ds \right).
\end{aligned} \tag{A4.3.12}$$

Recall that the defaultable bond price is evaluated as

$$P^d(t, T) = \hat{P}(t, T) \bar{Q}(t) \tag{A4.3.13}$$

$$:= \hat{P}(t, T) \prod_{i=1}^{n_p} \prod_{k=1}^{\eta_i(t)} (1 - q_i(\tau_{ik})), \tag{A4.3.14}$$

and by re-expressing $\prod_{i=1}^{n_p} \prod_{k=1}^{\eta_i(t)} (1 - q_i(\tau_{ik}))$ as

$$\exp \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})),$$

then

$$P^d(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp \left\{ \mathcal{M}^d(t, T) - \frac{1}{2} \sum_{i=1}^n \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^n \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) - \sum_{i=1}^n \sum_{j=1}^{m_i} \mathcal{N}_{\beta_{ij}}(t, T) \mathcal{D}_{\beta_{ij}}(t) \right\}, \quad (\text{A4.3.15})$$

where

$$\begin{aligned} \mathcal{M}^d(t, T) &= \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})) \\ &\quad - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [1 - [1 - q_i(s)] e^{-\xi_i^d(s, y)}] dy ds \quad (\text{A4.3.16}) \\ &\quad + \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - e^{-\xi_i^d(s, t)}] ds \right\}. \end{aligned}$$

Appendix 4.4. Finite Affine Realisations in Terms of Forward Rates

Equation (4.4.3) yields a system of n_d equations, which may be written in matrix form as

$$\begin{bmatrix} f^d(t, T_1) + \frac{\partial \mathcal{M}^d(t, T_1)}{\partial T_1} \\ f^d(t, T_2) + \frac{\partial \mathcal{M}^d(t, T_2)}{\partial T_2} \\ \vdots \\ f^d(t, T_{n_d}) + \frac{\partial \mathcal{M}^d(t, T_{n_d})}{\partial T_{n_d}} \end{bmatrix} = \mathbb{O}^d(t) \times \begin{bmatrix} \mathcal{E}_{\sigma_1}^d(t) \\ \vdots \\ \mathcal{E}_{\sigma_{n_w}}^d(t) \\ \mathcal{D}_{\sigma_1}^d(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}^d(t) \\ \mathcal{D}_{\beta_1}^d(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}^d(t) \end{bmatrix}. \quad (\text{A4.4.1})$$

Given that the elements of matrix $\mathbb{O}^d(t)$ are deterministic functions and assuming that the determinant of this matrix is non-zero then we can invert the matrix and express the state

variables $\mathcal{E}_{\sigma_i}^d(t)$, $\mathcal{D}_{\sigma_i}^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ in terms of forward rates of n_d distinct maturities, i.e.,

$$\begin{bmatrix} \mathcal{E}_{\sigma_1}^d(t) \\ \vdots \\ \mathcal{E}_{\sigma_{n_w}}^d(t) \\ \mathcal{D}_{\sigma_1}^d(t) \\ \vdots \\ \mathcal{D}_{\sigma_{n_w}}^d(t) \\ \mathcal{D}_{\beta_1}^d(t) \\ \vdots \\ \mathcal{D}_{\beta_{n_p}}^d(t) \end{bmatrix} = (\mathbb{O}^d)^{-1}(t) \times \begin{bmatrix} f^d(t, T_1) + \frac{\partial \hat{\mathcal{M}}^d(t, T_1)}{\partial T_1} \\ f^d(t, T_2) + \frac{\partial \hat{\mathcal{M}}^d(t, T_2)}{\partial T_2} \\ \vdots \\ f^d(t, T_{n_d}) + \frac{\partial \hat{\mathcal{M}}^d(t, T_{n_d})}{\partial T_{n_d}} \end{bmatrix}. \quad (\text{A4.4.2})$$

Then the state variables can be expressed as functions of the n_d benchmark forward rates $f^d(t, T_h)$, in the form

$$\mathcal{E}_{\sigma_i}^d(t) = \sum_{h=1}^{n_d} \varpi_{ih} \left[f^d(t, T_h) + \frac{\partial \hat{\mathcal{M}}^d(t, T_h)}{\partial T_h} \right], \quad (\text{A4.4.3})$$

$$\mathcal{D}_{\sigma_i}^d(t) = \sum_{h=1}^{n_d} \varpi_{lh} \left[f^d(t, T_h) + \frac{\partial \hat{\mathcal{M}}^d(t, T_h)}{\partial T_h} \right], \quad \text{with } l = n_w + i, \quad (\text{A4.4.4})$$

and

$$\mathcal{D}_{\beta_i}^d(t) = \sum_{h=1}^{n_d} \varpi_{kh} \left[f^d(t, T_h) + \frac{\partial \hat{\mathcal{M}}^d(t, T_h)}{\partial T_h} \right], \quad \text{with } k = 2n_w + i. \quad (\text{A4.4.5})$$

Further, we substitute expressions (A4.4.3), (A4.7.2) and (A4.7.3) for the state variables into the forward rate formula (4.4.3), to obtain (4.4.4), which expresses the forward rate of any maturity in terms of the n_d forward rates.

Appendix 4.5. Derivation of the Defaultable Spot Rate Dynamics under Stochastic Intensity

Recall that the Poisson volatilities are constant, whereas the Wiener volatilities satisfy (4.3.3). Taking the differential of (4.2.22), the defaultable instantaneous spot rate satisfies

the stochastic differential equation

$$dr^d(t) = \left[\frac{\partial f^d(0, t)}{\partial t} + \sum_{i=1}^{n_w} \frac{\partial}{\partial t} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds \right. \quad (\text{A4.5.1})$$

$$+ \sum_{i=1}^{n_p} \frac{\partial}{\partial t} \left(\int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - (1 - q_i(s)) e^{-\xi_i^d(s, t)}] ds \right. \\ \left. - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) \right] dt \quad (\text{A4.5.2})$$

$$+ \sum_{i=1}^{n_w} \sigma_{0i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}^d(t) [dQ_i(t) - \psi_i^d(t) dt],$$

Using the results of Appendix 2.3, as well as

$$\int_0^t \frac{\partial}{\partial t} (\psi^d(s) \beta_0 [1 - (1 - q(s)) e^{-\beta_0(t-s)}]) ds = \beta_0^2 \int_0^t \psi^d(s) (1 - q(s)) e^{-\beta_0(t-s)} ds, \quad (\text{A4.5.3})$$

to express the defaultable spot rate dynamics as

$$dr^d(t) = \left[\frac{\partial f^d(0, t)}{\partial t} + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, t, \bar{f}^d(s))^2 ds - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) \zeta_i^d(s, t, \bar{f}^d(s)) ds \right. \\ \left. + \sum_{i=1}^{n_p} \beta_i^{d^2} \int_0^t \psi_i^d(s) (1 - q_i(s)) e^{-\xi_i^d(s, t)} ds - \sum_{i=1}^{n_w} \kappa_{\sigma_i}(t) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) \right] dt \\ + \sum_{i=1}^{n_w} \sigma_{0i}^d(t, \bar{f}^d(t)) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}^d(t) [dQ_i(t) - \psi_i^d(t) dt]. \quad (\text{A4.5.4})$$

Substituting expressions (4.3.8), (4.3.9) and (4.3.10) into equation (A4.5.4), we obtain dynamics (4.5.6).

Appendix 4.6. Derivation of the Defaultable Bond Price Formula under Stochastic Intensity

Again using the Inui & Kijima (1998) approach, under the volatility specifications (4.3.1) and (4.5.4) and the stochastic intensities $\psi_i^d(t)$, the stochastic integral equation for the

forward rate dynamics under the risk neutral measure becomes

$$\begin{aligned}
f^d(t, T) &= f^d(0, T) + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, T, \bar{f}^d(s)) \zeta_i^d(s, T, \bar{f}^d(s)) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i^d(s, T, \bar{f}^d(s)) d\widetilde{W}_i(s) \\
&\quad - \sum_{i=1}^{n_p} \int_0^t \psi_i^d(s) \beta_{0i}^d [[1 - q_i(t)] e^{-\beta_{0i}^d(T-s)} - 1] ds + \sum_{i=1}^{n_p} \int_0^t \beta_{0i}^d [dQ_i(s) - \psi_i^d(s) ds].
\end{aligned} \tag{A4.6.1}$$

Using the fundamental relationship $\hat{P}(t, T) = \exp\left(-\int_t^T f^d(t, y) dy\right)$, we may write the price of the ‘pseudo’ bond as

$$\begin{aligned}
\hat{P}(t, T) &= \exp\left(-\int_t^T f^d(0, y) dy - \sum_{i=1}^{n_w} \int_t^T \int_0^t \sigma_i^d(s, y, \bar{f}^d(s)) \zeta_i^d(s, y, \bar{f}^d(s)) ds dy \right. \\
&\quad \left. - \sum_{i=1}^{n_w} \int_t^T \int_0^t \sigma_i^d(s, y, \bar{f}^d(s)) d\widetilde{W}_i(s) dy \right. \\
&\quad \left. - \sum_{i=1}^{n_p} \int_t^T \int_0^t \psi_i^d(s) \beta_{0i}^d [1 - [1 - q_i(t)] e^{-\beta_{0i}^d(y-s)}] ds dy - \sum_{i=1}^{n_p} \int_t^T \int_0^t \beta_{0i}^d [dQ_i(s) - \psi_i^d(s) ds] dy \right) \\
&= \exp\left(-\int_t^T f^d(0, y) dy - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) \zeta_i^d(s, y, \bar{f}^d(s)) dy ds - \sum_{i=1}^{n_w} \int_0^t \int_t^T \sigma_i^d(s, y, \bar{f}^d(s)) dy d\widetilde{W}_i(s) \right. \\
&\quad \left. - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_{0i}^d [1 - [1 - q_i(s)] e^{-\beta_{0i}^d(y-s)}] dy ds - \sum_{i=1}^{n_p} \int_0^t \int_t^T \beta_{0i}^d dy [dQ_i(s) - \psi_i^d(s) ds] \right).
\end{aligned} \tag{A4.6.2}$$

Under the constant volatility specifications (4.5.4) we have

$$\int_t^T \beta_{0i}^d dy = \beta_{0i}^d (T - t). \tag{A4.6.3}$$

Therefore, by integrating from 0 to t and for $i = 1, \dots, n_p$

$$\int_0^t \int_t^T \beta_{0i}^d dy [dQ_i(s) - \psi_i^d(s) ds] = \beta_{0i}^d (T - t) \int_0^t [dQ_i(s) - \psi_i^d(s) ds]. \tag{A4.6.4}$$

Following similar manipulations as in Appendix 4.3, the results (A4.3.6) and (A4.3.10) still hold.

For $i = 1, \dots, n_p$, we obtain

$$\begin{aligned}
&\int_0^t \int_t^T \psi_i^d(s) \beta_{0i}^d [1 - [1 - q_i(s)] e^{-\beta_{0i}^d(y-s)}] dy ds \\
&= \beta_{0i}^d (T - t) \int_0^t \psi_i^d(s) ds + \int_0^t \psi_i^d(s) [1 - q_i(s)] (e^{-\beta_{0i}^d(T-s)} - e^{-\beta_{0i}^d(t-s)}) ds.
\end{aligned} \tag{A4.6.5}$$

Substitute the results (A4.3.6), (A4.6.4) and (A4.6.5) into equation (A4.6.2), collect like terms, and the bond price formula will simplify to

$$\begin{aligned}
\hat{P}(t, T) = \exp & \left(- \int_t^T f^d(0, y) dy - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(t)) \zeta_i^d(s, t, \bar{f}^d(t)) ds \right. \\
& - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \int_0^t \sigma_i^{d^2}(s, t, \bar{f}^d(s)) ds - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \int_0^t \sigma_i^d(s, t, \bar{f}^d(s)) d\widetilde{W}_i(s) \\
& - \sum_{i=1}^{n_p} \beta_{0i}^d(T-t) \int_0^t [dQ_i(s) - \psi_i^d(s) ds] \\
& \left. + \sum_{i=1}^{n_p} \left(\beta_{0i}^d(T-t) \int_0^t \psi_i^d(s) ds + \int_0^t \psi_i^d(s) [1 - q_i(s)] (e^{-\beta_{0i}^d(T-s)} - e^{-\beta_{0i}^d(t-s)}) ds \right) \right).
\end{aligned} \tag{A4.6.6}$$

By using the definitions (4.3.8), (4.3.10) and (4.3.11), equation (A4.6.6) simplifies further to

$$\begin{aligned}
\hat{P}(t, T) = \frac{P^d(0, T)}{P^d(0, t)} \exp & \left(- \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) \right. \\
& \left. - \sum_{i=1}^{n_p} (T-t) \left\{ \mathcal{D}_{\beta_i}^d(t) + \frac{1}{\beta_{0i}^d} \mathcal{E}_{\beta_i}^d(t) - \frac{1}{\beta_{0i}^{d^2}} \sum_{i=1}^{n_p} \left(e^{-\beta_{0i}^d(T-t)} - 1 \right) \mathcal{E}_{\beta_i}^d(t) \right\} \right).
\end{aligned} \tag{A4.6.7}$$

Thus the defaultable bond price (recall equation (4.2.8)) is evaluated as

$$\begin{aligned}
P^d(t, T) = \exp & \left\{ \check{\mathcal{M}}^d(t, T) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T) \mathcal{E}_{\sigma_i}^d(t) - \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}(t, T) \mathcal{D}_{\sigma_i}^d(t) \right. \\
& \left. - (T-t) \sum_{i=1}^{n_p} \mathcal{D}_{\beta_i}^d(t) - \sum_{i=1}^{n_p} \mathcal{C}_{\beta_i}^d(t, T) \mathcal{E}_{\beta_i}^d(t) \right\},
\end{aligned} \tag{A4.6.8}$$

where,

$$\check{\mathcal{M}}^d(t, T) = \sum_{i=1}^{n_p} \sum_{k=1}^{\eta_i(t)} \ln(1 - q_i(\tau_{ik})) - \int_t^T f(0, y) dy, \tag{A4.6.9}$$

and

$$\mathcal{C}_{\beta_i}^d(t, T) = \frac{(T-t)\beta_{0i}^d + e^{-\beta_{0i}^d(T-t)} - 1}{\beta_{0i}^{d^2}}. \tag{A4.6.10}$$

Appendix 4.7. Simulation Details

We discretize the time interval $[0, 1]$ into N equal subintervals of length $\Delta t = 1/N$. Thus we set $t_\ell = \ell\Delta t$, for $\ell = 1, 2, \dots, N$.

4.7.1. Deterministic Intensity. The spot rate dynamics are (recall relationship (4.3.5))

$$r^d(t_\ell) = f^d(0, t_\ell) + \mathcal{D}_\sigma^d(t_\ell) + \sum_{i=1}^2 \mathcal{D}_{\beta_i}^d(t_\ell). \quad (\text{A4.7.1})$$

The stochastic variable $\mathcal{D}_\sigma^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$ evaluated in terms of the benchmark defaultable forward rates, by using the system (A4.4.2), thus can be expressed as functions of the $n_s = 4$ benchmark forward rates $f(t, T_h)$ ($h = 1, 2, 3, 4$) as

$$\mathcal{D}_\sigma^d(t_\ell) = \sum_{h=1}^4 \varpi_{2h} \left[f^d(t_\ell, T_h) + \frac{\partial \hat{\mathcal{M}}^d(t_\ell, T_h)}{\partial T_h} \right], \quad (\text{A4.7.2})$$

and

$$\mathcal{D}_{\beta_i}^d(t_\ell) = \sum_{h=1}^3 \varpi_{kh} \left[f^d(t_\ell, T_h) + \frac{\partial \hat{\mathcal{M}}^d(t_\ell, T_h)}{\partial T_h} \right], \quad (\text{A4.7.3})$$

with $k = i + 2$. For the elements ϖ_{xh} we need the matrix $\mathbb{O}^d(t)$ which is evaluated as

$$\begin{bmatrix} \frac{1}{\kappa}(1 - e^{-\kappa(T_1-t)})e^{-\kappa(T_1-t)} & e^{-\kappa(T_1-t)} & e^{-\kappa_1(T_1-t)} & e^{-\kappa_2(T_1-t)} \\ \frac{1}{\kappa}(1 - e^{-\kappa(T_2-t)})e^{-\kappa(T_2-t)} & e^{-\kappa(T_2-t)} & e^{-\kappa_1(T_2-t)} & e^{-\kappa_2(T_2-t)} \\ \frac{1}{\kappa}(1 - e^{-\kappa(T_3-t)})e^{-\kappa(T_3-t)} & e^{-\kappa(T_3-t)} & e^{-\kappa_1(T_3-t)} & e^{-\kappa_2(T_3-t)} \\ \frac{1}{\kappa}(1 - e^{-\kappa(T_4-t)})e^{-\kappa(T_4-t)} & e^{-\kappa(T_4-t)} & e^{-\kappa_1(T_4-t)} & e^{-\kappa_2(T_4-t)} \end{bmatrix},$$

with inverse

$$(\mathbb{O}^d)^{-1}(t) = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} \\ \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} \end{bmatrix}.$$

In addition

$$\begin{aligned} \hat{\mathcal{M}}^d(t, T) &= - \int_t^T f^d(0, y) dy - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i^d(s) \beta_i^d(s, y) [1 - [1 - q_i(s)] e^{-\xi_i^d(s, y)}] dy ds \\ &+ \sum_{i=1}^{n_p} \mathcal{N}_{\beta_i}(t, T) \left\{ \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - e^{-\xi_i^d(s, t)}] ds \right\}. \end{aligned}$$

thus

$$\begin{aligned} \frac{\partial \hat{\mathcal{M}}^d(t, T)}{\partial T} &= -f^d(0, T) - \sum_{i=1}^2 \psi_i^d \int_0^t \beta_i^d(s, T) [1 - [1 - q_i(s)] e^{-\xi_i^d(s, T)}] ds \\ &\quad + \sum_{i=1}^2 \frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} \left\{ \int_0^t \psi_i^d(s) \beta_i^d(s, t) [1 - e^{-\xi_i^d(s, t)}] ds \right\}, \end{aligned}$$

or

$$\begin{aligned} \frac{\partial \hat{\mathcal{M}}^d(t, T)}{\partial T} &= -f^d(0, T) - \sum_{i=1}^2 \psi_i^d (1 - e^{-\frac{\beta_{0i}^d}{\kappa_i} (1 - e^{-\kappa_i t})}) e^{-\kappa_i (T-t)} \\ &\quad - \sum_{i=1}^2 \frac{\psi_i^d \beta_{0i}^d}{\kappa_i} (e^{-\kappa_i (T-t)} - e^{-\kappa_i T}) + \sum_{i=1}^2 \psi_i^d (1 - q_i) \left[e^{\frac{\beta_{0i}^d}{\kappa_i} (e^{-\kappa_i (T-t)} - 1)} - e^{\frac{\beta_{0i}^d}{\kappa_i} (e^{-\kappa_i T} - 1)} \right]. \end{aligned}$$

From equation (4.4.9) and for $h = 1, 2, 3, 4$, the discretised version of the benchmark defaultable forward dynamics is

$$\begin{aligned} f^d(t_{\ell+1}, T_h) - f^d(t_\ell, T_h) &= \frac{\sigma_0^d(t, \bar{f}(t))^2}{k_\sigma} e^{-\kappa_\sigma (T_h - t_\ell)} (1 - e^{-\kappa_\sigma (T_h - t_\ell)}) \Delta t \\ &\quad - \sum_{i=1}^2 \psi_i^d \beta_{0i}^d e^{-k_{\beta_i} (T_h - t_\ell)} [[1 - q_i] e^{\frac{\beta_{0i}^d}{k_{\beta_i}} (e^{-k_{\beta_i} (T_h - t_\ell)} - 1)} - 1] \Delta t - \sum_{i=1}^2 \psi_i^d \beta_{0i}^d e^{-k_{\beta_i} (T_h - t_\ell)} \Delta t \\ &\quad + \sigma_0^d(t, \bar{f}(t)) e^{-\kappa_\sigma (T_h - t_\ell)} \Delta \widetilde{W}(t_\ell) + \sum_{i=1}^2 \beta_{0i}^d e^{-k_{\beta_i} (T_h - t_\ell)} \Delta Q_i(t_\ell). \end{aligned} \quad (\text{A4.7.4})$$

Note that the minimum set of the state variables considered in this example is the set of the 4 benchmark forward rates, by using the results of Section 4.4. Therefore, discretisation has only been performed on the dynamics driving these benchmark forward rates, see equations (A4.7.4), while all the other stochastic variables included in the Markovian system such as $r^d(t)$, $\mathcal{D}_\sigma^d(t)$ and $\mathcal{D}_{\beta_i}^d(t)$, have been expressed in terms of this set of state variables.

4.7.2. Stochastic Intensity. The spot rate dynamics are

$$r^d(t_\ell) = f^d(0, t_\ell) + \mathcal{D}_\sigma^d(t_\ell) + \sum_{i=1}^2 \mathcal{D}_{\beta_i}^d(t_\ell), \quad (\text{A4.7.5})$$

where

$$\mathcal{D}_\sigma^d(t_{\ell+1}) - \mathcal{D}_\sigma^d(t_\ell) = [\mathcal{E}_\sigma^d(t_\ell) - \kappa_\sigma(t_\ell)\mathcal{D}_\sigma^d(t_\ell)]\Delta t + \sigma_0^d(t, \bar{f}^d(t))\widetilde{W}(\Delta t), \quad (\text{A4.7.6})$$

$$\mathcal{D}_{\beta_i}^d(t_{\ell+1}) - \mathcal{D}_{\beta_i}^d(t_\ell) = [-\beta_{0i}(1 - q_i)\psi_i^d(t_\ell) + \mathcal{E}_{\beta_i}^d(t_\ell)]\Delta t + \beta_{0i}Q(\Delta t), \quad (\text{A4.7.7})$$

with

$$\mathcal{E}_\sigma^d(t_{\ell+1}) - \mathcal{E}_\sigma^d(t_\ell) = [\sigma_0^d(t, \bar{f}^d(t))^2 - 2\kappa_\sigma(t_\ell)\mathcal{E}_\sigma(t_\ell)]\Delta t, \quad (\text{A4.7.8})$$

$$\mathcal{E}_{\beta_i}^d(t_{\ell+1}) - \mathcal{E}_{\beta_i}^d(t_\ell) = [\psi_i^d(t_\ell)(1 - q_i)\beta_{0i}^{d^2} - \beta_{0i}^d\mathcal{E}_{\beta_i}^d(t_\ell)]\Delta t, \quad (\text{A4.7.9})$$

and

$$\psi_i^d(t_{\ell+1}) - \psi_i^d(t_\ell) = \theta_i[\bar{\psi}_i - \psi_i^d(t_\ell)]\Delta t + \sqrt{\psi_i^d(t_\ell)\sigma_{\psi_i^d}}\widetilde{W}(\Delta t). \quad (\text{A4.7.10})$$

The quantities $f(t_\ell, T_h)$, for $h = 1, 2, \dots, 6$ included in (A4.7.6) and (A4.7.8) depend on the state variables and by using equation (4.5.13) may be expressed as

$$\begin{aligned} f^d(t_\ell, T_h) = & f^d(0, T_h) + \frac{\partial \mathcal{N}_\sigma(t_\ell, T_h)}{\partial T_h} \mathcal{N}_\sigma(t_\ell, T_h) \mathcal{E}_\sigma^d(t_\ell) + \frac{\partial \mathcal{N}_\sigma(t_\ell, T_h)}{\partial T_h} \mathcal{D}_\sigma^d(t_\ell) \\ & + \sum_{i=1}^2 \mathcal{D}_{\beta_i}^d(t_\ell) - \sum_{i=1}^2 \frac{\partial \mathcal{C}_{\beta_i}(t_\ell, T)}{\partial T} \mathcal{E}_{\beta_i}^d(t_\ell), \end{aligned} \quad (\text{A4.7.11})$$

where

$$\mathcal{N}_\sigma(t, T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa},$$

$$\frac{\partial \mathcal{N}_\sigma(t, T)}{\partial T} = e^{-\kappa(T-t)},$$

$$\frac{\partial \mathcal{C}_{\beta_i}(t, T)}{\partial T} = \frac{1 - e^{-\beta_{0i}^d(T-t)}}{\beta_{0i}^d}.$$

The minimum set of the state variables considered in the stochastic intensity example is the set of the 8 stochastic quantities, $\mathcal{E}_\sigma^d(t)$, $\mathcal{D}_\sigma^d(t)$, $\mathcal{E}_{\beta_i}^d(t)$, $\mathcal{D}_{\beta_i}^d(t)$ and $\psi_i^d(t)$ with $i = 1, 2$. Therefore, discretisation has been only performed on the dynamics driving these state variables, while the 6 benchmark forward rates used in the volatility structure have been expressed in terms of these set of state variables.

Note that the set of state variables considered for these simulations is not the set of benchmark forward rates. Of course, using the same procedure as in Section 4.4, we can express the set of the stochastic terms (excluding the stochastic intensities) in terms of a set of benchmark defaultable forward rates and use as state variable vector this set of benchmark forwards and the stochastic intensities. However, here we proceed with the stochastic quantities obtained originally and express the state dependent volatilities in terms of these stochastic terms.

CHAPTER 5

Option Pricing under Jump-Diffusions

This chapter examines the pricing of interest rate derivatives when the interest rate dynamics experience infrequent jump shocks modelled as a Poisson process and within the Markovian HJM framework developed in Chapter 2. Closed form solutions for the price of a bond option under deterministic volatility specifications are derived and a control variate numerical method is developed under a more general state dependent volatility structure, a case in which closed form solutions are not trivial.

5.1. Introduction

Interest rate derivatives are securities, the payoffs of which depend in some way on the level of interest rates. The value of an interest-rate option is substantially affected by the presence of skewness and kurtosis in the interest rates. The kurtosis explains the smile effect¹ and results in fat-tailed distributions. The skewness results in asymmetric interest rate distributions that match with the empirically observed distributional profile of the interest rates². Jump-diffusion and stochastic volatility models demonstrate an ability to accommodate these features, providing a modelling setting which explicitly incorporates tail risk to more accurately reflect reality. However, these classes of models come at the expense of an increasing complexity that makes it impossible in most cases to derive computationally tractable solutions for derivative prices.

Most of the interest rate models under jump-diffusions are not computationally tractable even when the jump sizes are constant or drawn from well known distributions such as normal and log-normal. Therefore, most of the studies in this area use numerical approximate methods to evaluate interest rate instruments, including those of Ahn & Thompson (1988), Ahn (1988), Mercurio & Runggaldier (1993), Naik & Lee (1995), Baz & Das

¹The shape of the implied volatilities (extracted from traded option prices by inverting the Black & Scholes (1973) or Black (1976) option pricing formula, whichever is applicable) for a range of different strikes is called the smile. The smile implies that at-the-money options trade at lower volatilities while the options away from the money trade at higher volatilities.

²Table 2.1.1 demonstrates the leptokurtic feature of the empirical distributions of the spot rate.

(1996) and Das (1999). Bates (1996), Duffie & Kan (1996) and Chacko & Das (2002) have considered more advanced models of stochastic volatility with jumps. Research work by Jamshidian (1989), Shirakawa (1991), Heston (1993) (stochastic volatility setting), Das & Foresi (1996) and Glasserman & Kou (2003) provides closed form evaluation formulas for bonds and bond options.

This chapter presents two classes of term structure models that incorporate jump behavior of interest rates and more general volatility specifications and also maintain tractability in the pricing of interest rate derivatives. More specifically, this chapter derives closed form solutions for bond options under deterministic volatility specifications and a numerical solution under the more general stochastic volatility case, which is, however, numerically tractable and efficient due to the fact that the term structure model developed admits finite dimensional Markovian representations. The Markovianisation of the jump-diffusion version of the HJM model employed here, even under state dependent volatility specifications, has been achieved by a suitable choice of volatility functions, as Chapter 2 explains.

For the deterministic volatility setup, we consider a parameterisation of the Shirakawa (1991) model of the term structure of interest rates under jump-diffusions. Under an appropriate equivalent probability measure, we consider option pricing within this framework. We use Fourier transform techniques to obtain a representation of the solution. A tractable Black-Scholes type pricing formula is derived under the assumption of a constant jump volatility function.

An extension of the Shirakawa (1991) framework is considered in the second part of this chapter, in which the volatility evolves stochastically, by means of a volatility dependency on the state variables of the system. Again under an appropriate equivalent probability measure, we study the pricing of bond options. In this case, however, closed form valuation formulas are not available. Taking the state dependent volatility specifications of the type discussed in Chapter 2, the interest rate dynamics become Markovian in a finite dimensional state variable and thus all the quantities involved such as forward rates or bond prices can be expressed in terms of this state variable. Taking advantage of these Markovian representations, we employ these particular Markovian structures to obtain approximate bond option prices by use of Monte Carlo method. We further improve the

efficiency of the Monte Carlo method by using a control variate technique, taking into account the closed form solutions obtained in the deterministic volatility setting.

This chapter is planned as follows. In Section 5.2 we develop the deterministic volatility model. In Section 5.3 we solve the bond option pricing equation using Fourier Transform techniques and we obtain closed form solutions for European bond options under constant jump volatility specifications. In Section 5.5 the state dependent volatility model is considered and the volatility restrictions that lead to Markovian term structures are discussed. Section 5.6 deals with the numerical implementation of the two models developed. We test the accuracy of the Monte Carlo results in the deterministic volatility model, since closed form solutions are available in this case. In addition, we numerically evaluate bond options under the stochastic volatility model. Finally by combining both models and closed form solutions, we develop a control variate method that significantly reduces computational effort and improves accuracy. Section 5.7 concludes and provides future directions for research.

5.2. The Deterministic Volatility Model

Within the Shirakawa (1991) framework and by using the same notation as in Section 2.2, we present, in this section, a deterministic volatility HJM model under jump-diffusions and we discuss the pricing of bond options.

Recall that the dynamics of the instantaneous forward rate $f(t, T)$ are driven by both Gaussian and Poisson risk terms and given by (see equation (2.2.2))

$$df(t, T) = \alpha(t, T)dt + \sum_{i=1}^{n_w} \sigma_i(t, T)dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t, T)[dQ_i(t) - \lambda_i dt]. \quad (5.2.1)$$

In stochastic integral equation form, equation (5.2.1) may be written as

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, T)dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, T)[dQ_i(s) - \lambda_i ds]. \quad (5.2.2)$$

Setting $T = t$ in equation (5.2.2), the stochastic integral equation for the spot rate is given by

$$r(t) \equiv f(t, t) = f(0, t) + \int_0^t \alpha(s, t) ds + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, t) [dQ_i(s) - \lambda_i ds], \quad (5.2.3)$$

and the corresponding stochastic differential equation is

$$dr(t) = \vartheta(t) dt + \sum_{i=1}^{n_w} \sigma_i(t, t) dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t, t) [dQ_i(t) - \lambda_i dt], \quad (5.2.4)$$

where $\vartheta(t)$ is defined as

$$\begin{aligned} \vartheta(t) = & \frac{\partial}{\partial t} f(0, t) + \alpha(t, t) + \int_0^t \frac{\partial}{\partial t} \alpha(s, t) ds \\ & + \sum_{i=1}^{n_w} \int_0^t \frac{\partial}{\partial t} \sigma_i(s, t) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \frac{\partial}{\partial t} \beta_i(s, t) [dQ_i(s) - \lambda_i ds]. \end{aligned} \quad (5.2.5)$$

The corresponding dynamics for the bond price are (see equation (2.2.8))

$$\begin{aligned} \frac{dP(t, T)}{P(t^-, T)} = & [r(t) + H(t, T)] dt - \sum_{i=1}^{n_w} \zeta_i(t, T) dW_i(t) \\ & + \sum_{i=1}^{n_p} (e^{-\xi_i(t, T)} - 1) dQ_i(t), \end{aligned} \quad (5.2.6)$$

where $\zeta_i(t, T)$, $\xi_i(t, T)$, and $H(t, T)$ are defined by the expressions (2.2.9), (2.2.10) and (2.2.11) respectively. Note, however, that since here we are studying the deterministic volatility case, we omit the level dependency $\bar{f}(t)$ observed in the expressions (2.2.9) and (2.2.11).

Consider a European call option of maturity T_C written on a bond having maturity T ($T > T_C$) and denote by $C = C(r, t, T_C)$ the value of this bond option at time t .

Taking into account that the dynamics of the spot rate are given by (5.2.4) and using the jump-diffusion version of Ito's lemma we derive the stochastic differential equation for

the bond option price as

$$dC = \left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i \right) \frac{\partial C}{\partial r} + \frac{1}{2} \sum_{i=1}^{n_w} \sigma_i^2(t, t) \frac{\partial^2 C}{\partial r^2} \right) dt \quad (5.2.7)$$

$$+ \sum_{i=1}^{n_w} \sigma_i(t, t) \frac{\partial C}{\partial r} dW_i(t) + \sum_{i=1}^{n_p} [C(r + \beta_i(t, t), t, T_C) - C(r, t, T_C)] dQ_i(t).$$

In the next section, we develop the classical hedging portfolio argument in the bond option market, in the spirit of the original Black-Scholes hedging approach, to derive the bond option pricing partial differential-difference equation.

5.2.1. Hedging Argument in the Bond Option Market. We have $n_w + n_p$ sources of risk, n_w due to the Gaussian process $W_i(t)$ and n_p due to the Poisson process Q_i , thus we consider a hedging portfolio containing a bond with maturity T and $n_o = n_w + n_p$ bond options of maturities T_1, T_2, \dots, T_{n_o} . All these options are written on the bond having maturity T .

By taking an appropriate position in bonds and bond options, it is possible to eliminate both Gaussian and Poisson risks. The condition that the riskless hedged portfolio earns the risk-free rate of interest $r(t)$, of the Gaussian bond market, implies that there must exist³ a vector $\Phi = (\phi_1, \dots, \phi_{n_w})^\top$ and a vector $\Psi = (\psi_1, \dots, \psi_{n_p})^\top$ such that for bond options of any maturity T_C it must be the case that⁴

$$\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i + \sum_{i=1}^{n_w} \phi_i(t) \sigma_i(t, t) \right) \frac{\partial C}{\partial r} + \frac{1}{2} \sum_{i=1}^{n_w} \sigma_i^2(t, t) \frac{\partial^2 C}{\partial r^2} - rC \quad (5.2.8)$$

$$+ \sum_{i=1}^{n_p} \psi_i(t) [C(r + \beta_i(t, t), t) - C(r, t)] = 0.$$

Also for bonds of any maturity T the following drift restriction holds

$$\alpha(t, T) = \sum_{i=1}^{n_w} \sigma_i(t, T) (-\phi_i(t) + \zeta_i(t, T)) - \sum_{i=1}^{n_p} \beta_i(t, T) (\psi_i(t) e^{-\xi_i(t, T)} - \lambda_i). \quad (5.2.9)$$

Equation (5.2.8) is the same partial differential-difference equation as for the bond price but now is solved over $0 \leq t \leq T_c$ and under boundary conditions appropriate to the type

³Note that the underlying Gaussian and jump risks (dW_i, dQ_i) driving the option price dynamics are the same as those driving the bond price dynamics and the instantaneous spot rate dynamics, thus the market price of these risks will be the same as those in the bond hedging portfolio.

⁴See Appendix 5.1 for details.

of option being evaluated. The boundary conditions in the case of a call bond option price are

$$C(r(T_C), T_C, T_C) = \max[P(r(T_C), T_C, T) - E, 0], \quad (5.2.10)$$

and

$$C(\infty, t, T_C) = 0,$$

as a result of the condition on the bond price that $P(\infty, t, T) = 0$, where E is the exercise price.

Note that the gist of the argument is establishing no-arbitrage consistency among a set of instruments sufficient to complete the market. Thus we may derive the condition (5.2.8) by using alternative portfolios, for instance a portfolio consisting of a bond option and n_o bonds.

5.3. The Option Pricing Partial Differential-Difference Equation

In deriving the martingale representation of the bond option price, the money market account has been used as the numeraire. By changing the numeraire, the bond option pricing equation can be formulated within a framework similar to that used by Merton (1976) to evaluate stock options involving Gaussian-Poisson risk. The price of the zero-coupon bond with maturity T_C will be employed as the numeraire for bond option pricing.

For every fixed finite time horizon T , for the same reasons as in Section 2.2.1, we can obtain a unique equivalent probability measure $\tilde{\mathbb{P}}$, under which the $\tilde{W}_i(t) = -\int_0^t \phi_i(s) ds + W_i(t)$ are standard Wiener processes and the Q_i are Poisson processes associated with intensity $\psi_i(t)$. Thus imposing the drift restriction (5.2.9) on equation (5.2.6), the dynamics for $P(t, T_C)$, the zero coupon bond maturing at bond option maturity, under $\tilde{\mathbb{P}}$, are given by

$$\frac{dP(t, T_C)}{P(t^-, T_C)} = r(t)dt - \sum_{i=1}^{n_w} \zeta_i(t, T_C) d\tilde{W}_i(t) - \sum_{i=1}^{n_p} (1 - e^{-\xi_i(t, T_C)}) [dQ_i(t) - \psi_i(t)dt]. \quad (5.3.1)$$

Using the result (5.2.8), the dynamics (5.2.7) for $C(r, t, T_C)$ under $\tilde{\mathbb{P}}$ are given by

$$\begin{aligned} \frac{dC}{C} = & r(t)dt + \frac{1}{C} \frac{\partial C}{\partial r} \sum_{i=1}^{n_w} \sigma_i(t, t) d\tilde{W}_i(t) \\ & + \frac{1}{C} \sum_{i=1}^{n_p} [C(r + \beta_i(t, t), t, T_C) - C(r, t, T_C)] [dQ_i(t) - \psi_i(t)dt]. \end{aligned} \quad (5.3.2)$$

Define the relative option and bond prices

$$Y(t) = \frac{C(r, t, T_C)}{P(r, t, T_C)}, \quad (5.3.3)$$

and

$$X(t) = \frac{P(r, t, T)}{P(r, t, T_C)}, \quad (5.3.4)$$

respectively. An application of the jump-diffusion version of the Ito's lemma gives the dynamics for Y^5 as

$$\begin{aligned} \frac{dY}{Y} = & \sum_{i=1}^{n_w} \left(\zeta_i(t, T_C) + \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} \right) [d\tilde{W}_i(t) + \zeta_i(t, T_C)dt] \\ & + \sum_{i=1}^{n_p} \left(\frac{C(r + \beta_{0i})}{C(r)} e^{\xi_i(t, T_C)} - 1 \right) [dQ_i(t) - \psi_i(t) e^{-\xi_i(t, T_C)} dt], \end{aligned} \quad (5.3.5)$$

and the dynamics for X as

$$\begin{aligned} \frac{dX}{X} = & \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T)) [d\tilde{W}_i(t) + \zeta_i(t, T_C)dt] \\ & + \sum_{i=1}^{n_p} \left(\frac{e^{-\xi_i(t, T)}}{e^{-\xi_i(t, T_C)}} - 1 \right) [dQ_i(t) - \psi_i(t) e^{-\xi_i(t, T_C)} dt]. \end{aligned} \quad (5.3.6)$$

By application of the Girsanov's theorem, a new measure \mathbb{P}^* may be found under which the new processes (specified here in increment form)

$$dW_i^*(t) = d\tilde{W}_i(t) + \zeta_i(t, T_C)dt, \quad (5.3.7)$$

are standard Gaussian processes, and

$$dQ_i^*(t) = dQ_i(t) - \psi_i(t) e^{-\xi_i(t, T_C)} dt, \quad (5.3.8)$$

⁵See Appendix 5.2 for details.

are Poisson processes associated with the intensity vector $\Psi^* = (\psi_1(t)e^{-\xi_1(t,T_C)}, \psi_2(t)e^{-\xi_2(t,T_C)}, \dots, \psi_n(t)e^{-\xi_n(t,T_C)})^\top$.

It follows from (5.3.5) and (5.3.6) that the relative option price Y and the relative bond price X are martingales under \mathbb{P}^* and using the expectation operator \mathbb{E}^* under this new measure, we may write

$$Y(t) = \mathbb{E}^*[Y(T_C) \mid \mathcal{F}_t], \quad (5.3.9)$$

where the Wiener processes $W_i^*(t)$ and the Poisson process $Q_i(t)^*$ with intensity Ψ^* generate the \mathbb{P} -augmentation of the filtration \mathcal{F}_t . By using the definition of $Y(t)$,⁶ equation (5.3.9) expands to

$$\begin{aligned} \frac{C(r, t, T_C)}{P(r, t, T_C)} &= \mathbb{E}^* \left[\frac{C(r, T_C, T_C)}{P(r, T_C, T_C)} \mid \mathcal{F}_t \right] \\ &= \mathbb{E}^* [\max(0, P(r, T_C, T) - E) \mid \mathcal{F}_t] \\ &= \mathbb{E}^* [\max(0, X(T_C) - E) \mid \mathcal{F}_t]. \end{aligned} \quad (5.3.10)$$

Therefore, the relative option price Y can be expressed as a function of the relative bond price X , i.e.,

$$Y(X, t) = \mathbb{E}^* [(X(T_C) - E)^+ \mid \mathcal{F}_t]. \quad (5.3.11)$$

The value of the adjusted option⁷ $Y(X, t)$ is driven by the dynamics for X , which are given by equation (5.3.6). Given the assumption on the volatility function, this process reduces to a form that puts us essentially in the framework used by Merton to price stock options under a Geometric jump-diffusion process, the only difference being that the coefficients of the stochastic differential equation are time dependent. Application of the Feynman-Kac Theorem for processes with jumps to equation (5.3.11) leads to the partial

⁶Recall that $C(r, t, T_C)$ is the value of a European option written on a bond with maturity T thus $C(r, T_C, T_C) = \max(0, P(r, T_C, T) - E)$ and $P(r, T_C, T_C) = 1$.

⁷The value of the option under the new T_C -forward measure is given by

$$C(t, T_C) = P(t, T_C) \mathbb{E}^* [(P(T_C, T) - E)^+ \mid \mathcal{F}_t].$$

differential-difference equation

$$\begin{aligned} \frac{\partial Y}{\partial t} + \sum_{i=1}^{n_p} (e^{-\xi_i(t, T_C)} - e^{-\xi_i(t, T)}) \psi_i(t) X \frac{\partial Y}{\partial X} \\ + \frac{1}{2} \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T))^2 X^2 \frac{\partial^2 Y}{\partial X^2} \\ + \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t, T_C)} \left(Y(X(t) \frac{e^{-\xi_i(t, T)}}{e^{-\xi_i(t, T_C)}}) - Y(X(t)) \right) = 0, \end{aligned} \quad (5.3.12)$$

subject to the boundary condition

$$\lim_{t \rightarrow T_C} Y(X, t) = (X(T_C) - E)^+. \quad (5.3.13)$$

In the next section, a technique to solve the partial differential-difference equation (5.3.12) is proposed, by employing Fourier Transform methods, that will lead to a pricing formula for the bond option. The Fourier Transform provides a quite general framework for solving partial differential equations of financial economics, since it handles a variety of pricing frameworks such as the jump-diffusion setting or the American option problem.

5.3.1. Solution to the Option Pricing Equation by Fourier Transform Techniques.

By changing the variable X to the logarithmic variable $Z = \ln X$ and defining the new function

$$\Upsilon(Z, t) = Y(e^Z, t)$$

the partial differential-difference equation (5.3.12) becomes

$$\begin{aligned} \frac{\partial \Upsilon}{\partial t} + \left[\sum_{i=1}^{n_p} \psi_i(t) (e^{-\xi_i(t, T_C)} - e^{-\xi_i(t, T)}) - \frac{1}{2} \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T))^2 \right] \frac{\partial \Upsilon}{\partial Z} \\ + \frac{1}{2} \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T))^2 \frac{\partial^2 \Upsilon}{\partial Z^2} \\ + \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t, T_C)} \left(\Upsilon \left(Z(t) + \ln \left[\frac{e^{-\xi_i(t, T)}}{e^{-\xi_i(t, T_C)}} \right] \right) - \Upsilon(Z(t)) \right) = 0, \end{aligned} \quad (5.3.14)$$

subject to the boundary condition

$$\lim_{t \rightarrow T_C} \Upsilon(Z, t) = (e^{Z(T_C)} - E)^+. \quad (5.3.15)$$

Define the Fourier transform of the solution $\Upsilon = \Upsilon(Z, t)$ to the partial differential-difference equation (5.3.14) by

$$\bar{\Upsilon}(\omega, t) = \int_{-\infty}^{\infty} \Upsilon(Z, t) e^{-i\omega Z} dZ, \quad (5.3.16)$$

where $i = \sqrt{-1}$ is the imaginary unit. By employing Fourier transform techniques, as Appendix 5.3 shows, the function $\bar{\Upsilon}(\omega, t)$ satisfies an ordinary differential equation with complex coefficients having solution

$$\bar{\Upsilon}(\omega, t) = \bar{\Upsilon}(\omega, T_C) \exp \left\{ (T_C - t) \left(-\bar{c}(t, T_C) + i\omega[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)] - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C) + \bar{\xi}(\omega, t, T_C) \right) \right\}, \quad (5.3.17)$$

where

$$\bar{c}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} ds, \quad (5.3.18)$$

$$\bar{v}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) (e^{-\xi_i(s, T_C)} - e^{-\xi_i(s, T)}) ds, \quad (5.3.19)$$

$$\bar{\sigma}^2(t, T_C) = \frac{1}{T_C - t} \int_t^{T_C} \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds, \quad (5.3.20)$$

$$\bar{\xi}(\omega, t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} \left(\frac{e^{-\xi_i(s, T)}}{e^{-\xi_i(s, T_C)}} \right)^{i\omega} ds. \quad (5.3.21)$$

By the Fourier inversion theorem, we have that

$$\Upsilon(Z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\Upsilon}(\omega, t) e^{i\omega Z} d\omega. \quad (5.3.22)$$

Thus, by substituting (5.3.17) into (5.3.22) we obtain

$$\begin{aligned} \Upsilon(Z, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\Upsilon}(\omega, T_C) \\ &\exp \left\{ (T_C - t) \left(-\bar{c}(t, T_C) + i\omega[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)] - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C) + \bar{\xi}(\omega, t, T_C) \right) + i\omega Z \right\} d\omega \\ &= \frac{e^{-(T_C-t)\bar{c}(t, T_C)}}{2\pi} \int_{-\infty}^{\infty} \bar{\Upsilon}(\omega, T_C) e^{i\omega([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C-t) + Z) - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C)(T_C-t) + \bar{\xi}(\omega, t, T_C)(T_C-t)} d\omega. \end{aligned}$$

Note that by changing the variable Z back to the variable X (recall that $Z = \ln X$), we obtain

$$\begin{aligned}\bar{\Upsilon}(\omega, T_C) &= \int_{-\infty}^{\infty} \Upsilon(Z, T_C) e^{-i\omega Z} dZ \\ &= \int_{-\infty}^{\infty} Y(e^Z, T_C) e^{-i\omega Z} dZ,\end{aligned}\quad (5.3.23)$$

and so

$$\begin{aligned}Y(X, t) &= \frac{e^{-\bar{c}(t, T_C)(T_C-t)}}{2\pi} \\ &\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} Y(e^Z, T_C) e^{-i\omega Z} dZ \right) e^{i\omega([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C-t) + \ln X) - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C)(T_C-t) + \bar{\xi}(\omega, T_C, t)(T_C-t)} d\omega \\ &= \frac{e^{-\bar{c}(t, T_C)(T_C-t)}}{2\pi} \\ &\int_{-\infty}^{\infty} Y(e^Z, T_C) \left(\int_{-\infty}^{\infty} e^{i\omega([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C-t) + \ln X - Z) - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C)(T_C-t) + \bar{\xi}(\omega, t, T_C)(T_C-t)} d\omega \right) dZ \\ &= e^{-\bar{c}(t, T_C)(T_C-t)} \int_{-\infty}^{\infty} Y(e^Z, T_C) \mathcal{K}(Z, X, t) dZ,\end{aligned}\quad (5.3.24)$$

where the kernel \mathcal{K} is defined by

$$\mathcal{K}(Z, X, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C-t) + \ln X - Z) - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C)(T_C-t) + \bar{\xi}(\omega, t, T_C)(T_C-t)} d\omega.\quad (5.3.25)$$

Thus, the value of the bond option can be expressed as

$$C(r, t, T_C) = e^{-\bar{c}(t, T_C)(T_C-t)} P(r, t, T_C) \int_{\ln E}^{\infty} (e^Z - E) \mathcal{K}(Z, X, t) dZ.\quad (5.3.26)$$

Unlike the corresponding result in Merton's jump-diffusion stock option model, it does not seem possible to proceed further with (5.3.26) and obtain a closed form solution under the more general volatility specifications. This is apparently due to the term $\left(\frac{e^{-\xi_i(s, T)}}{e^{-\xi_i(s, T_C)}} \right)^{i\omega}$ in the expression for $\bar{\xi}(\omega, t, T_C)$ in equation (5.3.25) for the kernel \mathcal{K} . Since in Merton's analysis the coefficients of his integro partial differential equation were not time varying this term reduces to 1 and so he was able to obtain closed form solutions.

5.3.2. Constant Jump Volatility Case. We will show in this section that restricting the model to constant jump volatilities will provide closed form solutions for the option

price. Assume constant jump volatilities, i.e., set $\beta_i(t, T) = \beta_{0i}$, where the β_{0i} are constant. Then the quantity $\bar{\xi}(\omega, t, T_C)$ in (5.3.21) simplifies to

$$\bar{\xi}(\omega, t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \frac{\psi_i e^{-\beta_{0i}(T-T_C)\mathbf{i}\omega}}{\beta_{0i}} (1 - e^{-\beta_{0i}(T_C-t)}), \quad (5.3.27)$$

which simplifies further the term $e^{\bar{\xi}(\omega, t, T_C)(T_C-t)}$ and allows us to complete the inversion of the Fourier Transform. Thus, by an application of the Taylor expansion of $e^x = \sum_{p=0}^{\infty} \frac{x^p}{p!}$ we write

$$\begin{aligned} e^{\bar{\xi}(\omega, t, T_C)(T_C-t)} &= \exp \left[\sum_{i=1}^{n_p} \frac{\psi_i e^{-\beta_{0i}(T-T_C)\mathbf{i}\omega}}{\beta_{0i}} (1 - e^{-\beta_{0i}(T_C-t)}) \right] \\ &= \prod_{i=1}^{n_p} \exp \left[\frac{\psi_i (1 - e^{-\beta_{0i}(T_C-t)})}{\beta_{0i}} e^{-\beta_{0i}(T-T_C)\mathbf{i}\omega} \right] \\ &= \prod_{i=1}^{n_p} \sum_{p=0}^{\infty} \frac{\zeta_i^p}{p!} e^{p\mu_i \mathbf{i}\omega} \\ &= \sum_{p=0}^{\infty} \frac{\zeta_1^p}{p!} e^{p\mu_1 \mathbf{i}\omega} \sum_{p=0}^{\infty} \frac{\zeta_2^p}{p!} e^{p\mu_2 \mathbf{i}\omega} \dots \sum_{p=0}^{\infty} \frac{\zeta_{n_p}^p}{p!} e^{p\mu_{n_p} \mathbf{i}\omega} \\ &= \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \dots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1}}{p_1!} e^{p_1\mu_1 \mathbf{i}\omega} \frac{\zeta_2^{p_2}}{p_2!} e^{p_2\mu_2 \mathbf{i}\omega} \dots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} e^{p_{n_p}\mu_{n_p} \mathbf{i}\omega} \\ &= \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \dots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1}}{p_1!} \frac{\zeta_2^{p_2}}{p_2!} \dots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} e^{(p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p})\mathbf{i}\omega} \end{aligned} \quad (5.3.28)$$

where $\zeta_i = \frac{\psi_i(1 - e^{-\beta_{0i}(T_C-t)})}{\beta_{0i}}$ and $\mu_i = -\beta_{0i}(T - T_C)$.

Substituting the expression (5.3.28) into the kernel function (5.3.25) and simplifying, we obtain

$$\begin{aligned} \mathcal{K}(Z, X, t) &= \frac{1}{2\pi} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \dots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1}}{p_1!} \frac{\zeta_2^{p_2}}{p_2!} \dots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &\int_{-\infty}^{\infty} e^{\mathbf{i}\omega([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C-t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p})) - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C)(T_C-t)} d\omega. \end{aligned} \quad (5.3.29)$$

By using the result

$$\int_{-\infty}^{\infty} e^{-q\omega - \rho\omega^2} d\omega = \sqrt{\frac{\pi}{\rho}} e^{\frac{q^2}{4\rho}}, \quad (5.3.30)$$

we are able to evaluate the integral expression in (5.3.29), to derive

$$\begin{aligned} \mathcal{K}(Z, X, t) &= \frac{1}{\sqrt{2\pi(T_C - t)\bar{\sigma}(t, T_C)}} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1} \zeta_2^{p_2}}{p_1! p_2!} \cdots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &\exp \left\{ -\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p}))^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)} \right\}. \end{aligned} \quad (5.3.31)$$

Thus equation (5.3.24) reduces to

$$\begin{aligned} Y(X, t) &= \frac{e^{-\bar{c}(t, T_C)(T_C - t)}}{\sqrt{2\pi(T_C - t)\bar{\sigma}(t, T_C)}} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1} \zeta_2^{p_2}}{p_1! p_2!} \cdots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &\int_{-\infty}^{\infty} Y(e^Z, T_C) e^{-\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p}))^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)}} dZ \\ &= e^{-\bar{c}(t, T_C)(T_C - t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1} \zeta_2^{p_2}}{p_1! p_2!} \cdots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &\int_{\ln E}^{\infty} \frac{e^Z - E}{\sqrt{2\pi(T_C - t)\bar{\sigma}(t, T_C)}} e^{-\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p}))^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)}} dZ. \end{aligned}$$

Further by evaluating separately the integrals⁸ in the last expression and using the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt,$$

we obtain

$$\begin{aligned} Y(X, t) &= e^{-\bar{c}(t, T_C)(T_C - t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1} \zeta_2^{p_2}}{p_1! p_2!} \cdots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &[e^{\bar{v}(t, T_C)(T_C - t) + \ln X + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p})} \Phi(d_1(\mathbf{p})) - E\Phi(d_2(\mathbf{p}))] \\ &= e^{-\bar{c}(t, T_C)(T_C - t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1} \zeta_2^{p_2}}{p_1! p_2!} \cdots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ &[X e^{\bar{v}(t, T_C)(T_C - t) + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p})} \Phi(d_1(\mathbf{p})) - E\Phi(d_2(\mathbf{p}))], \end{aligned} \quad (5.3.32)$$

⁸See Appendix 5.4 for detailed evaluation of the integrals.

where $\mathbf{p} = (p_1, p_2, \dots, p_{n_p})$ and we define

$$d_1(\mathbf{p}) = \frac{\ln \frac{X}{E} + (p_1\mu_1 + p_2\mu_2 + \dots + p_{n_p}\mu_{n_p}) + [\bar{v}(T_C, t) + \frac{1}{2}\bar{\sigma}^2(T_C, t)](T_C - t)}{\bar{\sigma}(T_C, t)\sqrt{T_C - t}}, \quad (5.3.33)$$

and

$$d_2(\mathbf{p}) = d_1(\mathbf{p}) - \bar{\sigma}(T_C, t)\sqrt{T_C - t}. \quad (5.3.34)$$

By recalling the definitions (5.3.3) and (5.3.4) of Y and X , we derive the value of the bond option price as

$$C(r, t, T_C) = e^{-\bar{c}(t, T_C)(T_C - t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \dots \sum_{p_{n_p}=0}^{\infty} \frac{\zeta_1^{p_1}}{p_1!} \frac{\zeta_2^{p_2}}{p_2!} \dots \frac{\zeta_{n_p}^{p_{n_p}}}{p_{n_p}!} \left[P(r, t, T) e^{\bar{v}(t, T_C)(T_C - t) + (\sum_{i=1}^{n_p} p_i \mu_i)} \Phi(d_1(\mathbf{p})) - EP(r, t, T_C) \Phi(d_2(\mathbf{p})) \right]. \quad (5.3.35)$$

The closed form bond option pricing formula derived is in the spirit of Shirakawa's (1991) closed form bond option evaluation results, in which the Poisson risk was assumed to be binomial, however here we provide a pricing formula allowing for multi-factor Poisson risk.

5.4. Markovian Spot Rate Dynamics under a Deterministic Volatility Structure

Within the jump-diffusion framework and under certain volatility specifications, the above term structure model admits a finite dimensional Markovian representation. These results are obtained and presented in Chapter 2, however in this section, we summarise the main results obtained under deterministic Wiener volatilities and constant jump volatilities, which are the volatility specifications that lead to a closed form solution for the bond option price as we have seen in the previous section.

Substitution of the condition (5.2.9) into (5.2.3), leads to the spot rate dynamics under the risk neutral measure, which are of the form (recall (2.2.15) where we omit the state

dependency on $\bar{f}(t)$)

$$\begin{aligned} r(t) = & f(0, t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t) \zeta_i(s, t) ds + \sum_{i=1}^{n_p} \int_0^t \psi_i(s) \beta_i(s, t) [1 - e^{-\xi_i(s, t)}] ds \\ & + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s, t) d\widetilde{W}_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s, t) [dQ_i(s) - \psi_i(s) ds]. \end{aligned} \quad (5.4.1)$$

Assume the case of deterministic volatility specifications with constant jump volatilities, determined as

ASSUMPTION 5.4.1. For $i = 1, \dots, n_p$, the deterministic Wiener volatility structure is of the form (2.3.1) and for $i = 1, \dots, n_w$, the Poisson volatility functions are of the form

$$\beta_i(s, t) = \beta_{0i}, \quad (5.4.2)$$

where β_{0i} are constant.

Essentially the volatility specifications used here are the same as the ones implied by Assumption 2.3.1 for $\kappa_{\beta_i}(t) = 0$ and $\beta_{0i}(t) = \beta_{0i}$ constant. Using results from Chapter 2, Markovian spot rate dynamics can be obtained under these volatility specifications. For volatility specifications satisfying Assumption 5.4.1, the dynamics for the spot rate (5.4.1) can be expressed as (2.3.5) in stochastic integral equation form, or (recall (2.3.6) for $\kappa_{\beta_i}(t) = 0$ and $\beta_{0i}(t) = \beta_{0i}$ constant)

$$\begin{aligned} dr(t) = & \left[D(t) - \sum_{i=2}^{n_w} \hat{\kappa}_{\sigma_i}(t) \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \kappa_{\sigma_1}(t) \mathcal{D}_{\beta_i}(t) - \kappa_{\sigma_1}(t) r(t) \right] dt \\ & + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i} [dQ_i(t) - \psi_i(t) dt], \end{aligned} \quad (5.4.3)$$

in stochastic differential equation form, where $\mathcal{E}_{\beta_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ reduce to

$$\mathcal{E}_{\beta_i}(t) = \int_0^t \psi_i(s) \beta_{0i}^2 e^{-\beta_{0i}(t-s)} ds, \quad (5.4.4)$$

$$\mathcal{D}_{\beta_i}(t) = \int_0^t \psi_i(s) \beta_{0i} [1 - e^{-\beta_{0i}(t-s)}] ds + \int_0^t \beta_{0i} (dQ_i(s) - \psi_i(s) ds). \quad (5.4.5)$$

The stochastic quantities $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ satisfy stochastic differential equations with drifts and diffusion terms that depend on the current value of these quantities, as the

Proposition 2.3.2 shows. Since the stochastic quantities $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ display Markovian dynamics, the instantaneous spot rate dynamics (5.4.3) are Markovian under the forward rate volatility specifications of Assumption 5.4.1. The corresponding multi-factor bond price formula in terms of $r(t)$ and the stochastic quantities $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$ assumes the multi-factor exponential affine representation (recall Proposition 2.3.3)

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \mathcal{M}(t, T) - \mathcal{N}_{\sigma_1}(t, T)r(t) - \sum_{i=2}^{n_w} (\mathcal{N}_{\sigma_i}(t, T) - \mathcal{N}_{\sigma_1}(t, T))\mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p} ((T - t) - \mathcal{N}_{\sigma_1}(t, T))\mathcal{D}_{\beta_i}(t) \right\}, \quad (5.4.6)$$

where, $\mathcal{M}(t, T)$ reduces to

$$\begin{aligned} \mathcal{M}(t, T) = & \mathcal{N}_{\sigma_1}(t, T)f(0, t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t, T)\mathcal{E}_{\sigma_i}(t) \\ & - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s)\beta_{0i}[1 - e^{-\beta_{0i}(y-s)}]dyds + \sum_{i=1}^{n_p} (T - t) \int_0^t \psi_i(s)\beta_{0i}[1 - e^{-\beta_{0i}(t-s)}]ds. \end{aligned} \quad (5.4.7)$$

These Markovian representations of the jump-diffusion version of the HW type model developed in this section, will be used in Section 5.6 where the simulated bond option prices are compared to the closed form solution (5.3.35) for the bond option price obtained in Section 5.3.2.

5.5. The State Dependent Volatility Model

In this section we consider state dependent forward rate volatility specifications. Generalising the basic assumption of Shirakawa (1991), on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The stochastic differential equation for the instantaneous forward rate $f(t, T)$ driven by both Gaussian and Poisson risk is given by equation (2.2.2) and the corresponding stochastic differential equation for the instantaneous spot rate is given by (2.2.6).

Consider a European call option of maturity T_C written on a bond having maturity T ($T > T_C$) and denote by $C = C(\bar{f}(t), t, T_C)$ the value of this bond option at time t . Now however the underlying variables include not only $r(t)$ but also a number of forward rates of fixed maturities, so that, $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_{\bar{n}_s}))^\top$. For notational convenience set $f_j = f(t, T_j)$, with $j = 1, 2, \dots, \bar{n}_s$.

Thus using the jump-diffusion version of Ito's lemma to derive the stochastic differential equation for the bond option price, we have to take into account the dynamics of all the underlying factors, namely (2.2.2) for the $f(t, T_i)$, ($i = 1, 2, \dots, \bar{n}_s$) and (2.2.6) for the $r(t)$. The multi-dimensional version of Ito's lemma is applied to derive

$$dC = \left(\frac{\partial C}{\partial t} + KC \right) dt + \sum_{i=1}^{n_w} D_i C dW_i(t) + \sum_{i=1}^{n_p} J_i C dQ_i(t), \quad (5.5.1)$$

where

$$\begin{aligned} KC \equiv & \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i \right) \frac{\partial C}{\partial r} + \sum_{j=1}^{\bar{n}_s} (\alpha(t, T_j) - \sum_{i=1}^{n_p} \beta_i(t, T_j) \lambda_i) \frac{\partial C}{\partial f_j} \\ & + \frac{1}{2} \frac{\partial^2 C}{\partial r^2} \sum_{i=1}^{n_w} \sigma_i^2(t, t) + \frac{1}{2} \sum_{j=1}^{\bar{n}_s} \frac{\partial^2 C}{\partial r \partial f_j} \sum_{i=1}^{n_w} \sigma_i(t, t, \bar{f}(t)) \sigma_i(t, T_j, \bar{f}(t)) \\ & + \frac{1}{2} \sum_{k=1}^{\bar{n}_s} \sum_{j=1}^{\bar{n}_s} \frac{\partial^2 C}{\partial f_k \partial f_j} \sum_{i=1}^{n_w} \sigma_i(t, T_k, \bar{f}(t)) \sigma_i(t, T_j, \bar{f}(t)), \end{aligned} \quad (5.5.2)$$

$$D_i C \equiv \sigma_i(t, t, \bar{f}(t)) \frac{\partial C}{\partial r} + \sum_{j=1}^{\bar{n}_s} \sigma_i(t, T_j, \bar{f}(t)) \frac{\partial C}{\partial f_j}, \quad (5.5.3)$$

for $i = 1, 2, \dots, n_w$, and,

$$J_i C = C(r + \beta_i(t, t), f_1 + \beta_i(t, T_1), \dots, f_{\bar{n}_s} + \beta_i(t, T_{\bar{n}_s}), t, T_C) - C(\bar{f}(t), t, T_C), \quad (5.5.4)$$

for $i = 1, 2, \dots, n_p$.

5.5.1. Hedging Argument in the Bond Option Market. We have $n_w + n_p$ sources of risk, n_w due to the Gaussian processes $W_i(t)$ and n_p due to the Poisson processes Q_i , thus we consider a hedging portfolio containing a bond with maturity T and $n_o = n_w + n_p$ bond options of maturities T_1, T_2, \dots, T_{n_o} . All these options are written on the bond having maturity T .

By taking an appropriate position in the bond and various bond options, it is possible to eliminate both Gaussian and Poisson risks. The condition that the riskless hedged portfolio earns the risk-free rate of interest $r(t)$, of the Gaussian bond market, implies

that there must exist⁹ a vector $\Phi = (\phi_1, \dots, \phi_{n_w})^\top$ and a vector $\Psi = (\psi_1, \dots, \psi_{n_p})^\top$ such that for bond options of any maturity T_C it must be the case that¹⁰

$$\frac{\partial C}{\partial t} + KC + \sum_{i=1}^{n_w} \phi_i(t) D_i C + \sum_{i=1}^{n_p} \psi_i(t) J_i C - rC = 0. \quad (5.5.5)$$

Also for bonds of any maturity T the following drift restriction holds

$$\alpha(t, T) = \sum_{i=1}^{n_w} \sigma_i(t, T, \bar{f}(t)) (-\phi_i(t) + \zeta_i(t, T, \bar{f}(t))) - \sum_{i=1}^{n_p} \beta_i(t, T) (\psi_i(t) e^{-\xi_i(t, T)} - \lambda_i). \quad (5.5.6)$$

The partial differential equation (5.5.5) is solved over $0 \leq t \leq T_C$ and under boundary conditions appropriate to the type of option being evaluated. The boundary conditions in the case of a call bond option equation are

$$C(r(T_C), T_C, T_C) = \max[P(r(T_C), T_C, T) - E, 0], \quad (5.5.7)$$

and

$$C(\infty, t, T_C) = 0,$$

as a result of the condition on the bond price that $P(\infty, t, T) = 0$, where E is the exercise price. It does not seem possible to solve the bond option pricing partial differential-difference equation (5.5.5), thus we settle for a numerical approach to obtain bond option prices under the state dependent volatility framework. As is shown in Chapter 2, under the specific state dependent volatility structures of Assumption 2.3.2, the term structure model considered here admits finite dimensional Markovian realisations. The Markovian representations of the state dependent term structure model under jump-diffusions are presented in Section 2.3.2.

Using the exponential affine term structure of interest rates (2.3.35), we can express the instantaneous forward rate in terms of $r(t)$ and the stochastic quantities $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and

⁹Note that the underlying Gaussian and jump risks (dW, dQ_i) driving the option price dynamics are the same as those driving the bond price dynamics and the instantaneous spot rate dynamics, thus the market price of these risks will be the same as those in the bond hedging portfolio.

¹⁰See Appendix 5.1 for details.

$\mathcal{D}_{\beta_i}(t)$, as

$$\begin{aligned}
 f(t, T) = & f(0, T) - \frac{\partial \bar{\mathcal{M}}(t, T)}{\partial T} + \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} + \sum_{i=1}^{n_w} \frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} \mathcal{N}_{\sigma_i}(t, T) \mathcal{E}_{\sigma_i}(t) \\
 & + \sum_{i=2}^{n_w} \left(\frac{\partial \mathcal{N}_{\sigma_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\sigma_i}(t) + \sum_{i=1}^{n_p} \left(\frac{\partial \mathcal{N}_{\beta_i}(t, T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t, T)}{\partial T} \right) \mathcal{D}_{\beta_i}(t).
 \end{aligned} \tag{5.5.8}$$

The relationship (5.5.8) can be used to express the benchmark forward rates $f(t, T_j)$, with $j = 1, 2, \dots, \bar{n}_s$ of the state dependent volatility functions in terms of the stochastic state variables $r(t)$, $\mathcal{E}_{\sigma_i}(t)$, $\mathcal{D}_{\sigma_i}(t)$ and $\mathcal{D}_{\beta_i}(t)$. In addition, by taking the number \bar{n}_s of the fixed forward rate maturities used in the state dependent volatility structure $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_{\bar{n}_s}))^\top$, equal to the number of stochastic quantities ($2n_w + n_p$), we have a closed system with the forward rates of any maturity evaluated by (5.5.8). See Section 2.4 for more discussion on how finite dimensional affine realisations in terms of forward rates may be obtained. These Markovian representations of the jump-diffusion version of the state dependent term structure model developed here, will be used in Section 5.6 where the above Markovian term structure of interest rates is simulated in order to obtain bond option prices. In particular, we use the representations in which the state variables may be a set of stochastic quantities rather than a set of benchmark forward rates. We decided on this approach due to the fact that in the case of constant jump volatilities, it is not possible to apply the results of Section 2.4 in which finite dimensional affine realisations in terms of forward rates were obtained. The matrix (2.4.6) is not invertible in the constant jump sizes case since the quantities $\frac{\partial \mathcal{N}_{\beta_i}(t, T_h)}{\partial T_h}$ reduce to 1 for all values of i . Thus for the model with constant jump sizes it is not possible to obtain representations of the stochastic quantities of the system in terms of economic quantities observed in the market, in contrast to the model with exponential decaying jump sizes studied in Chapter 2.

5.6. Monte Carlo Simulations

Monte Carlo simulation for derivative pricing, when the underlying asset follows a state dependent volatility jump-diffusion process(multivariate), is extremely intensive computationally, as the variance of the sampled variable is usually large and for N sample paths

the standard errors of the Monte Carlo simulations decreases as only $1/\sqrt{N}$. To improve the Monte Carlo efficiency, one should employ either a variance reduction methodology, namely antithetic variable, control variates, stratified sampling and importance sampling and/or use low discrepancy sequences. A control variate technique was developed by Chiarella, Clewlow, Musti (2003) for a state dependent volatility HJM model when the forward rate dynamics are driven by diffusion processes. We extend this model to accommodate our jump-diffusion setting, taking also advantage of the Markovian representations that have been obtained under the particular volatility specifications.

For the one Wiener/two Poisson case we have examined two classes of models. The first one is the deterministic volatility (HW) model with volatilities

$$\sigma(t, T) = \sigma_0 e^{-\kappa_\sigma(T-t)}, \quad (5.6.1)$$

and

$$\beta_i(t, T) = \beta_{0i}, \quad \text{with } i = 1, 2. \quad (5.6.2)$$

This model yields closed form solutions of the form (5.3.35) for bond option prices.¹¹

The second model is the stochastic volatility (SV) model, with state dependent volatilities

$$\sigma(t, T, \bar{f}(t)) = \sigma_0(t, \bar{f}(t)) e^{-\kappa_\sigma(T-t)},$$

¹¹For the volatility specifications

$$\sigma_i(s, t) = \sigma_{0i} e^{-\kappa_{\sigma_i}(t-s)}, \quad (5.6.3)$$

and

$$\beta_i(s, t) = \beta_{0i}, \quad (5.6.4)$$

the quantities (5.3.18), (5.3.19) and (5.3.20) simplify

$$\bar{c}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \frac{\psi_i}{\beta_{0i}} \left(1 - e^{-\beta_{0i}(T_C-t)}\right), \quad (5.6.5)$$

$$\bar{v}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \frac{\psi_i}{\beta_{0i}} \left(1 - e^{-\beta_{0i}(T_C-t)} + e^{-\beta_{0i}(T-t)} - e^{-\beta_{0i}(T-T_C)}\right), \quad (5.6.6)$$

$$\bar{\sigma}^2(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_w} \frac{\sigma_0^2}{2\kappa_i^3} \left(e^{\kappa_i T_C} - e^{\kappa_i T}\right)^2 \left(e^{-2\kappa_i T} - e^{2\kappa_i t - 2\kappa_i T - 2\kappa_i T_C}\right). \quad (5.6.7)$$

where

$$\sigma_0(t, \bar{f}(t)) = \begin{cases} 0.05 \sigma_0(t), & L_f(t) < 0.005; \\ \sigma_0(t)[(L_f(t) - 0.005)^\gamma + 0.05], & L_f(t) \geq 0.005; \end{cases} \quad (5.6.8)$$

with $L_f(t) = c_0 r(t) + \sum_{h=1}^3 c_h f(s, T_h)$ and $\gamma = \frac{1}{2}$. Also we consider $\beta_i(t, T) = \beta_{0i} e^{-k_{\beta_i}(T-t)}$ and constant ψ_i . Recall that in the current setup $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), f(t, T_3))^\top$. Also note that, since these Markovian structures may drive the forward rate to negative values, the state dependent volatilities (5.6.8) have been selected so as to provide the model with a well defined state dependent volatility function.

5.6.1. Simulation Scheme. Let t be time, T be maturity, and \mathbb{T} be the time horizon where $0 \leq t \leq T \leq \mathbb{T}$. The time horizon $(0, \mathbb{T})$ is subdivided into N intervals of length $\Delta t = \frac{\mathbb{T}}{N}$ so that $t = n\Delta t$ and $T = t + m\Delta t$. This scheme requires the knowledge of the initial forward curve $f(0, T)$. The initial forward rate curve considered is the same as the one used in the numerical examples of Chapter 3, that it has the functional form $f(0, t) = (a_0 + a_1 t + a_2 t^2) e^{-vt}$ with parameters being estimated as $a_0 = 0.033287$, $a_1 = 0.014488$, $a_2 = -0.000117$, and $v = 0.0925$. Recall that the data used for interpolation are the US zero yields on July 20, 2001, with maturities up to 10 years including the overnight rate.

5.6.2. The initial bond price. The initial bond price $P(0, T)$ is given by relationship

$$P(0, T) = \exp \left(- \int_0^T f(0, s) ds \right). \quad (5.6.9)$$

Thus given the initial forward structure $f(0, T)$, the initial bond price is calculated.

The bond price $P(t, T)$ can be also expressed in terms of a risk neutral expectation as (See equation (2.2.14))

$$P(t, T) = \tilde{\mathbb{E}} \left[\exp \left(- \int_t^T r(s) ds \right) \mid \mathcal{F}_t \right], \quad (5.6.10)$$

thus the initial bond price $P(0, T)$ may be evaluated as

$$P(0, T) = \tilde{\mathbb{E}} \left[\exp \left(- \int_0^T r(s) ds \right) \mid \mathcal{F}_0 \right]. \quad (5.6.11)$$

Performing simulations over Π paths, the initial bond price may be estimated as

$$P(0, T) = \frac{1}{\Pi} \sum_{i=0}^{\Pi} \exp \left(- \sum_{j=0}^N r_i(j\Delta t) \Delta t \right). \quad (5.6.12)$$

Equation (5.6.12) can be used to provide a check on the accuracy of our simulation schemes, in particular giving an indication of the size of the discretisation bias. Table 5.6.1 provides the simulated initial bond prices for the deterministic volatility HW type of models when the parameter values set as $\sigma_0 = 1.5\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 2\%$, $\beta_{02} = -3\%$, $\psi_1 = 1$ and $\psi_2 = 1.5$. The discretised spot rate dynamics used in the simulation scheme are the Markovian dynamics presented in Section 5.4, recall equation (5.4.3).

$P(0, 1) = \exp \left(- \int_0^1 f(0, s) ds \right) = 0.9381583$				
N	Π	P(0,1)	St. Dev.	St. Err.
200	5,000	0.938044	0.000336	0.34033
	50,000	0.938009	0.000108	1.37195
	500,000	0.937833	0.000034	8.81883
400	5,000	0.938262	0.000341	-0.30571
	50,000	0.938035	0.000108	1.13973
	500,000	0.937984	0.000034	5.10233
800	5,000	0.938087	0.000336	0.21052
	50,000	0.938086	0.000108	0.66752
	500,000	0.938094	0.000034	1.86966

TABLE 5.6.1. Initial Bond Prices - HW models.

Table 5.6.2 presents the simulated initial bond prices, of a bond maturing in 1 year, for the state dependent volatility type of models, and when the parameter values set as $\sigma_0 = 1.5\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 2\%$, $\beta_{02} = -3\%$, $\kappa_{\beta_1} = 0.31$, $\kappa_{\beta_2} = 0.17$, $\psi_1 = 1$ and $\psi_2 = 1.5$. In addition, we set $c_0 = 1$, $c_1 = 2$, $c_2 = 1$, $c_3 = 2$. The discretised spot rate dynamics are the Markovian dynamics described in Section 2.3.2, see in particular equation (2.3.25), with the benchmark forward rates expressed in terms of the stochastic factors of the system, by using equation (5.5.8).

Discretisation error¹² becomes evident when the distance between the true and simulated bond prices exceeds two standard deviations. In Table 5.6.1, this is the case for 500,000

¹²Standard error is defined as the difference between exact price and simulated price divided by the standard deviation of the simulated prices.

$P(0, 1) = \exp\left(-\int_0^1 f(0, s)ds\right) = 0.9381583$				
N	Π	P(0,1)	St. Dev.	St. Err.
200	5,000	0.938080	0.000312	0.25150
	50,000	0.937930	0.000098	2.32676
	500,000	0.937856	0.000031	9.28062
400	5,000	0.938430	0.000310	-0.87725
	50,000	0.938091	0.000098	0.68753
	500,000	0.937999	0.000033	4.86024
800	5,000	0.938161	0.000311	-0.00876
	50,000	0.938026	0.000098	1.34415
	500,000	0.938108	0.000031	1.62363

TABLE 5.6.2. Initial Bond Prices - SV models.

paths or more, with the error appearing to be somewhat reduced when Δt is reduced. In particular, discretisation error is no longer evident when the discretisation level is increased to 800. It is important to keep the magnitude of the error from this source in mind when interpreting the results from the simulations in the subsequent sections.

The initial bond price results obtained by the simulations for both models (deterministic volatility and stochastic volatility model) are consistent, to four decimal place accuracy, (especially when we reduce the discretisation bias by setting the discretisation level to $N = 800$) with the value obtained from the analytical bond price (5.6.9), providing evidence of the effectiveness of this numerical scheme.

5.6.3. Bond Option Price Evaluation. Denote with $C(t, T_c, \mathbb{T})$ the time t -value of a European call option maturing at T_c on the zero-coupon bond with maturity \mathbb{T} , where $0 \leq t \leq T_c \leq \mathbb{T}$. The current value of a European call option $C(0, T_c, \mathbb{T})$ can be evaluated, under the risk neutral measure as the expected discounted payoff of the option at the option's maturity

$$C(0, T_c, \mathbb{T}) = \tilde{\mathbb{E}} \left[\exp \left\{ - \int_0^{T_c} r(s) ds \right\} (P(T_c, \mathbb{T}) - E)^+ | \mathcal{F}_0 \right], \quad (5.6.13)$$

or, alternatively, under the T_c -forward measure, as

$$C(0, T_c, \mathbb{T}) = P(0, \mathbb{T}) \mathbb{E}^* \left[(P(T_c, \mathbb{T}) - E)^+ | \mathcal{F}_0 \right]. \quad (5.6.14)$$

For simulation based approaches to bond option pricing, we have found that the use of one or the other probability measure does not seem to provide any significant advantage. Here we report the simulations under the risk neutral probability measure.

Given the Markovian spot rate dynamics under the risk neutral measure, which are equation (5.4.3) for the deterministic volatility models and equation (2.3.25) for the state dependent volatility models, the bond option price is evaluated using formula (5.6.13), by using the Euler-Maruyama scheme for the integration, as

$$C(0, T_c, \mathbb{T}) = \frac{1}{K} \sum_{k=1}^K \exp \left\{ - \sum_{i=1}^N r_k(i\Delta t) \Delta t \right\} (P_k(T_c, \mathbb{T}) - E)^+. \quad (5.6.15)$$

The bond price $P_k(T_c, \mathbb{T})$ is computed by the exponential affine term structure (5.4.6) for the deterministic volatility model and (2.3.35) for the state dependent volatility model.

Table 5.6.3 shows the simulated bond option prices for the deterministic volatility HW type of models under the following choice of parameter values $\kappa_\sigma = 0.18$, $\psi_1 = 1$, $\psi_2 = 1.5$, $\sigma_0 = 1.5\%$, $\beta_{01} = 2\%$, $\beta_{02} = -3\%$. The exercise price is set $E = 0.95$ and the exact value for the bond option price is evaluated from (5.3.35).

Exact Option Price		$C_{exact}^{HW}(0, 0.5, 1) = 0.018181443925$		
N	II	$C_{MC}^{HW}(0, 0.5, 1)$	St. Dev.	St. Err.
200	5,000	0.018285	0.000200	-0.51624
	50,000	0.018121	0.000063	0.96650
	500,000	0.018157	0.000020	1.24335
400	5,000	0.018245	0.000200	-0.31898
	50,000	0.018179	0.000063	0.03544
	500,000	0.018180	0.000020	0.05745

TABLE 5.6.3. Call Bond Option Prices - HW models.

The bond option prices obtained by the Monte Carlo simulations for the deterministic volatility model are consistent with the value obtained from the analytical bond option price (5.3.35) with accuracy reaching three significant figures at the 50,000 simulated paths and over. This provides evidence that the numerical scheme employed here is effective.

Table 5.6.4 shows the simulated bond option prices for the stochastic volatility type of models under the following choice of parameter values; $\sigma_0 = 1.5\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 2\%$, $\beta_{02} = -3\%$, $\kappa_{\beta_1} = 0.31$, $\kappa_{\beta_2} = 0.17$, $\psi_1 = 1$, $\psi_2 = 1.5$. Recall that the Wiener state

dependent volatilities have the functional form (5.6.8), with $c_0 = 1$, $c_1 = 2$, $c_2 = 1$ and $c_3 = 2$. The exercise price is set to be $E = 0.95$.

N	Π	$C_{MC}^{SV}(0, 0.5, 1)$	St. Dev.
200	5,000	0.022210	0.000182
	50,000	0.022240	0.000058
	500,000	0.022303	0.000018
400	5,000	0.022107	0.000181
	50,000	0.022414	0.000058
	500,000	0.022280	0.000018

TABLE 5.6.4. Call Bond Option Prices - SV models.

The bond option prices obtained by the Monte Carlo simulations for the stochastic volatility model are consistent to at least two significant figures, however in the next section we will attempt to improve convergence of the stochastic volatility numerical scheme by an application of a control variate method.

5.6.4. Control Variate Method. The application of Monte Carlo simulations to evaluate bond option prices under the HJM framework comes at the expense of significant computational effort. To improve convergence we propose to use a control variate method. The HW model under deterministic Wiener volatilities and constant jump sizes accommodates a closed form option pricing formula and thus we can compute the option price C_{exact}^{HW} . The SV model (state dependent Wiener volatilities and deterministic jump volatilities) can only be evaluated numerically. Running simulations of these two models, the option prices C_{MC}^{HW} under the deterministic volatility model and the option prices C_{MC}^{SV} under the stochastic volatility model are estimated. The control variate adjustment proposes that the approximated option value of the stochastic volatility model is evaluated by

$$C^{SV} = C_{MC}^{SV} - C_{MC}^{HW} + C_{exact}^{HW}. \quad (5.6.16)$$

The rationale of the control variate method is that the known error imposed by the Monte Carlo simulations in the case of deterministic volatility model

$$C_{MC}^{HW} - C_{exact}^{HW},$$

is assumed to be close to the error of the Monte Carlo estimation for the case of stochastic volatility model, namely

$$C_{MC}^{SV} - C^{SV}.$$

Evaluating (5.6.16) can be time consuming since it requires the results of two simulations. However, use of the Markovian representations of the models considered have considerably simplify and speeded up the calculation. As Table 5.6.5 shows the standard errors of the option values estimated by the control variate method are of the order of approximately one seventh with respect to the values obtained by the standard Monte Carlo simulation of the Markovian stochastic volatility term structure model. This reduction is uniform across the order of discretisation and the number of simulated paths. Additionally, the accuracy on the bond option price has increased to three significant figures by the application of the control variate scheme compared to the two significant figures accuracy obtained by the application of the standard Monte Carlo simulation.

N	Π	$C^{SV}(0, 0.5, 1)$	St. Dev.	$C^{CV}(0, 0.5, 1)$	St. Dev.
200	5,000	0.022484	<i>0.000186</i>	0.022340	<i>0.000026</i>
	50,000	0.022220	<i>0.000058</i>	0.022347	<i>0.000008</i>
	500,000	0.022313	<i>0.000019</i>	0.022323	<i>0.000003</i>
400	5,000	0.022325	<i>0.000186</i>	0.022315	<i>0.000026</i>
	50,000	0.022299	<i>0.000059</i>	0.022325	<i>0.000008</i>
	500,000	0.022298	<i>0.000019</i>	0.022326	<i>0.000003</i>

TABLE 5.6.5. Call Bond Option Prices - SV models; Control Variate Method.

To justify the efficiency of the control variate method, we ensure firstly that

$$\mathbb{E}[C_{MC}^{HW} - C_{exact}^{HW}] = 0. \quad (5.6.17)$$

From Table 5.6.3, we conclude that condition (5.6.17) holds since insignificant discretisation error exists in particular when we increase the order of the discretisation to 400.

The control variate method is employed here to price the same product - bond options - under two different models. This is a somewhat unorthodox and - to our knowledge - new perspective on control variate methods in pricing derivatives. Typically a control variate method is applied to another (closely related) instrument priced in the same model, whereas here the control variate is the same instrument priced in a closely related model. Under these model specifications the state variables evolve differently. However, the state

variables can be seen as simply two different sets of functions of the driving Wiener and Poisson processes, which are the same in both models. Therefore, the control variate method will be correctly used if the state variables of the two models considered are highly correlated. Table 5.6.6 presents the correlation coefficients of the state variables, which are common in these two classes of models, and these are the $r(t)$, $\mathcal{D}_{\beta_1}(t)$ and $\mathcal{D}_{\beta_2}(t)$. The state variables are clearly highly positively correlated, thus the control variate method may be safely used for the pricing of bond options under a stochastic volatility model.

N	Π	$r(t)$	$\mathcal{D}_{\beta_1}(t)$	$\mathcal{D}_{\beta_2}(t)$
200	5,000	0.995985	0.998992	0.999704
	50,000	0.995856	0.999007	0.999700
	500,000	0.995885	0.999002	0.999699
400	5,000	0.995788	0.998948	0.999712
	50,000	0.995895	0.998994	0.999705
	500,000	0.995902	0.998998	0.999699

TABLE 5.6.6. Correlation Coefficients.

Thus, combination of these Markovian structures with a control variate method provides an efficient numerical scheme that may yield good results in Monte Carlo simulation even with a relatively small number of simulated paths. Given that the control variate method improves the standard error by seven times relative to the standard error obtained from the standard Monte Carlo simulation on the stochastic volatility model, one may obtain the same order of accuracy with forty-nine times less the number of simulated paths. An important contribution to the efficiency of this numerical scheme must be attributed to the fact that the models developed here possess Markovian dynamics. All the parameters used in the simulations such as bond prices, benchmark forward rates used in the volatility structure could be expressed in terms of the state variables of the Markovian system. Discretisation has only been applied to the dynamics of the state variables of the system, therefore a lot of numerical evaluations have been avoided.

5.7. Conclusions

This chapter develops two models to price bond options when interest rates are subject to jumps. In the first model, both Wiener and Poisson volatilities are time dependent, and working within the Shirakawa general HJM model, we have derived the partial

differential-difference equation for the pricing of bond options. In addition, by employing Fourier transform techniques, bond option prices have been evaluated and an easily tractable Black-Scholes type bond option pricing formula under the assumption of constant jump volatility has been derived. In the second model, the volatility structure is more general, by allowing for state dependent Wiener volatilities and time dependent Poisson volatilities. In this second model, it is difficult to explicitly solve the bond option pricing problem, therefore Monte Carlo simulation techniques are used to evaluate bond options. However, under appropriate volatility functions, the term structures obtained for both models display Markovian dynamics. These Markovian representations contribute to increase the efficiency and accuracy of the application of the Monte Carlo simulations. Additionally, taking advantage of the closed form solutions obtained under the deterministic volatility setting, we employ a control variate method that significantly improves the efficiency of the numerical procedure.

The main contributions of this chapter are:

- ★ Bond option pricing models are developed, under two forward rate volatility settings, one deterministic and the other stochastic, when interest rates are subject to jumps. Under the deterministic volatility setting, closed form solutions are obtained, whereas a Monte Carlo simulation method is proposed for the stochastic volatility model.
- ★ Under suitable volatility specifications, both models admit finite dimensional Markovian structures.
- ★ For the deterministic volatility model, Monte Carlo simulations of the Markovian term structures in which bond options prices are compared to the exact closed form solutions provide evidence of the effectiveness of the numerical schemes developed.
- ★ Application of a control variate method improves significantly the efficiency and accuracy of the numerical approximation scheme used to price bond options under the state dependent volatility model with jumps.

The important characteristic of the solutions for bond option prices proposed in this chapter is that they incorporate the complexity of a stochastic volatility and/or jump-diffusion

model although they enjoy computational tractability due to the Markovian structures used.

Appendix 5.1. The No-Arbitrage Condition in the Bond Option Market

Recall the stochastic differential equation for the spot rate

$$dr(t) = \vartheta(t)dt + \sum_{i=1}^{n_w} \sigma_i(t, t)dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t, t)[dQ_i(t) - \lambda_i dt], \quad (\text{A5.1.1})$$

where $\vartheta(t)$ is defined as

$$\begin{aligned} \vartheta(t) = & \frac{\partial}{\partial t} f(0, t) + \alpha(t, t) + \int_0^t \frac{\partial}{\partial t} \alpha(s, t) ds \\ & + \sum_{i=1}^{n_w} \int_0^t \frac{\partial}{\partial t} \sigma_i(s, t) dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \frac{\partial}{\partial t} \beta_i(s, t) [dQ_i(s) - \lambda_i ds]. \end{aligned} \quad (\text{A5.1.2})$$

Using the jump-diffusion version of Ito's lemma we derive the stochastic differential equation for the bond option price

$$\begin{aligned} dC = & \left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i \right) \frac{\partial C}{\partial r} + \frac{1}{2} \sum_{i=1}^{n_w} \sigma_i^2(t, t) \frac{\partial^2 C}{\partial r^2} \right) dt \\ & + \sum_{i=1}^{n_w} \sigma_i(t, t) \frac{\partial C}{\partial r} dW_i(t) + \sum_{i=1}^{n_p} [C(r + \beta_i(t, t), t, T_C) - C(r, t, T_C)] dQ_i(t), \end{aligned} \quad (\text{A5.1.3})$$

We consider a hedging portfolio containing a bond with maturity T and $n_o = n_w + n_p$ bond options of maturities T_1, T_2, \dots, T_{n_o} in proportions w_1, w_2, \dots, w_{n_o} with $w_1 + w_2 + \dots + w_{n_o+1} = 1$, where w_{n_o+1} is the proportion corresponding to the bond. All these options are written on the bond having maturity T . If we denote with $C_i(t) = C(t, T_i)$ ($i = 1, 2, \dots, (n_o)$) the value of the i^{th} bond option, we may write the stochastic differential equation for $C_i(t)$ in the general form

$$\frac{dC_i(t)}{C_i(t)} = \mu_{C_i}(t)dt + \sum_{j=1}^{n_w} \nu_{C_i,j}(t)dW_j(t) + \sum_{j=1}^{n_p} \chi_{C_i,j}(t)dQ_j(t),$$

where

$$\begin{aligned}\mu_{C_i}(t) &= \frac{1}{C_i} \left(\frac{\partial C_i}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i \right) \frac{\partial C_i}{\partial r} + \frac{1}{2} \frac{\partial^2 C_i}{\partial r^2} \sum_{j=1}^{n_w} \sigma_{ij}^2(t, t) \right), \\ \nu_{C_{i,j}}(t) &= \frac{1}{C_i} \frac{\partial C_i}{\partial r} \sum_{j=1}^{n_w} \sigma_{ij}(t, t), \\ \chi_{C_{i,j}}(t) &= \frac{1}{C_i} \sum_{j=1}^{n_p} [C_i(r + \beta_{ij}(t, t), t) - C_i(r, t)].\end{aligned}$$

Also recall the stochastic differential equation for the bond price P

$$\frac{dP(t)}{P(t)} = \mu_P(t)dt + \sum_{i=1}^{n_w} \nu_{P_i}(t)dW_i(t) + \sum_{i=1}^{n_p} \chi_{P_i}(t)dQ_i(t),$$

where

$$\mu_P(t) = r(t) + H(t, T), \quad \nu_{P_i}(t) = -\zeta_i(t, T), \quad \text{and} \quad \chi_{P_j}(t) = \eta_j(t, T) - 1.$$

Let V be the value of the hedging portfolio then the return on the portfolio is given by

$$\begin{aligned}\frac{dV}{V} &= w_1 \frac{dC_1}{C_1} + w_2 \frac{dC_2}{C_2} + \cdots + w_{n_o} \frac{dC_{n_o}}{C_{n_o}} + w_{n_o+1} \frac{dP}{P} \\ &= \sum_{i=1}^{n_o} w_i \mu_{C_i} dt + w_{n_o+1} \mu_P dt + \sum_{i=1}^{n_o} w_i \sum_{j=1}^{n_w} \nu_{C_{i,j}} dW_j(t) + w_{n_o+1} \sum_{j=1}^{n_w} \nu_{P_j} dW_j(t) \\ &\quad + \sum_{i=1}^{n_o} w_i \sum_{j=1}^{n_p} \chi_{C_{i,j}} dQ_j(t) + w_{n_o+1} \sum_{j=1}^{n_p} \chi_{P_j} dQ_j(t).\end{aligned}$$

In order to eliminate both Gaussian and Poisson risks we need to choose $w_1, w_2, \dots, w_{n_o+1}$

so that

$$\sum_{i=1}^{n_o} w_i \nu_{C_{i,j}} + w_{n_o+1} \nu_{P_j} = 0, \quad \text{when } j = 1, 2, \dots, n_w \quad (\text{A5.1.4})$$

$$\sum_{i=1}^{n_o} w_i \chi_{C_{i,j}} + w_{n_o+1} \chi_{P_j} = 0, \quad \text{when } j = 1, 2, \dots, n_p. \quad (\text{A5.1.5})$$

The hedging portfolio then becomes riskless, thus, it should earn the risk-free rate of interest $r(t)$, of the Gaussian bond market, i.e.,

$$\frac{dV}{V} = \sum_{i=1}^{n_o} w_i \mu_{C_i} dt + w_{n_o+1} \mu_P dt = r(t)dt,$$

that can be simplified to

$$\sum_{i=1}^{n_o} w_i(\mu_{C_i} - r(t)) + w_{n_o+1}(\mu_P - r(t)) = 0, \quad (\text{A5.1.6})$$

using also the fact that $w_1 + w_2 + \dots + w_{n_o+1} = 1$. Equations (A5.1.4), (A5.1.5) and (A5.1.6) form a system of $(n_o + 1)$ equations with $(n_o + 1)$ unknowns $w_1, w_2, \dots, w_{n_o+1}$.

This system can only have a non-zero solution if the determinant

$$\begin{vmatrix} \nu_{C_{1,1}}(t) & \nu_{C_{2,1}}(t) & \cdots & \nu_{C_{n_o,1}}(t) & \nu_{P_1}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{C_{1,n_w}}(t) & \nu_{C_{2,n_w}}(t) & \cdots & \nu_{C_{n_o,n_w}}(t) & \nu_{P_{n_w}}(t) \\ \chi_{C_{1,1}}(t) & \chi_{C_{2,1}}(t) & \cdots & \chi_{C_{n_o,1}}(t) & \chi_{P_1}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_{C_{1,n_p}}(t) & \chi_{C_{2,n_p}}(t) & \cdots & \chi_{C_{n_o,n_p}}(t) & \chi_{P_{n_p}}(t) \\ \mu_{C_1}(t) - r(t) & \mu_{C_2}(t) - r(t) & \cdots & \mu_{C_{n_o}}(t) - r(t) & \mu_P(t) - r(t) \end{vmatrix}$$

is equal to zero. The above equation implies that for $h = 1, 2, \dots, n_o$ there exist $\phi_1(t), \phi_2(t), \dots, \phi_{n_w}(t)$ and $\psi_1(t), \psi_2(t), \dots, \psi_{n_p}(t)$ such that

$$\mu_{C_i}(t) - r(t) = - \sum_{j=1}^{n_w} \phi_j(t) \nu_{C_{i,j}}(t) - \sum_{j=1}^{n_p} \psi_j(t) \chi_{C_{i,j}}(t), \quad (\text{A5.1.7})$$

and

$$\mu_P(t) - r(t) = - \sum_{i=1}^{n_w} \phi_i(t) \nu_{P_i}(t) - \sum_{i=1}^{n_p} \psi_i(t) \chi_{P_i}(t). \quad (\text{A5.1.8})$$

Thus using equations (A5.1.7) for bond options of any maturity T_C we must have that

$$\mu_C(t) - r(t) = - \sum_{i=1}^{n_w} \phi_i(t) \nu_{C_i}(t) - \sum_{i=1}^{n_p} \psi_i(t) \chi_{C_i}(t),$$

and substituting the expressions for $\mu_C(t)$, $\nu_{C_i}(t)$ and $\chi_{C_i}(t)$, we have that

$$\begin{aligned} & \frac{1}{C} \left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t) \lambda_i \right) \frac{\partial C}{\partial r} + \frac{1}{2} \sum_{i=1}^{n_w} \sigma_i^2(t, t) \frac{\partial^2 C}{\partial r^2} \right) - r(t) \\ & = - \frac{1}{C} \frac{\partial C}{\partial r} \sum_{i=1}^{n_w} \phi_i(t) \sigma_i(t, t) - \sum_{i=1}^{n_p} \psi_i(t) \frac{1}{C} [C(r + \beta_i(t, t), t) - C(r, t)], \end{aligned} \quad (\text{A5.1.9})$$

or after further manipulations

$$\begin{aligned} \frac{\partial C}{\partial t} + (\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t, t)\lambda_i + \sum_{i=1}^{n_w} \phi_i(t)\sigma_i(t, t))\frac{\partial C}{\partial r} + \frac{1}{2} \sum_{i=1}^{n_w} \sigma_i^2(t, t)\frac{\partial^2 C}{\partial r^2} - rC \\ + \sum_{i=1}^{n_p} \psi_i[C(r + \beta_i(t, t), t) - C(r, t)] = 0. \end{aligned} \quad (\text{A5.1.10})$$

By substituting the expressions for $\mu_{P_i}(t)$, $\nu_{P_i}(t)$ and $\chi_{P_i}(t)$ in equation (A5.1.8), we derive the drift restriction

$$\alpha(t, T) = \sum_{i=1}^{n_w} \sigma_i(t, T)(-\phi_i(t) + \zeta_i(t, T)) - \sum_{i=1}^{n_p} \beta_i(t, T)(\psi_i(t)e^{-\xi_i(t, T)} - \lambda_i). \quad (\text{A5.1.11})$$

Appendix 5.2. Application of Ito's lemma on \mathbf{Y}

The dynamics for $P(t, T_C)$ are given by

$$\begin{aligned} dP(t, T_C) = r(t)P(t, T_C)dt - \sum_{i=1}^{n_w} \zeta_i(t, T_C)P(t, T_C)d\widetilde{W}_i(t) \\ + P(t, T_C) \sum_{i=1}^{n_p} (e^{-\xi_i(t, T_C)} - 1)[dQ_i(t) - \psi_i dt], \end{aligned} \quad (\text{A5.2.1})$$

and the dynamics for $C(r, t, T)$ are

$$\begin{aligned} dC = r(t)Cdt + \frac{\partial C}{\partial r} \sum_{i=1}^{n_w} \sigma_{0i}(t)d\widetilde{W}_i(t) \\ + \sum_{i=1}^{n_p} [C(r + \beta_{0i}, t, T) - C(r, t, T)][dQ_i(t) - \psi_i dt]. \end{aligned} \quad (\text{A5.2.2})$$

Define the new quantity

$$Y(C, P) = \frac{C(r, t, T)}{P(r, t, T_C)} \quad (\text{A5.2.3})$$

then

$$\begin{aligned} \frac{\partial Y}{\partial C} = \frac{1}{P}, \quad \frac{\partial Y}{\partial P} = -\frac{C}{P^2} \\ \frac{\partial^2 Y}{\partial C^2} = 0, \quad \frac{\partial^2 Y}{\partial P^2} = \frac{2C}{P^3}, \quad \frac{\partial^2 Y}{\partial P \partial C} = -\frac{1}{P^2}. \end{aligned} \quad (\text{A5.2.4})$$

Application of the multi-dimensional jump-diffusion version of Ito's Lemma leads to

$$\begin{aligned}
dY = & \left[\frac{\partial Y}{\partial t} + (rP - \sum_{i=1}^{n_p} (e^{-\xi_i(t, T_C)} - 1) \psi_i P) \frac{\partial Y}{\partial P} + (rC - \sum_{i=1}^{n_p} [C(r + \beta_{0i}) - C(r)] \psi_i) \frac{\partial Y}{\partial C} \right. \\
& + \frac{1}{2} \sum_{i=1}^{n_w} \left(\zeta_i^2(t, T_C) P^2 \frac{\partial^2 Y}{\partial P^2} - 2\zeta_i(t, T_C) P \sigma_{0i} \frac{\partial C}{\partial r} \frac{\partial^2 Y}{\partial P \partial C} + \sigma_{0i}^2 \left(\frac{\partial C}{\partial r} \right)^2 \frac{\partial^2 Y}{\partial C^2} \right) \Big] dt \\
& + \sum_{i=1}^{n_w} \left(-\zeta_i(t, T_C) P \frac{\partial Y}{\partial P} + \sigma_{0i} \frac{\partial C}{\partial r} \frac{\partial Y}{\partial C} \right) d\widetilde{W}_i(t) \\
& + \sum_{i=1}^{n_p} [\{Y(C(r) + C(r + \beta_{0i}) - C(r), P + P(e^{-\xi_i(t, T_C)} - 1)) - Y(C(r), P)\}] dQ_i(t),
\end{aligned} \tag{A5.2.5}$$

and after substitution of the partial derivatives it is simplified to

$$\begin{aligned}
dY = & \left[-rY + rY + \frac{1}{2} \sum_{i=1}^{n_w} \left(2\zeta_i^2(t, T_C) Y + 2\zeta_i(t, T_C) \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} Y + 0 \right) \right] dt \\
& + \sum_{i=1}^{n_p} \left((e^{-\xi_i(t, T_C)} - \frac{C(r + \beta_{0i})}{C}) \psi_i Y \right) dt \\
& + \sum_{i=1}^{n_w} \left(\zeta_i(t, T_C) Y + \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} Y \right) d\widetilde{W}_i(t) \\
& + \sum_{i=1}^{n_p} Y \left(\frac{C(r + \beta_{0i})}{e^{-\xi_i(t, T_C)} C(r)} - 1 \right) dQ_i(t),
\end{aligned} \tag{A5.2.6}$$

and further to

$$\begin{aligned}
\frac{dY}{Y} = & \sum_{i=1}^{n_w} \left(\zeta_i(t, T_C) + \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} \right) [d\widetilde{W}_i(t) + \zeta_i(t, T_C) dt] \\
& + \sum_{i=1}^{n_p} \left(\frac{C(r + \beta_{0i})}{C(r)} e^{\xi_i(t, T_C)} - 1 \right) [dQ_i(t) - \psi_i e^{-\xi_i(t, T_C)} dt].
\end{aligned} \tag{A5.2.7}$$

Appendix 5.3. Fourier Transform Technique

Define the Fourier transform of the solution $\Upsilon = \Upsilon(Z, t)$ to the partial differential equation (5.3.14) by

$$\overline{\Upsilon}(\omega, t) = \int_{-\infty}^{\infty} \Upsilon(Z, t) e^{-i\omega Z} dZ, \tag{A5.3.1}$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit. Then

$$\int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial t} e^{-\mathbf{i}\omega Z} dZ = \frac{\partial \bar{\Upsilon}(\omega, t)}{\partial t}, \quad (\text{A5.3.2})$$

while¹³

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} dZ &= \Upsilon e^{-\mathbf{i}\omega Z} \Big|_{-\infty}^{\infty} + \mathbf{i}\omega \int_{-\infty}^{\infty} \Upsilon e^{-\mathbf{i}\omega Z} dZ \\ &= \mathbf{i}\omega \bar{\Upsilon}(\omega, t), \end{aligned} \quad (\text{A5.3.3})$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial^2 \Upsilon}{\partial Z^2} e^{-\mathbf{i}\omega Z} dZ &= \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} \Big|_{-\infty}^{\infty} + \mathbf{i}\omega \int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} dZ \\ &= \Upsilon e^{-\mathbf{i}\omega Z} \Big|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} \Upsilon e^{-\mathbf{i}\omega Z} dZ \\ &= -\omega^2 \bar{\Upsilon}(\omega, t). \end{aligned} \quad (\text{A5.3.4})$$

Also note that

$$\begin{aligned} \int_{-\infty}^{\infty} \Upsilon(Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}) e^{-\mathbf{i}\omega Z} dZ \\ &= e^{\mathbf{i}\omega \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}} \int_{-\infty}^{\infty} \Upsilon(Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}) e^{-\mathbf{i}\omega(Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}})} dZ \\ &= \left(\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}} \right)^{\mathbf{i}\omega} \bar{\Upsilon}(\omega, t), \end{aligned} \quad (\text{A5.3.5})$$

Using the results (A5.3.1)-(A5.3.5), the partial differential equation (5.3.14) for $\Upsilon(Z, t)$ becomes an ordinary differential equation with complex coefficients for $\bar{\Upsilon}(\omega, t)$, i.e.,

$$\begin{aligned} \frac{\partial \bar{\Upsilon}(\omega, t)}{\partial t} &= \left(-\mathbf{i}\omega \left[\sum_{i=1}^{n_p} \psi_i(t) (e^{-\xi_i(t,T_C)} - e^{-\xi_i(t,T)}) - \frac{1}{2} \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T))^2 \right] \right. \\ &\quad \left. + \frac{\omega^2}{2} \sum_{i=1}^{n_w} (\zeta_i(t, T_C) - \zeta_i(t, T))^2 \right. \\ &\quad \left. + \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t,T_C)} - \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t,T)} \left(\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}} \right)^{\mathbf{i}\omega} \right) \bar{\Upsilon}(\omega, t). \end{aligned} \quad (\text{A5.3.6})$$

¹³Note that we assume $\Upsilon e^{-\mathbf{i}\omega Z} \Big|_{-\infty}^{\infty} = 0$ and $\frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} \Big|_{-\infty}^{\infty} = 0$. It is necessary later to verify that the solution obtained based on these assumptions satisfies the partial differential equation (5.3.14). The assumption is then justified on the basis of uniqueness of the solution.

Equation (A5.3.6) may be expressed as

$$\begin{aligned} \frac{\partial}{\partial t} \left[\bar{\Upsilon}(\omega, t) \exp \left\{ - \sum_{i=1}^{n_p} \int_0^t \psi_i(s) e^{-\xi_i(s, T_C)} ds \right. \right. \\ \left. \left. + i\omega \left[\sum_{i=1}^{n_p} \int_0^t \psi_i(s) (e^{-\xi_i(s, T_C)} - e^{-\xi_i(s, T)}) ds - \frac{1}{2} \int_0^t \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds \right] \right. \right. \\ \left. \left. - \frac{\omega^2}{2} \int_0^t \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds \right. \right. \\ \left. \left. + \sum_{i=1}^{n_p} \int_0^t \psi_i(s) e^{-\xi_i(s, T_C)} \left(\frac{e^{-\xi_i(s, T)}}{e^{-\xi_i(s, T_C)}} \right)^{i\omega} ds \right\} \right] = 0. \end{aligned} \quad (\text{A5.3.7})$$

Integrating from t to T_C

$$\begin{aligned} \bar{\Upsilon}(\omega, t) = \bar{\Upsilon}(\omega, T_C) \exp \left\{ - \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} ds \right. \\ \left. + i\omega \left[\sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) (e^{-\xi_i(s, T_C)} - e^{-\xi_i(s, T)}) ds - \frac{1}{2} \int_t^{T_C} \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds \right] \right. \\ \left. - \frac{\omega^2}{2} \int_t^{T_C} \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds \right. \\ \left. + \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} \left(\frac{e^{-\xi_i(s, T)}}{e^{-\xi_i(s, T_C)}} \right)^{i\omega} ds \right\}. \end{aligned} \quad (\text{A5.3.8})$$

Let

$$\bar{c}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} ds, \quad (\text{A5.3.9})$$

$$\bar{v}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) (e^{-\xi_i(s, T_C)} - e^{-\xi_i(s, T)}) ds, \quad (\text{A5.3.10})$$

$$\bar{\sigma}^2(t, T_C) = \frac{1}{T_C - t} \int_t^{T_C} \sum_{i=1}^{n_w} (\zeta_i(s, T_C) - \zeta_i(s, T))^2 ds, \quad (\text{A5.3.11})$$

$$\bar{\xi}(\omega, t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \int_t^{T_C} \psi_i(s) e^{-\xi_i(s, T_C)} \left(\frac{e^{-\xi_i(s, T)}}{e^{-\xi_i(s, T_C)}} \right)^{i\omega} ds, \quad (\text{A5.3.12})$$

then equation (A5.3.8) is simplified to

$$\begin{aligned} \bar{\Upsilon}(\omega, t) = \\ \bar{\Upsilon}(\omega, T_C) e^{(T_C - t) [-\bar{c}(t, T_C) + i\omega[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)] - \frac{\omega^2}{2}\bar{\sigma}^2(t, T_C) + \bar{\xi}(\omega, t, T_C)]}. \end{aligned} \quad (\text{A5.3.13})$$

Appendix 5.4. Derivation of Black-Scholes type Integral

We set as I the integral

$$I = \int_{\ln E}^{\infty} (e^Z - E) e^{-\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + \sum_{i=1}^{n_p} p_i \mu_i)^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)}} dZ, \quad (\text{A5.4.1})$$

then by performing further manipulations

$$\begin{aligned} I &= \int_{\ln E}^{\infty} e^Z e^{-\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + \sum_{i=1}^{n_p} p_i \mu_i)^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)}} dZ \\ &\quad - E \int_{\ln E}^{\infty} e^{-\frac{([\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + \sum_{i=1}^{n_p} p_i \mu_i)^2}{2\bar{\sigma}^2(t, T_C)(T_C - t)}} dZ. \end{aligned} \quad (\text{A5.4.2})$$

The change of the variable

$$u = \frac{[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + \sum_{i=1}^{n_p} p_i \mu_i}{\bar{\sigma}(t, T_C)\sqrt{T_C - t}},$$

and the setting of

$$d_2 = \frac{[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln \frac{X}{E} + \sum_{i=1}^{n_p} p_i \mu_i}{\bar{\sigma}(t, T_C)\sqrt{T_C - t}},$$

leads to

$$\begin{aligned} I &= \int_{d_2}^{-\infty} e^Z e^{-\frac{u^2}{2}} (-\bar{\sigma}(t, T_C)\sqrt{T_C - t}) du - E \int_{d_2}^{-\infty} e^{-\frac{u^2}{2}} (-\bar{\sigma}(t, T_C)\sqrt{T_C - t}) du \\ &= \bar{\sigma}(t, T_C)\sqrt{T_C - t} \left(\int_{-\infty}^{d_2} e^{-\frac{u^2}{2} + [\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X + \sum_{i=1}^{n_p} p_i \mu_i - u\bar{\sigma}(t, T_C)\sqrt{T_C - t}} du - E \int_{-\infty}^{d_2} e^{-\frac{u^2}{2}} du \right) \\ &= \bar{\sigma}(t, T_C)\sqrt{T_C - t} \left(e^{[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X + \sum_{i=1}^{n_p} p_i \mu_i} \int_{-\infty}^{d_2} e^{-\frac{u^2}{2} - u\bar{\sigma}(t, T_C)\sqrt{T_C - t}} du - E \int_{-\infty}^{d_2} e^{-\frac{u^2}{2}} du \right) \\ &= \bar{\sigma}(t, T_C)\sqrt{T_C - t} \left(e^{[\bar{v}(t, T_C) - \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln X + \sum_{i=1}^{n_p} p_i \mu_i + \frac{1}{2}\bar{\sigma}^2(t, T_C)(T_C - t)} \right. \\ &\quad \left. \int_{-\infty}^{d_2} e^{-\frac{1}{2}(u + \bar{\sigma}(t, T_C)\sqrt{T_C - t})^2} du - E \int_{-\infty}^{d_2} e^{-\frac{u^2}{2}} du \right). \end{aligned} \quad (\text{A5.4.3})$$

By changing the variable to $U = u + \bar{\sigma}(t, T_C)\sqrt{T_C - t}$, defining d_1 as

$$d_1 = d_2 + \bar{\sigma}(t, T_C)\sqrt{T_C - t} = \frac{[\bar{v}(t, T_C) + \frac{1}{2}\bar{\sigma}^2(t, T_C)](T_C - t) + \ln \frac{X}{E} + \sum_{i=1}^{n_p} p_i \mu_i}{\bar{\sigma}(t, T_C)\sqrt{T_C - t}},$$

and using the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt,$$

we obtain

$$\begin{aligned}
 I &= \bar{\sigma}(t, T_C) \sqrt{T_C - t} \left(e^{\bar{v}(t, T_C)(T_C - t) + \ln X + \sum_{i=1}^{n_p} p_i \mu_i} \int_{-\infty}^{d_1} e^{-\frac{1}{2} U^2} dU - E \int_{-\infty}^{d_2} e^{-\frac{u^2}{2}} du \right) \\
 &= \sqrt{2\pi} \bar{\sigma}(t, T_C) \sqrt{T_C - t} \left(e^{\bar{v}(t, T_C)(T_C - t) + \ln X + \sum_{i=1}^{n_p} p_i \mu_i} \Phi(d_1) - E \Phi(d_2) \right). \quad (\text{A5.4.4})
 \end{aligned}$$

Conclusion and Further Directions for Research

The introduction of jump-diffusion processes into the modelling of the term structure of interest rates allows one to more efficiently model interest rate movements and to better capture the standardized empirical statistical features of interest rates dynamics, thus providing an improved setting to overcome some of the mispricing of derivative securities that arises in the well-known pure diffusion models. The short rate trajectories, observed in bond markets, exhibit jumps from time to time, furthermore the empirical distributions of the short rate show considerable skewness and kurtosis. Further, the combination of jump-diffusion models with stochastic volatility models, as empirical studies demonstrate, provides a class of models that better fit empirical facts. Additionally, the stochastic volatility jump-diffusion setting may well be adapted to interest rate derivative models sensitive to credit risk as the time of default is mathematically described by the first jump time of a jump (e.g. Poisson, Cox, point) process.

The HJM framework constitutes the most general and adaptable framework for the study of interest rate dynamics that accommodates, by construction, consistency with the currently observed yield curve within an arbitrage free environment. However, the general HJM model is Markovian in the entire yield curve thus requiring an infinite number of state variables to determine the future evolution of the yield curve. By imposing appropriate conditions on the forward rate volatility, the HJM model can admit finite dimensional Markovian structures, a feature that makes this class of models computationally tractable.

Within jump-diffusion versions of the HJM framework, this thesis has investigated specific stochastic volatility term structure models (by the means of state dependent forward rate volatility specifications) that lead to finite dimensional Markovian representations. Further, by seeking an expression of the state variable in terms of market observed quantities, finite dimensional affine realisations of the term structure in terms of forward rates and yields have been derived for both default-free and defaultable market conditions. The thesis provides some applications of these general Markovian term structure models, by

developing jump-diffusion extensions of the Hull & White (1990), (1994) class of models and the Ritchken & Sankarasubramanian (1995) class of models. In addition, by performing Monte Carlo simulations of the Markovian spot rate dynamics, the thesis has demonstrated the ability of these classes of models to efficiently capture the stylised empirical facts of interest rate movements such as excess skewness and kurtosis. The thesis has finished with the pricing of bond options within the proposed jump-diffusion framework. Under deterministic volatility specification, closed form bond option prices are derived, whereas under the more general stochastic volatility set-up, numerical schemes are employed.

6.1. Markovianisation of Jump-Diffusion Versions of the HJM Model

By imposing appropriate conditions on the forward rate volatility, the HJM model can admit finite dimensional Markovian structures where the generality of the HJM models coexists with the computational tractability of Markovian structures. This thesis has built on earlier work on the Markovianisation of HJM models such as Cheyette (1992), Carverhill (1994), Ritchken & Sankarasubramanian (1995), Bhar & Chiarella (1997), Inui & Kijima (1998), de Jong & Santa-Clara (1999), Björk & Landèn (2002), Björk & Svensson (2001) and Chiarella & Kwon (2001b),(2003). The thesis focuses in particular on the Markovianisation of jump-diffusion versions of the HJM model.

Chapter 2 has considered a multi-factor jump-diffusion model of the HJM term structure of interest rates. In particular, a generalisation of the Shirakawa (1991) framework was developed, where the instantaneous forward rate displayed discontinuous dynamics driven by multiple Wiener and Poisson noise terms. The study has started, for completeness purposes, with the case of deterministic volatilities, followed by the more general case of state dependent volatility structures. Markovian spot rate and bond price dynamics were obtained under both of these two volatility settings. In particular, the state dependent volatility structures consist of state dependent Wiener and deterministic Poisson volatility specifications, since as was explained later in the thesis, under state dependent Poisson volatility specifications, it becomes difficult to obtain Markovian representations of the system. In the latter case, we have settled for an “approximate” Markovian structure. In addition, finite dimensional affine realisations of the term structure in terms of forward

rates and yields were obtained, making the model suitable for calibration and leading to an exponential affine bond price formula in terms of these forward rates. The interaction of the Markovian representations with the capability of the models to incorporate economically interpretable state variables are the features that makes these classes of models suitable for calibration applications and for more precise parameter estimations. The full exploitation of these applications are left to future research.

These two volatility settings have allowed us in Chapter 3 to create what we believe is the natural extension of the Hull & White (1990), (1994) class of models and the Ritchken & Sankarasubramanian (1995) class of models to the jump-diffusion case. Chapter 4 has shown how the framework developed here can be extended to deal with defaultable term structure models.

Chapter 3 has shown that the state dependent volatility models (RS) have a tendency to generate forward rate curves with sharper curvature changes than the equivalent deterministic volatility models (HW). This is probably due to the fact that the state dependent volatility models incorporate a larger number of state variables, which has made the model more flexible and able to capture more realistic forward rate behavior. By performing Monte Carlo simulations of the Markovian instantaneous spot rate dynamics, Chapter 3 has investigated the properties and distributional profiles of the HW and RS class of models when they are extended to incorporate jumps. Both jump-diffusion models, HW and RS, exhibit asymmetric distributions (with a long tail to the right if positive jump sizes dominate), a feature that becomes more pronounced, as jump volatility levels increase. The state dependent volatility models capture more effectively the asymmetric feature of the empirical spot rate distribution compared to deterministic volatility models.

By way of conclusion, the outcome of the Monte Carlo simulations of the Markovian spot rate dynamics have demonstrated the ability of this class of models to efficiently capture the stylised empirical facts of interest rate movements such as excess skewness and kurtosis, thus this class of models could provide more accurate derivative security pricing and econometric estimation. Additionally and more importantly, the tractability of the Markovian structures obtained provides an efficient and more accurate basis for Monte Carlo simulations, that may be employed for derivative pricing as is presented in Chapter 5.

6.2. Markovianisation of Defaultable HJM Models

In the reduced form credit models, such as the ones developed by Jarrow & Turnbull (1995), Das & Tufano (1996), Jarrow et al. (1997), Duffie & Singleton (1997), Madan & Unal (1998), Schönbucher (1998), Schönbucher (2000) and Bielecki & Rutkowski (2002) default is triggered by exogenous sources in an unpredictable manner, providing a more realistic modelling set-up, however the empirical implementation of such models is still quite limited. The extension of the jump-diffusion versions of the HJM framework to the defaultable case may be regarded as an excellent modelling platform within the intensity type of models that would generate a tractable class of defaultable models appropriate for numerical applications.

In Chapter 4, a parameterisation of the Schönbucher (2000), (2003) general HJM framework has been examined where jumps in the defaultable term structure cause jumps and defaults to defaultable bond prices. A specific formulation of state and time dependent volatility specifications, under deterministic default intensity, lead to Markovian defaultable spot rate and defaultable bond price dynamics. The state variables of this model can be expressed in terms of a finite number of benchmark defaultable forward rates. To make the model more realistic, the case of a stochastic credit spread has been investigated, in which case it becomes difficult to obtain Markovian representations of the system. Then an “approximate” Markovian structure or constant Poisson volatilities were proposed.

In order to investigate the impact of the volatility (in particular the jump volatility) specifications on the distributional profile of the defaultable spot rate, Monte Carlo simulations were performed, under both the assumptions of deterministic default intensities and stochastic default intensities. As the simulations have demonstrated, the stochastic intensity models display more pronounced leptokurtic effects compared to the deterministic intensity models. Also, as jump volatilities increase, under both settings for the model, the spot rate distributions become more asymmetric with a long tail to the right for positive jump sizes or a long tail to the left for negative jump sizes. We have explained also how this model extends the RS default-free stochastic volatility model to a defaultable stochastic volatility term structure model under jump-diffusions.

The proposed defaultable term structure developed in this chapter combines the tractability of Markovian representations and the richness of stochastic volatility jump-diffusion

models and as the numerical simulations show succeeds in capturing the stylised empirical features of the distributions of defaultable interest rates. Therefore, this Markovian class of defaultable models that incorporates the realistic characteristics of stochastic volatility jump-diffusion defaultable forward rate dynamics combined with stochastic default intensities, may be employed for more accurate credit derivative pricing and hedging as well as model calibration and econometric estimation techniques. Further development of these research directions is left for further research.

Another matter of ongoing research is the feature of the model which allows non-default events to trigger jumps in defaultable interest rates which seems particularly well-suited for an extension to a multi-obligor framework, when the default of one obligor triggers a jump in credit spreads faced by other obligors.

6.3. Bond Option Pricing under a Markovian HJM Term Structure with Jumps

The price of an interest-rate option is substantially affected by the presence of skewness and kurtosis in the interest rates. The empirically observed features of the smile effect in option prices and of the leptokurtic distributions of the interest rates, can be captured by jump-diffusion and stochastic volatility models that however accommodate an increasing complexity that makes it impossible to derive computationally tractable solutions in most cases of interest.

Chapter 5 has contributed to this area by developing two models to price bond options that incorporate the richness of a stochastic volatility and/or jump-diffusion model whilst remaining computationally tractable. In the first model, both Wiener and Poisson volatilities are time dependent. Within the Shirakawa general HJM model and by employing Fourier transform techniques, bond option prices have been evaluated and a tractable Black-Scholes type bond option pricing formula under the assumption of constant jump volatility has been derived. In the second model, in which we allow for state dependent Wiener volatilities and time dependent Poisson volatilities, it seems difficult if not impossible to explicitly solve the bond option pricing problem, therefore Monte Carlo simulation techniques are used to evaluate bond options. However, under suitable volatility specifications as discussed in Chapter 2, both models admit finite dimensional Markovian

structures. Performing Monte Carlo simulations of these Markovian structures, bond options prices are compared to the exact closed form solutions for the deterministic volatility model. These Markovian representations have contributed to increase the efficiency and accuracy of the application of the Monte Carlo simulations. Further, the availability of closed form solutions under the deterministic volatility model facilitates the application of a control variate method to evaluate the bond option prices of the state dependent volatility model, that has improved even further the convergence and has reduced significantly the computational effort of the numerical scheme.

These classes of models provide an appealing modelling setup for calibration applications and econometric estimations. A fit to empirical information may developed to calibrate the model parameters as well as the volatility smile. As jumps and stochastic volatility capture the stylized facts of interest rate markets, the bond option prices obtained from these models should reflect these features, and thus result in a more accurate parameter estimation.

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