Choosing Your Own Luck: Strategic Risk Taking and Effort in Contests*

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Abstract

We study contests where, in addition to exerting effort, agents can design fair risk to strategically add noise to their performance. Strategic risk taking removes inefficiencies associated with randomization of effort in the contest, but introduces moral hazard. We show that Pareto improvements over contests without risk taking can be achieved. Moreover, under an appropriate "stop-loss" restriction imposed on risk taking, maximum feasible effort can be extracted from the agents.

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Keywords: Contest; Gambling; Strategic Risk Taking; Noise

1 Introduction

Contests—pay schemes based on ordinal performance comparisons—are used extensively to motivate agents in organizations and other settings.[1] A popular model of contests is the *all-pay contest*, where agents exert effort at a cost, and are then rewarded on the basis of their rank of output. In such models, output is a deterministic function of effort. On the other hand, a distinct but nonetheless useful approach was pioneered by Lazear and Rosen (1981), where once again agents are compensated on the basis of the rank of their output, but where the link between effort and output is stochastic.

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[1] Examples include promotions and bonuses (Bognanno, 2001; Baker et al., 1988), sales contests (Lim et al., 2009), forced ranking systems (Bretz Jr. et al., 1992), and R&D competition (Terwiesch and Ulrich, 2009).

The assumption of noisy output is both natural and compelling. For instance, in a contest for a contract to build a public utility, say, the competing firms will (obviously) spend a lot of time perfecting their design, but the final choice will not depend on the amount of time spent, or any particular detail necessarily. Instead, it will depend on the decision-maker's preferences (which itself may consist of a committee), and so may be random in the sense that certain aspects of each proposal may appeal to the decision maker in different—and from the point of view of the firm, *random*—ways.

An important caveat to the justification above is that how the decision maker reacts to a proposal depends on the proposal itself. That is, by choosing the particulars of the proposal, be it the size or the shape or the functionality and style of certain elements of the design, the firm is effectively *choosing* the nature of the randomness in the choice. This is tantamount to letting the firm *gamble*, in addition to choosing a base level of effort. Thus, a common feature in many real world contests is that, in addition to exerting effort, agents have an opportunity to take calculated risks that may further increase or decrease their performance. For another example, consider a researcher who can choose to spend resources to pursue a conventional project with predictable returns or a risky project with the same expected returns, but a much larger variance in possible outcomes. Similarly, investors and portfolio managers can manipulate asset allocations to choose how much and what types of risk to take.

In this paper, we study contests where agents can choose both effort and risk. Specifically, agents can add fair gambles to modify their effort. As compared to the existing literature, we introduce two important and realistic innovations: (i) risk taking is restricted by a "stop-loss" non-negativity constraint on output; and (ii) risk is otherwise unrestricted. In other words, agents can *design* arbitrary fair gambles around their effort, as long as their output remains non-negative.[2] These two ingredients allow us to resolve the puzzle outlined above. We show that allowing agents to gamble can be Pareto-improving, i.e., lead to higher effort *and* higher earnings for the agents as compared to conventional contests without gambling.[3] Moreover, by further restricting risk taking it is possible to extract maximum feasible effort from the agents.

As a starting point, and for a stark comparison, we consider a symmetric winner-take-all contest where agents compete for a fixed prize only by exerting effort. In equilibrium, agents randomize effort and earn no rents. When the cost of effort is convex, such randomization is inefficient. Indeed, the same (expected) effort exerted deterministically would be less costly or, equivalently, a

[2] Unrestricted risk taking in contests has been considered in the literature (e.g., Myerson, 1993; Seel and Strack, 2013); however, those models study "pure" risk taking without effort choice. For a detailed review of the literature, see below.

[3] Relatedly, Morgan et al. (2018) show that "too much meritocracy"—having too little noise in the winner determination process—can be detrimental for aggregate effort in contests. In their setting, noise is exogenous and the effect is driven by agents dropping out of the contest when noise becomes small.

higher effort can be reached at the same expected cost. Randomization, however, is necessary for the agents to remain in equilibrium in this complete information environment.

Suppose now that agents can gamble with their performance by adding zero-mean noise to their—in general, still stochastic—effort. Suppose also that the resulting output is restricted to be non-negative; that is, downside risk can be at most the agent's realization of effort. The introduction of gambling has two effects. First, agents no longer need to randomize their effort in equilibrium, which raises efficiency. Second, the game turns into a rank-order tournament where the symmetric equilibrium effort is deterministic, and the distribution of noise is determined endogenously following a concavification argument similar to the one used in the information design literature (e.g., Kamenica and Gentzkow, 2011). The noise introduces moral hazard and, in general, allows agents to earn positive rents. Yet, we show that in sufficiently large contests a higher expected effort can be achieved as compared to the contest without gambling, i.e., the introduction of gambling is Pareto-improving. We also obtain a number of stochastic dominance results showing, in particular, that expected top performance is also large in the contest with gambling.

The effect of endogenous gambling in such contests has another important feature. It simplifies, tremendously, the analysis of the *optimal* schedule of rank-dependent rewards for a principal allocating a fixed budget. The analysis of the all-pay contest model suggests that more equitable prize schedules induce more effort when costs are convex (Fang et al., 2020).[4] For rank-order tournaments with exogenous noise, the optimal allocation of prizes changes between winner-take-all and equitable prize sharing depending on the details of noise (Drugov and Ryvkin, 2020). In contrast, we show that when risk taking is endogenous, effort is independent of the allocation of prizes, and under an appropriate stop-loss restriction (so that output must be above some minimum levels), the maximum feasible effort can be achieved under any prize schedule.

To put our results in context, observe that risk taking adds an element of luck, or noise, to performance, which is generally believed to have a detrimental effect on incentives to exert effort in contests. For example, Lazear and Oyer (2012), p 485, write: "An important variable in the Lazear and Rosen (1981) model is the amount of noise—that is, the degree to which luck affects the probability of winning. When there is more noise (so that luck becomes relatively more important and effort relatively less important), workers will try less hard to win, because effort has a reduced effect on whether they win." Thus, received wisdom suggests that when agents can *choose* how much noise to inject into their performance, they should, in equilibrium, take the largest possible amount of risk and reduce effort to a minimum. Indeed, this is the central finding of Hvide (2002). Yet, contest-like incentives are very common in settings such as R&D competition and the financial sector, where high effort coexists with significant and *endogenous* risk taking. We resolve this puzzle by observing that real world competitions feature *both* stop-loss restrictions (which are

[4]Similar results are obtained by Moldovanu and Sela (2001) for all-pay contests with private information.

crucially absent in Hvide (2002)) as well as endogenous gambles. This suggests that real world contests are more efficient than previous analyses might suggest.

Our stop-loss restriction requires that gambles can only result in output above a certain, pre-specified level, which may typically be thought to be zero. Of course, the restriction of gambling to non-negative performance is only one possibility. As noted above, we show that if the downside risk of gambling is further restricted to a *positive* threshold performance, the principal can extract all the rents and push the equilibrium effort to the largest achievable level satisfying the agents' participation constraint. This optimal gambling scheme is rather intuitive, and also practical: The agents are allowed to gamble with the output they produce, but only up to a point.[5]

Related literature Our paper contributes to the vast literature on contests and tournaments utilizing the complete information all-pay auction model (e.g., Hillman and Riley, 1989; Siegel, 2009; Fang et al., 2020) and the noisy tournament model (e.g., Lazear and Rosen, 1981; Ryvkin and Drugov, 2020; Drugov and Ryvkin, 2020). To the best of our knowledge, we are the first to explore a unified framework combining both models and to provide a direct comparison of effort in the two settings, albeit with endogenously selected noise.

We also contribute to the literature on strategic risk-taking, or gambling, in contests. The most relevant strand of this literature is the one dealing with (partially) "unrestricted" risk-taking, where agents compete by choosing fair gambles with arbitrary distributions subject to some constraints. Such gambles arise naturally in equilibrium in models of political competition (Myerson, 1993; Lizzeri, 1999) and competition for status (Becker et al., 2005; Ray and Robson, 2012). Fang and Noe (2018) explore risk-taking in contests of heterogeneous agents where the principal's goal is to select a high-ability employee. Seel and Strack (2013) consider a dynamic contest where each agent's score is a Brownian motion with a drift and an absorbing state at zero. The agents privately observe their states and decide when to stop; the agent with the highest score at stopping wins. In this setting, agents effectively choose the distribution of their stopping time, and its feasibility is demonstrated using a result on Skorokhod embeddings. Importantly, these papers focus on gambling only and abstract from costly effort choices as the additional strategic dimension. A number of authors look at the effects of "restricted" risk-taking, whereby agents can only choose a parameter, such as variance, of a fixed distribution of noise. Here, too, most authors focus on pure risk-taking Gaba and Kalra (1999); Hvide and Kristiansen (2003); Taylor (2003); Gaba et al. (2004), but two papers—Hvide (2002) and Gilpatric (2009)—allow agents to choose both effort and the variance of noise. Hvide (2002) shows that the largest possible variance, and lowest effort, is chosen in equilibrium. Gilpatric (2009) considers a model where increasing variance is costly

[5]We also consider an alternative setup where agents can gamble as long as their performance is non-negative, but only receive a prize if it exceeds a positive threshold. In this case, the equilibrium distribution of output acquires a mass at zero, and efficiency is lost.

and shows that first-best effort and flexible levels of variance can be implemented by an appropriate choice of prizes that includes a reward for top performers and a punishment for bottom performers.

The rest of the paper is structured as follows. In Section 2, we set up a model of contests with strategic gambling, characterize the equilibrium, and compare it to the baseline model without gambling. In Section 3, we study the effects of additional restrictions on gambling and characterize optimal contest design. Several extensions and robustness checks are discussed in Section 4, and Section 5 concludes.

2 The Gambling Contest

2.1 Setup

As a starting point, we consider the canonical winner-take-all contest. There are $n \geqslant 2$ risk-neutral players (contestants) competing for a prize whose value is normalized to 1. They simultaneously choose their effort subject to symmetric convex costs.[6] We use $x_i \in \mathbb{R}_+$ to denote player i's effort choice and $c(x_i)$ to denote the associated cost.[7] We assume that $c(\cdot)$ is twice differentiable, strictly increasing from zero, and strictly convex (i.e., $c(0) = 0$ and $c'(x), c''(x) > 0$ for all $x > 0$). We also impose a mild technical restriction that the elasticity of the cost function is bounded at zero (i.e., $\limsup_{x \to 0} xc'(x)/c(x) < \infty$).[8]

Given x_i, each player i can run any fair gamble to obtain a distribution H_i of non-negative *output*. Specifically, each player i can add a random noise ε_i to his effort x_i, so that his final output is a random variable $Y_i \equiv x_i + \varepsilon_i$. Unlike in the canonical stochastic-output model (e.g., Lazear and Rosen, 1981), but as in models of strategic risk taking (e.g., Myerson, 1993), each player can choose *any* random noise ε_i subject to the constraints that $\mathbb{E}_i[\varepsilon_i] = 0$ (fair gamble) and $Y_i = x_i + \varepsilon_i \geqslant 0$ almost surely (no bankruptcy). Note that this flexible risk-taking situation naturally arises if the set of available fair gambles is sufficiently rich, or a player can design a gamble and propose it to a risk-neutral third party.

The player who produces the highest realization of output y_i wins the contest and earns the prize. For completeness, we assume that ties are broken through fair randomization; that is, if multiple players produce the same highest output then each of them is selected as the winner with equal probability. However, as is often the case and clarified shortly, ties occur with probability zero in equilibrium. In what follows, we simplify the notation by ignoring the possibility of ties.

[6] As we formally explain in Section 4.2, strategic risk taking plays no role if effort costs are concave.

[7] In general, players are allowed to randomize their effort; however, such randomization is never optimal when costs are strictly convex.

[8] This condition holds for any analytic function or when $c(x) = Ax^k$ (with $A, k > 0$). It fails, for example, for $c(x) = e^{-1/x}$, which rises at zero slower than *any* polynomial.

Specifically, we assume that each individual player chooses his effort x_i and a fair gamble ε_i so as to maximize

$$\mathbb{P}\{Y_i = x_i + \varepsilon_i > Y_j = x_j + \varepsilon_j \text{ for all } j \neq i\} - c(x_i) \text{ s.t. } \mathbb{E}[Y_i] = x_i \text{ and } Y_i \geqslant 0 \text{ almost surely.}$$

2.2 Equilibrium Characterization

Suppose all players choose the same deterministic effort x.[9] Then, since effort costs are sunk, our model reduces to the canonical strategic risk-taking model, in which each player i chooses a random variable Y_i (equivalently, a distribution of output G_i) subject to the mean constraint $\mathbb{E}[Y_i] = \int_0^\infty y dG_i(y) = x$ in order to maximize $\mathbb{P}\{Y_i > Y_j \text{ for all } j \neq i\}$. The following characterization is well-known in the literature (see, e.g., Theorem 2 of Myerson, 1993).

Lemma 1 *Consider a game in which each player i independently chooses a non-negative random variable Y_i subject to $\mathbb{E}[Y_i] = x(> 0)$ and his payoff is given by $\mathbb{P}\{Y_i > Y_j \text{ for all } j \neq i\}$. The game has a unique symmetric Nash equilibrium in which each player chooses Y_i with the distribution G such that $G(y)^{n-1} = \min\{y/(nx), 1\}$.*

To understand Lemma 1, consider an individual player's problem when all other players follow G. The player's problem can be written as

$$\max_{G_i \in \Delta(\mathbb{R}_+)} \int G(y)^{n-1} dG_i(y) \text{ s.t. } \int y dG_i(y) = x.$$

This is a linear programming problem familiar in the literature on Bayesian persuasion (Kamenica and Gentzkow, 2011; Aumann and Maschler, 1995) and strategic risk taking (e.g., Myerson, 1993; Fang and Noe, 2018), for which the method of concavification can be used to identify an optimal solution.[10] Given the concave piece-wise linear structure of $G(y)^{n-1} = \min\{y/(nx), 1\}$ (see the left panel of Figure 1), it is immediate that G_i is optimal if and only if it assigns all probability to $[0, nx]$. Clearly, the given distribution G satisfies this property, so G is indeed a symmetric equilibrium.[11]

[9] It will be shown that in our gambling contest with convex costs, the equilibrium effort choice is indeed deterministic. In other words, there is no symmetric equilibrium in which the players randomize over efforts. As shown in Section 4.2, this result does not hold if effort costs are concave. In that case, however, gambling plays no role in our gambling contest, so the resulting equilibrium is identical to that of the standard contest without gambling.

[10] Any optimal distribution should assign probability only to those y's that lie on the concave upper envelope of the value function $G(y)^{n-1}$, and the maximized value coincides with the value of the concave envelope at the mean x.

[11] The uniqueness of symmetric equilibrium follows from the fact that if $G(y)^{n-1}$ is not linear over its support, then an individual player's optimal distribution, denoted by G_i^*, cannot coincide with the given distribution. For example, if $G(y)^{n-1}$ is strictly concave over its support, then G_i^* is the degenerate distribution on x, which cannot coincide with the given G. If $G(y)^{n-1}$ is strictly convex over its support, then G_i^* is a binary distribution that puts probability mass on the lower and the upper bounds of $\text{supp}(G)$, which also cannot coincide with the given G.

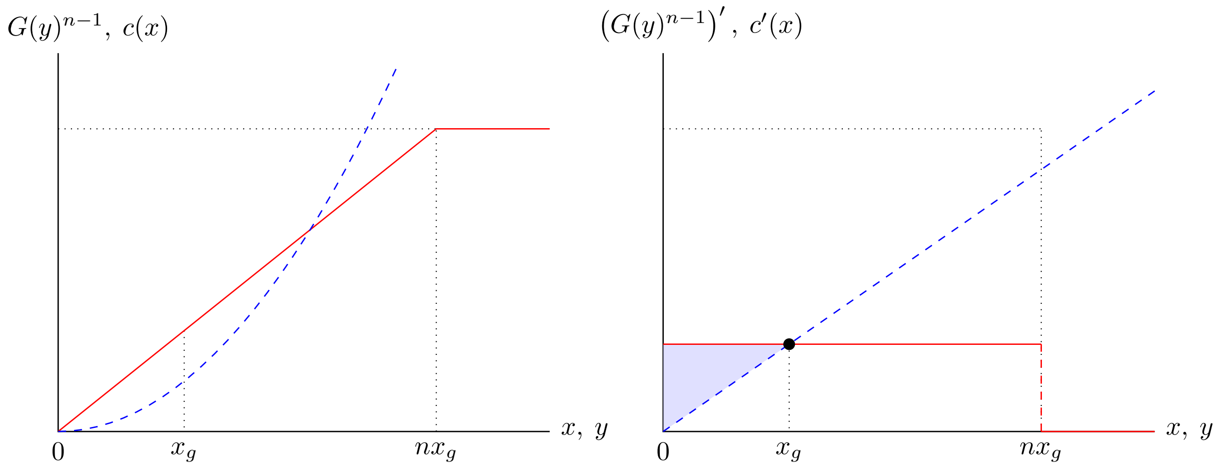

Figure 1: The left panel depicts the value function $G(y)^{n-1} = \min\{y/(nx), 1\}$ in Lemma 1 and Proposition 1 (solid) and the cost function $c(x)$ (dashed), while the right panel shows their first derivatives. This is based on the following parametric specification: $n = 3$ and $c(x) = x^2$.

The main innovation of our gambling contest, as compared to existing models of (flexible) strategic risk taking, is that the players' choice of efforts x_i is endogenous. To find a symmetric equilibrium, suppose all players $j \neq i$ choose effort x and follow the gambling strategy as in Lemma 1. Then, player i's effort choice problem is given by

$$\max_{x_i \in \mathbb{R}_+} u(x_i) - c(x_i), \text{ where } u(x_i) \equiv \max_{G_i \in \Delta(\mathbb{R}_+)} \int G(y)^{n-1} dG_i(y) \text{ s.t. } \int y dG_i(y) = x_i.$$

In other words, if $u(x_i)$ represents player i's indirect (maximized) expected benefit with effort x_i then his problem is to choose x_i that maximizes $u(x_i) - c(x_i)$. The concave structure of $G(y)^{n-1}$ implies that the degenerate distribution δ_{x_i} is *always* optimal, so we have $u(x_i) = G(x_i)^{n-1} = \min\{x_i/(nx), 1\}$ for *all* x_i. It then follows that the symmetric equilibrium value of x should satisfy

$$u'(x) - c'(x) = \frac{1}{nx} - c'(x) = 0 \Leftrightarrow xc'(x) = \frac{1}{n}.$$

Proposition 1 *In the gambling contest, there exists a unique symmetric equilibrium, in which each player i chooses effort x_g such that $x_g c'(x_g) = 1/n$ and adds noise ε_i so that the distribution of output $Y_i = x_g + \varepsilon_i$ satisfies $G(y)^{n-1} = \min\{y/(nx_g), 1\}$.*

Proof. The previous argument suffices to prove that the given strategy profile is indeed an equilibrium. In Appendix A, we prove that there does not exist any other equilibrium (in particular, the one in which the players randomize over efforts). ■

The symmetric equilibrium of the gambling contest has two notable properties. First, it induces a deterministic effort, which, as illustrated in the next section, is the most crucial difference from the pure-effort model. Second, the players obtain positive rents: either see the shaded region in the right panel of Figure 1, or observe that each player's equilibrium expected payoff is equal to

$$\frac{1}{n} - c(x_g) > \frac{1}{n} - x_g c'(x_g) = 0,$$

where the inequality is because $c(x)$ is strictly convex, so $c(x_g)/x_g < c'(x_g)$. Both of these are driven by the equilibrium "linearization" effect of strategic risk taking, namely, that in equilibrium, each player should face a value function that is linear over a relevant region.

2.3 Pure Effort vs. Strategic Risk Taking

In this section, we compare the equilibrium effort x_g from the gambling contest to the equilibrium (expected) effort from the pure-effort contest (i.e., the standard all-pay contest). The following characterization is well known in the literature.

Proposition 2 *In the pure-effort contest, there exists a unique symmetric equilibrium, in which each player randomizes effort over* $[0, c^{-1}(1)]$ *according to the distribution* F *such that*

$$F(x)^{n-1} = \min\{c(x), 1\}.$$

Effort randomization is necessary because a player has an incentive to outpace the other players as long as his effort is below $c^{-1}(1)$, but he also has an incentive to minimize his effort. As is well-known, in a symmetric contest this dissipates all rents to the players, i.e., the players' expected payoffs are equal to zero. Provided all other players randomize according to F, if a player chooses effort x then he wins the contest with probability $F(x)^{n-1}$. Combining this with the structure of F in Proposition 2, his effort choice problem becomes

$$\max_{x \in \mathbb{R}_+} F(x)^{n-1} - c(x) = \min\{c(x), 1\} - c(x) = \begin{cases} 0 & \text{if } x \leqslant c^{-1}(1) \\ 1 - c(x) < 0 & \text{if } x > c^{-1}(1). \end{cases}$$

Therefore, the player is indifferent over all efforts in $[0, c^{-1}(1)]$—each yielding zero expected payoff—and prefers them to any effort above $c^{-1}(1)$.

In order to evaluate the incentive-provision effect of strategic risk taking, we compare the (deterministic) equilibrium effort x_g of our gambling contest to the expected equilibrium effort x_e of the pure-effort contest. By Proposition 2, the latter is given as follows:

$$x_e \equiv \int x dF(x) = \int_0^{c^{-1}(1)} x dc(x)^{\frac{1}{n-1}}.$$

At first glance it may seem that $x_e > x_g$ is likely to hold. Intuitively, gambling dampens an agent's incentive to exert effort, as he can compensate lower effort with more aggressive gambling. Furthermore, as shown above, the players receive zero rents in the pure-effort contest, while they earn strictly positive rents in the gambling contest. Since the prize is fixed at 1 and each player wins with probability $1/n$ in both contests, this seems to suggest that the players exert higher efforts in the pure-effort contest.

The following result—the main economic result of this section—shows that the above intuition clearly fails. As shown below with specific examples, the comparison is ambiguous for n relatively small. However, if n is sufficiently large then the gambling contest necessarily induces higher efforts than the pure-effort contest.[12]

Proposition 3 *For any strictly convex $c(\cdot)$, $x_g > x_e$ for n sufficiently large.*

Proof. Recall that $x_e = \int_0^{c^{-1}(1)} x dc(x)^{1/(n-1)}$, while $x_g c'(x_g) = 1/n$. Clearly, both x_e and x_g converge to zero as $n \to \infty$; therefore, to analyze their large-n behavior it is convenient to consider aggregate efforts, nx_e and nx_g. For the pure-effort contest, we have

$$\lim_{n\to\infty} nx_e = \lim_{n\to\infty} n \int_0^{c^{-1}(1)} x dc(x)^{\frac{1}{n-1}} = \lim_{n\to\infty} \frac{n}{n-1} \int_0^{c^{-1}(1)} \frac{xc'(x)}{c(x)^{\frac{n-2}{n-1}}} dx = \int_0^{c^{-1}(1)} \frac{xc'(x)}{c(x)} dx < \infty,$$

where we used that the elasticity of the cost function is bounded over $[0, c^{-1}(1)]$.

For x_g, let $\phi(x) \equiv xc'(x)$. Since $\phi'(x) = c'(x) + xc''(x) > 0$ for all $x > 0$, its inverse ϕ^{-1} is well-defined. Then, we have

$$\lim_{n\to\infty} \frac{1}{nx_g} = \lim_{n\to\infty} \frac{1}{n\phi^{-1}(1/n)} = \lim_{z\to 0} \frac{z}{\phi^{-1}(z)} = \lim_{x\to 0} \frac{\phi(x)}{x} = \lim_{x\to 0} c'(x) = c'(0). \tag{1}$$

The above implies that for n sufficiently large,

$$x_g > x_e \Leftrightarrow c'(0) \int_0^{c^{-1}(1)} \frac{xc'(x)}{c(x)} dx < 1.$$

This inequality trivially holds if $c'(0) = 0$. For the case when $c'(0) > 0$, note that strict convexity of $c(\cdot)$ implies that $c'(0)x < c(x)$ for all $x > 0$. Then, we have

$$c'(0) \int_0^{c^{-1}(1)} \frac{xc'(x)}{c(x)} dx < \int_0^{c^{-1}(1)} \frac{c(x)c'(x)}{c(x)} dx = \int_0^{c^{-1}(1)} c'(x) dx = c(c^{-1}(1)) = 1.$$

■

[12] The simplified proof below uses the assumption that elasticity $c'(x)x/c(x)$ is bounded at zero. A more general proof is in Appendix A.

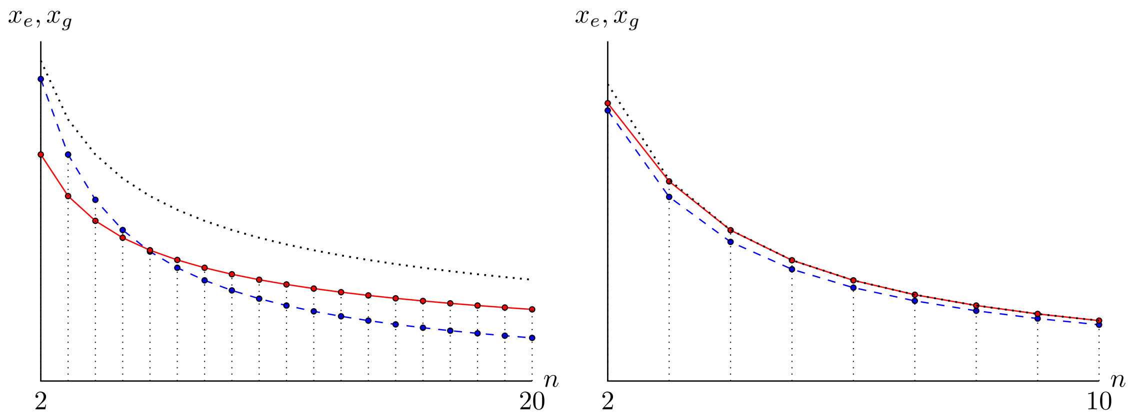

Figure 2: Both panels compare x_g (solid red) to x_e (dashed blue) and $\overline{x}$ (dotted black). In the left panel $c(x) = x^2$, while $c(x) = 0.75x + 0.25x^8$ in the right panel.

Example 1 *Suppose $c(x) = x^k$ for some $k > 1$. In this case, x_g and x_e can be obtained in closed form as follows:*

$$x_e = \frac{k}{k+n-1} \text{ and } x_g = \left(\frac{1}{nk}\right)^{\frac{1}{k}}.$$

It can be shown that for any $k > 1$, if $n = 2$ then $x_e > x_g$.[13] *On the other hand, for any k, there exists $\overline{n}(k)(> 2)$ such that $x_e < x_g$ if and only if $n \geqslant \overline{n}(k)$. For example, $\overline{n}(k) = 5$ for $k = 2, ..., 8$; $\overline{n}(1.5) = \overline{n}(9) = 6$; and $\overline{n}(50) = 8$.*

To understand Proposition 3, first notice that the pure-effort equilibrium in Proposition 2 is inefficient because each player's randomization itself produces a deadweight loss: if the players simply chose x_e deterministically then they would obtain strictly positive rents, because $c(\cdot)$ is strictly convex and F is a non-degenerate distribution, so by Jensen's inequality and Proposition 2,

$$\frac{1}{n} - c(x_e) = \frac{1}{n} - c\left(\int x dF(x)\right) > \frac{1}{n} - \int c(x) dF(x) = 0.$$

In contrast, in the gambling equilibrium in Proposition 1, the players choose a deterministic effort x_g, so there is no deadweight loss due to effort randomization. However, they obtain positive rents, implying that they incur insufficient effort costs.

Which contest induces a higher expected effort depends on which of the above two types of

[13] This result is non-generic. Even for $n = 2$, there are cost functions such that $x_g > x_e$. For example, consider $c(x) = \alpha x + (1-\alpha)x^k$ for some $\alpha \in (0,1)$ and $k > 1$. In this case,

$$x_g c'(x_g) = \alpha x_g + (1-\alpha)k x_g^k = \frac{1}{2} \text{ and } x_e = \int_0^1 x dc(x) = \frac{\alpha}{2} + \frac{(1-\alpha)k}{k+1}.$$

It can be directly shown that if k is sufficiently large then $x_g > x_e$.

inefficiency prevails. This immediately explains why the comparison between x_e and x_g is ambiguous in general. The clear result for n sufficiently large holds because the inefficiency due to insufficient effort costs asymptotically vanishes as n tends to infinity. To see this clearly, let $x^\circ = c^{-1}(1/n)$ denote the highest symmetric effort level the players can choose subject to their participation constraint—namely, that their expected payoffs should be non-negative. Now observe that

$$\lim_{n\to\infty} \frac{1}{nx^\circ} = \lim_{n\to\infty} \frac{1/n}{c^{-1}(1/n)} = \lim_{z\to 0} \frac{z}{c^{-1}(z)} = c'(0) = \lim_{n\to\infty} \frac{1}{nx_g},$$

where the last equality is from (1) in the proof of Proposition 3. Thus, when $c'(0) > 0$, all three efforts—x_e, x_g and x°—approach zero at the same rate $O(1/n)$ as n tends to infinity. However, the ratio x_e/x_g (and x_e/x°) remains less than 1, whereas x_g/x° converges to 1, i.e., the gambling contest asymptotically achieves the maximum effort while the pure-effort contest never does. The difference is even starker when $c'(0) = 0$. In this case x_e still tends to zero as $O(1/n)$, but x_g and x° converge to zero at a slower rate.[14]

In fact, an even stronger result can be established.

Proposition 4 *For any strictly increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(0)$ is finite and the integral $\int_0^a \frac{u(x)}{x}dx$ exists for any $a > 0$,*

$$\int u(x)dG(x) > \int u(x)dF(x) \tag{2}$$

for n sufficiently large.

Proof. Recall that $\text{supp}(F) = [0, c^{-1}(1)]$ and $\text{supp}(G) = [0, nx_g]$. First, we show that $nx_g > c^{-1}(1)$ for n sufficiently large. For $c'(0) = 0$ the result follows immediately because nx_g is unbounded for $n \to \infty$. Suppose $c'(0) > 0$. In this case,

$$\lim_{n\to\infty} nx_g = \frac{1}{c'(0)} = (c^{-1})'(0) > c^{-1}(1),$$

where the inequality is due to the strict concavity of $c^{-1}(\cdot)$.

[14]Exactly what rate it is depends on the shape of $c(\cdot)$. For example, $x_g, x^\circ = O(n^{-1/k})$ for $c(x) = x^k$, $k > 1$. To see how x_g and x° converge relative to each other, note that

$$\frac{x_g}{x^\circ} = \frac{\phi^{-1}(1/n)}{c^{-1}(1/n)} \to \lim_{z\to 0} \frac{\phi^{-1}(z)}{c^{-1}(z)} = \lim_{z\to 0} \frac{c'(c^{-1}(z))}{\phi'(\phi^{-1}(z))} \geqslant \lim_{z\to 0} \frac{c'(\phi^{-1}(z))}{\phi'(\phi^{-1}(z))} = \lim_{x\to 0} \frac{1}{1 + xc''(x)/c'(x)}.$$

Therefore, if the elasticity of marginal cost, $xc''(x)/c'(x)$, does not tend to infinity for $x \to 0$, x_g and x° converge to zero at the same rate.

Now fix an admissible $u(x)$. For $c'(0) > 0$, we can show that $G(x) \leqslant F(x)$ for any x when n is sufficiently large. We only need to check the inequality within the support of F, where $F(x)^{n-1} = c(x)$ and $G(x)^{n-1} = \frac{x}{nx_g}$, i.e., it is sufficient to show that asymptotically $c(x) \geqslant \frac{x}{nx_g}$ for all $x \in [0, c^{-1}(1)]$. For $c'(0) > 0$, we have

$$\lim_{n\to\infty} \frac{x}{nx_g c(x)} = \frac{c'(0)x}{c(x)} \leqslant 1,$$

where the inequality is due to the strict convexity of $c(\cdot)$.

Consider now the case $c'(0) = 0$. Note that the usual FOSD order cannot be established here because for any n there exists a unique $x_n > 0$ such that $c(x_n) = \frac{x_n}{nx_g}$ and hence F and G cross at x_n. However, we still have

$$\lim_{n\to\infty} n \int u(x)dF(x) = \lim_{n\to\infty} \frac{n}{n-1} \int_0^{c^{-1}(1)} u(x)c(x)^{-\frac{n-2}{n-1}} c'(x)dx = \int_0^{c^{-1}(1)} \frac{u(x)c'(x)}{c(x)} dx < \infty.$$

To show the finiteness of the last expression, we used that $\frac{u(x)}{x}$ is integrable on $[0, c^{-1}(1)]$ and the elasticity of the cost function $\frac{c'(x)x}{c(x)}$ is bounded.

Further, define $a_n = \int_0^{nx_g} \frac{u(x)}{x} dx$ and obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{a_n} \int u(x)dG(x) &= \lim_{n\to\infty} \frac{n}{(n-1)a_n} \int_0^{nx_g} u(x) \left(\frac{x}{nx_g}\right)^{-\frac{n-2}{n-1}} \frac{1}{nx_g} dx \\
&= \lim_{n\to\infty} \frac{n}{(n-1)a_n} \frac{1}{(nx_g)^{\frac{1}{n-1}}} \int_0^{nx_g} \frac{u(x)}{x^{\frac{n-2}{n-1}}} dx = 1.
\end{aligned}$$

We used that $(nx_g)^{\frac{1}{n-1}} \to 1$. Indeed, $(nx_g)^{\frac{1}{n-1}} = e^{-\frac{\ln(nx_g)}{n-1}}$, and

$$\lim_{n\to\infty} \frac{\ln(nx_g)}{n-1} = \lim_{t\to\infty} \frac{\ln(t\phi^{-1}(\frac{1}{t}))}{t-1} = \lim_{t\to\infty} \frac{\phi^{-1}(\frac{1}{t}) - (\phi^{-1})'(\frac{1}{t})\frac{1}{t^2}}{t\phi^{-1}(\frac{1}{t})} = 0,$$

where we used that ϕ^{-1} has a bounded elasticity.[15]

It is easy to see that a_n is unbounded. For any $a \in (0, nx_g)$, we have

$$a_n = \int_0^{nx_g} \frac{u(x)}{x} dx \geqslant \int_a^{nx_g} \frac{u(x)}{x} dx \geqslant u(a) \int_a^{nx_g} \frac{dx}{x} = u(a) \ln \frac{nx_g}{a},$$

[15]This holds because for any $z \geqslant 0$

$$0 \leqslant \frac{(\phi^{-1})'(z)z}{\phi^{-1}(z)} = \frac{z}{\phi^{-1}(z)[c'(\phi^{-1}(z)) + \phi^{-1}(z)c''(\phi^{-1}(z))]} = \frac{z}{z + \phi^{-1}(z)^2 c''(\phi^{-1}(z))} \leqslant 1.$$

and the last expression is unbounded as $n \to \infty$. Therefore, $n \int u(x)dG(x)$ is also unbounded, and the result follows. ■

The requirement that $\frac{u(x)}{x}$ is integrable in bounded intervals $[0, a]$ is very mild. It holds, for example, for any function of the form $u(x) = x^\alpha$, $\alpha > 0$. An example of $u(x)$ for which it oes not hold is $u(x) = -1/\ln x$.

2.4 Ordering Output across Contests

In winner take all contests, recall that the equilibrium distribution of effort is $F(x) = c(x)^{1/(n-1)}$ for $x \in [0, \bar{x} = c^{-1(1)}]$, while the equilibrium distribution of *output* in the risk-taking model is $G(x) = (x/nx_g)^{1/(n-1)}$ for $x \in [0, nx_g]$, where x_g satisfies $x_g c'(x_g) = 1/n$. Because c is convex, it follows that $nx_g \to 1/c'(0)$ as $n \to \infty$, and so $c^{-1}(1) < nx_g$ for sufficiently large n.

The next proposition tells us that G is larger than F in some stochastic sense.

Proposition 5 *For all n large enough, the following hold:*

1. *G dominates F in the* convex transform order *whereby $G^{-1}(F(x))$ is convex in $x \in \operatorname{supp}(F)$.*
2. *Moreover, G dominates F in the* increasing convex order *whereby $\int udF \leqslant \int udG$ for all increasing and convex $u \in \mathbb{R}^{\mathbb{R}_+}$.*

Proof. To establish part (i), elementary calculations show that $G^{-1}(z) = z^{n-1}nx_g$, so that $G^{-1}\big(F(x)\big) = \big(c(x)^{1/(n-1)}\big)^{n-1}nx_g = c(x)nx_g$, which is convex in x.

To see part (ii), note that by Theorem 3 above, $\int xdF \leqslant \int xdG$ for all n large enough. Therefore, by Theorem 4.B.4 of Shaked and Shanthikumar (2007), which says that if G dominates F in the convex transform order and has a greater mean, then G dominates F in the increasing convex order, we prove our claim. ■

Another consequence of dominance in the convex transform order is that $\operatorname{Var}(F) \leqslant \operatorname{Var}(G)$.

The next result shows that the order statistics can also be ordered stochastically.

Corollary 1 *Let X_i denote the i-th player's realised effort (ie, output) in the pure effort model, and Y_i denote the i-th player's realised output in the risk-taking model, and let $X_{(i)}$ and $Y_{(i)}$ denote the corresponding order statistics.*[16] *Then, for all n sufficiently large, $Y_{(j)}$ dominates $X_{(j)}$ in the convex transform order, and hence also in the increasing convex order, for all $j = 1, \ldots, n$.*

[16]Thus, $X_{(1)} \geqslant \cdots \geqslant X_{(n)}$, and similarly for Y. We will often write $X_{(j:n)}$ when we want to be explicit about the number of players.

Proof. By Proposition 5, X_i is dominated by Y_i in the convex transform order for all $i = 1, \ldots, n$. The claim now follows from Theorem 4.B.15 in Shaked and Shanthikumar (2007). The proof of Proposition 5 shows that if $E[X_i] \leqslant E[Y_i]$ (which it is for sufficiently large n, by Theorem 3), it follows that $Y_{(j)}$ dominates $X_{(j)}$ in the increasing convex order for all $j = 1, \ldots, n$. ■

Dominance in the increasing convex order has a useful implication for the *maximal* levels of output in the two models.

Corollary 2 *We have* $\max\{Y_1, \ldots, Y_n\}$ *dominates* $\max\{X_1, \ldots, X_n\}$ *in the convex transform order, and hence for all* n *sufficiently large, also in the increasing convex order. Moreover, the ratio* $\mathbb{E}[Y_{(1:n)}]/\mathbb{E}[X_{(1:n)}]$ *is increasing in* n.

Proof. Taking $j = 1$ as the highest (first) order statistic in Corollary 1 establishes the first part of the claim. The second part follows from Theorem 4.B.18(c) in Shaked and Shanthikumar (2007). ■

3 Contest Design: Modified Gambling Contests

In this section, we consider two natural modifications of the baseline gambling contest that can be pursued by a principal designing the contest. In the first one, the principal can impose a minimum output *requirement*. That is, agents are allowed to design any fair gambling strategy around their efforts x_i subject to a "stop-loss" requirement that their output does not fall below some level $\underline{y}$. We show that by appropriately choosing $\underline{y}$ the principal can extract any effort from the agents up to the maximum individually rational level x°.

In the second modification, the principal sets a minimum output *standard*, $\hat{y}$, which the agents must surpass in order to be considered for a prize. However, there is no stop-loss mechanism to preclude the agents from producing output below $\hat{y}$ if they so choose. In this setting, we show that increasing $\hat{y}$ is counterproductive and hence the baseline gambling contest is optimal.

3.1 Minimum Output Requirement

Suppose the principal can enforce a minimum output level, denoted by $\underline{y}$, provided that a player participates (i.e., his expected payoff is non-negative). Our baseline model can be interpreted as a special case where $\underline{y} = 0$. Note that $\underline{y}$ can be either positive or negative.

Proposition 6 *In the gambling contest with the minimum output requirement* $\underline{y} \in \mathbb{R}$*, assuming that all players participate in the contest, there exists a unique symmetric equilibrium, in which each* *player chooses effort $\underline{x}_g$ and adds noise so that $G(y)^{n-1} = \min\{(y - \underline{y})/(\overline{y} - \underline{y}), 1\}$ for all $y \geqslant \underline{y}$, where*

$$(\underline{x}_g - \underline{y})c'(\underline{x}_g) = \frac{1}{n} \quad \textit{and} \quad \overline{y} = \underline{y} + n(\underline{x}_g - \underline{y}). \tag{3}$$

In this equilibrium, the equilibrium effort $\underline{x}_g$ is strictly increasing, while the players' expected payoffs are strictly decreasing, in $\underline{y}$.

Proof. As for Proposition 1, we focus on verifying that the given G indeed yields a symmetric equilibrium. The uniqueness proof is effectively identical to that of Proposition 1 and so omitted.

Given his effort choice x_i and the other players' strategies $G(y)^{n-1}$, player i solves

$$\max_{G_i \in \Delta([\underline{y},\infty))} \int G(y)^{n-1} dG_i(y) = \int_{\underline{y}}^{\infty} \min\left\{\frac{y - \underline{y}}{\overline{y} - \underline{y}}, 1\right\} dG_i \text{ s.t. } \int y dG_i(y) = x_i.$$

As in the proof of Proposition 1, $G(y)^{n-1}$ is concave, so no gambling (i.e., the degenerate distribution at x_i) is always the player's optimal strategy. His maximized payoff depends only on x_i as follows:

$$u(x_i) = \min\left\{\frac{x_i - \underline{y}}{\overline{y} - \underline{y}}, 1\right\} - c(x_i).$$

Again, as in the proof of Proposition 1, it can be shown that $u(x)$ is maximized by $\underline{x}_g$ (as defined in (3)) and the player is indifferent between $\delta_{\underline{x}_g}$ and G (so G is a best response to G^{n-1}).

The result that $\underline{x}_g$ is strictly increasing in $\underline{y}$ follows from the first equation in (3): since $c(\cdot)$ is strictly convex, as $\underline{y}$ increases, the left-hand side stays equal to $1/n$ only when $\underline{x}_g$ increases. Given this, the payoff result is immediate, because a player's expected payoff is equal to $1/n - c(\underline{x}_g)$. ■

Corollary 3 *The maximum feasible effort $x^\circ = c^{-1}(1/n)$—the highest implementable effort level subject to the players' participation constraint—can be implemented by setting the minimum output requirement to $\underline{y}^* = x^\circ - 1/(nc'(x^\circ)) > 0$.*

Proof. In a symmetric equilibrium, each player wins the contest with probability $1/n$, and hence the highest expected effort that can be induced while giving a non-negative payoff to the players is $x^\circ = c^{-1}(1/n)$. By Proposition 6, this effort level can be implemented when the minimum output requirement is such that

$$(x^\circ - \underline{y})c'(x^\circ) = \frac{1}{n} \Leftrightarrow \underline{y} = x^\circ - \frac{1}{nc'(x^\circ)}.$$

The inequality $\underline{y}^* > 0$ follows from the strict convexity of $c(\cdot)$ that gives $nc'(x^\circ)x^\circ > nc(x^\circ) = 1$. ■

Proposition 6 helps understand the discrepancy between our results and those of Hvide (2002), who imposes no restrictions on $\underline{y}$ (i.e., $\underline{y} = -\infty$). In this case, regardless of the value of n, the equilibrium effort $\underline{x}_g = 0$. The following results are also straightforward.

Corollary 4 *(a) For any minimum requirement $\underline{y} \geqslant 0$, if n is sufficiently large then $\underline{x}_g > x_e$.*

(b) For any n, if $\underline{y} \in \mathbb{R}$ is sufficiently low then $\underline{x}_g < x_e$.

3.2 Minimum Output Standard (To Be Considered for a Prize)

Now suppose that the principal considers a player for a prize (e.g., promotion) only when the player's output exceeds a standard $\widehat{y}(\geqslant 0)$, but cannot punish players or otherwise enforce the standard if their output falls short of $\widehat{y}$. We also maintain the non-negativity constraint $y_i \geqslant 0$.

Proposition 7 *In the gambling contest with the minimum output standard $\widehat{y} \in \mathbb{R}_+$, there exists a unique symmetric equilibrium, in which each player chooses effort $\widehat{x}_g$ and adds noise so that $G(y)^{n-1} = \min\{\max\{y, \widehat{y}\}/\overline{y}, 1\}$ for all $y \in \mathbb{R}_+$, where*

$$\widehat{x}_g c'(\widehat{x}_g) = \frac{1}{n}\left(1 - (c'(\widehat{x}_g)\widehat{y})^{\frac{n}{n-1}}\right) \text{ and } \overline{y} = \frac{1}{c'(\widehat{x}_g)}. \tag{4}$$

In this equilibrium, the equilibrium effort $\widehat{x}_g$ is strictly decreasing, while the players' expected payoffs are increasing in, $\widehat{y}$.

Proof. As in the proof of Proposition 6, we only verify that the given G yields a symmetric equilibrium. Given his effort x_i and $G(y)^{n-1}$, player i solves

$$\max_{G_i \in \Delta(\mathbb{R}_+)} \int G(y)^{n-1}\chi_{\{y \geqslant \widehat{y}\}}(y) dG_i(y) = \int_{\widehat{y}}^{\infty} \min\left\{\frac{y}{\overline{y}}, 1\right\} dG_i(y) \text{ s.t. } \int y dG(y) = x_i,$$

where $\chi_A(x)$ denotes the indicator function ($\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$). The value function $G(y)^{n-1}\chi_{\{y \geqslant \widehat{y}\}}(y)$ is *not concave*, so the degenerate distribution δ_{x_i} is not necessarily optimal. Nevertheless, by the usual concavificatioon argument, it is easy to see that the player's maximized payoff is still

$$u(x_i) = \min\left\{\frac{x_i}{\overline{y}}, 1\right\} - c(x_i),$$

which he can achieve, for example, by mixing between 0 and $\widehat{y}$ if $x_i < \widehat{y}$ and adding no noise (choosing δ_{x_i}) if $x_i \geqslant \widehat{y}$. Again, $u(x)$ is maximized at $\widehat{x}_g$, and the given G (on whose support $G(y)^{n-1} = y/\overline{y}$) also yields $u(x)$ to the player. Therefore, G is a best response to G^{n-1}.

The result that $\widehat{x}_g$ is strictly decreasing in $\widehat{y}$ follows from the first equation in (4): an increase of $\widehat{y}$ lowers the right-hand side. For the equation to continue to hold, $\widehat{x}_g$ should decrease. Given this, the payoff result is immediate, because the players' expected payoffs are equal to $1/n - c(\widehat{x}_g)$. ■

4 Discussion

4.1 General prize schedules

Suppose the players are rewarded based on the ranking of their performance according to a prize schedule $v = (v_1, \ldots, v_n)$. We assume that prizes are non-negative, monotone, and satisfy a budget constraint: $v_1 \geqslant \ldots \geqslant v_n \geqslant 0$, $\sum_{i=1}^n v_i = 1$. The winner-take-all prize structure is a special case with $v_i = \chi_{\{i=1\}}$; and the prize scheme with m equal top prizes is $v_i = \frac{1}{m}\chi_{\{i \leqslant m\}}$.

Pure effort Without loss, we set $v_n = 0$, which is the player's outside option. The resulting unique symmetric equilibrium involves mixing in $[0, c^{-1}(v_1)]$ with distribution F satisfying

$$c(x) = \sum_{i=1}^{n} \binom{n-1}{i-1} F(x)^{n-i}[1 - F(x)]^{i-1} v_i.$$

As shown by Fang et al. (2020), for strictly convex costs, effort is FOSD-increasing when prizes become more equitable, in the sense of the majorization order. The effort-maximizing prize schedule is, therefore, $v_i = \frac{1}{n-1}\chi_{\{i \leqslant n-1\}}$, which awards the same prize to everyone except the player ranked last. This scheme is sometimes referred to as "punishment at the bottom," as opposed to the winner-take-all rewarding at the top. The resulting equilibrium distribution of effort has support $[0, c^{-1}(\frac{1}{n-1})]$ and satisfies

$$c(x) = \frac{1 - [1 - F(x)]^{n-1}}{n-1},$$

which gives

$$F(x) = 1 - [1 - (n-1)c(x)]^{\frac{1}{n-1}}.$$

Suppose first that $c'(0) > 0$. The expected effort can be written as

$$\begin{aligned} x_e &= \int_0^{c^{-1}(\frac{1}{n-1})} x dF(x) = -\int_0^{c^{-1}(\frac{1}{n-1})} x d[1-(n-1)c(x)]^{\frac{1}{n-1}} \\ &= \int_0^{c^{-1}(\frac{1}{n-1})} [1-(n-1)c(x)]^{\frac{1}{n-1}} dx = \frac{1}{n-1}\int_0^1 \frac{(1-z)^{\frac{1}{n-1}}}{c'(c^{-1}(\frac{z}{n-1}))} dz. \end{aligned}$$

In the limit, we obtain

$$\lim_{n\to\infty} n x_e = \lim_{n\to\infty} \frac{n}{n-1}\int_0^1 \frac{(1-z)^{\frac{1}{n-1}}}{c'(c^{-1}(\frac{z}{n-1}))} dz = \frac{1}{c'(0)}.$$

Note that the efficient level of effort satisfies $c(\bar{x}) = \frac{1}{n}$, and in the limit we have

$$\lim_{n\to\infty} n\bar{x} = \lim_{n\to\infty} n c^{-1}(\frac{1}{n}) = \lim_{z\to 0}\frac{c^{-1}(z)}{z} = \lim_{x\to 0}\frac{x}{c(x)} = \frac{1}{c'(0)}.$$

Suppose now that $c'(0) = 0$. For this case, we write

$$x_e = \int_0^{c^{-1}(\frac{1}{n-1})} x dF(x) = \int_0^{c^{-1}(\frac{1}{n-1})} \frac{x c'(x) dx}{[1-(n-1)c(x)]^{\frac{n-2}{n-1}}} = \frac{1}{n-1}\int_0^1 \frac{c^{-1}(\frac{z}{n-1}) dz}{(1-z)^{\frac{n-2}{n-1}}}.$$

This gives

$$\frac{x_e}{\bar{x}} = \frac{1}{n-1}\int_0^1 \frac{c^{-1}(\frac{z}{n-1}) dz}{c^{-1}(\frac{1}{n})(1-z)^{\frac{n-2}{n-1}}} = \int_0^1 \frac{c^{-1}(\frac{1-z}{n-1})}{c^{-1}(\frac{1}{n})} dz^{\frac{1}{n-1}} \geqslant \int_0^1 (1-z) dz^{\frac{1}{n-1}} = 1 - \frac{1}{n}.$$

The last inequality follows from the concavity of $c^{-1}(\cdot)$ whereby

$$c^{-1}\left(\frac{1-z}{n-1}\right) \geqslant c^{-1}\left(\frac{1-z}{n}\right) \geqslant (1-z)c^{-1}\left(\frac{1}{n}\right).$$

We can also show using Jensen's inequality that this ratio is below 1 for any finite n:

$$x_e = \int_0^1 c^{-1}\left(\frac{1-z}{n-1}\right) dz^{\frac{1}{n-1}} < c^{-1}\left(\int_0^1 \frac{1-z}{n-1} dz^{\frac{1}{n-1}}\right) = c^{-1}\left(\frac{1}{n}\right) = \bar{x},$$

due to strict concavity of $c^{-1}(\cdot)$. Together, the two bounds imply $\lim_{n\to\infty}\frac{x_e}{\bar{x}} = 1$.

Thus, under the optimal allocation of prizes full efficiency is asymptotically achieved with pure effort. In contrast, with optimal gambling efficiency can be achieved for any finite n and any allocation of prizes, as shown next.

Gambling Following the same approach as above, we have

$$\beta(y - \underline{y}) = \sum_{i=1}^{n} \binom{n-1}{i-1} H(y)^{n-i}[1 - H(y)]^{i-1} v_i$$

for some $\beta > 0$ and $y \in [\underline{y}, \overline{y}]$. Then $\beta(x_g - \underline{y}) = \int_{\underline{y}}^{\overline{y}} \beta(y - \underline{y}) dH(y) = \frac{1}{n} \sum_{i=1}^{n} v_i = \frac{1}{n}$ as before, and $\beta = c'(x_g)$, which gives $(x_g - \underline{y})c'(x_g) = \frac{1}{n}$, and the results above go through.

4.2 Concave Costs

Pure effort = Gambling!

4.3 Gambling Costs

In this section, we introduce gambling costs into our baseline model and construct a sequence of symmetric equilibria that converges to the gambling equilibrium in Proposition 2 as the unit cost of gambling vanishes.

Output-separable gambling costs. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ denote a strictly concave function. We assume that it is twice continuously differentiable and strictly decreasing from 0; this latter assumption incurs no loss of generality, because our analysis below depends only on strict concavity of Φ. Given Φ, we assume that the gambling cost of a player's inducing a distribution G_i from his effort x_i is given by

$$c_g(G_i) \equiv \lambda \int (\Phi(x_i) - \Phi(y))\, dG_i(y) = \lambda \left(\Phi(x_i) - \int \Phi(y) dG_i(y) \right),$$

where $\lambda (> 0)$ represents the unit cost of gambling. This modeling of gambling costs is effectively identical to that of posterior-separable costs in the literature on information costs (see, e.g., Caplin et al., 2019; Bloedel and Zhong, 2020) and so inherits all desirable properties from the latter. Among other things, it satisfies an essential property of gambling costs: if G_i is a mean-preserving spread of G'_i then

$$c_g(G_i) - c_g(G'_i) = \lambda \left(\int \Phi(y) dG'_i(y) - \int \Phi(y) dG_i(y) \right) \geqslant 0.$$

In addition, it can accommodate various forms of gambling costs. In particular, the usual variance cost–gambling costs being proportional to the variance of the induced distribution–is a special case with $\Phi(y) = -y^2$.

Candidate equilibrium structure. As in Section 2.2, consider the game in which each player chooses a distribution G_i given the common mean x. If all other players follow G then, due to gambling costs, an individual player's problem is given by

$$\max_{G_i \in \Delta(\mathbb{R}_+)} \int G(y)^{n-1} dG_i(y) - c_g(G_i) = \int \left[G(y)^{n-1} + \lambda\Phi(y)\right] dG_i(y) - \lambda\Phi(x).$$

In other words, the only difference from the baseline model is that now $G(y)^{n-1} + \lambda\Phi(y)$ (not $G(y)^{n-1}$) serves as the value function for an individual player. Then, by effectively the same logic as for the baseline model, in symmetric equilibrium, $G(y)^{n-1} + \lambda\Phi(y)$ must be affine in y over $supp(G)$. Furthermore, if λ is sufficiently small then it would be that for some $\beta > 0$ and $\overline{y} > 0$,

$$G(y)^{n-1} + \lambda\Phi(y) = \beta y \text{ for all } y \in \text{supp}(G) = [0, \overline{y}]. \tag{5}$$

Given the above result, the rest can be characterized just as in the baseline model. Let $v(x_i)$ denote a player's indirect payoff of exerting effort x_i. Since $G(y)^{n-1} + \lambda\Phi(y)$ is concave, $v(x_i) = G(x_i)^{n-1} + \lambda\Phi(x_i) = \max\{\beta x_i, \beta\overline{y}\}$. The equilibrium (deterministic) effort x should maximize $v(x_i) - c(x_i)$, leading to

$$\beta - \lambda\phi(x) - c'(x) = 0 \Leftrightarrow \beta = c'(x) + \lambda\phi(x). \tag{6}$$

Equilibrium existence. The three equations connecting x, β and $\overline{y}$ can be considered as the system of equations

$$\begin{aligned}
A_1(x, \beta, \overline{y}; \lambda) &= c'(x) - \beta + \lambda\phi(x) = 0, \\
A_2(x, \beta, \overline{y}; \lambda) &= \beta\overline{y} - 1 - \lambda\Phi(\overline{y}) = 0, \\
A_3(x, \beta, \overline{y}; \lambda) &= x - \int y dG(y) = x - \frac{1}{n\beta} - \frac{\lambda}{\beta}\int \Phi(y) dG(y) = 0
\end{aligned}$$

that defines implicit functions $x(\lambda)$, $\beta(\lambda)$ and $\overline{y}(\lambda)$ such that $x(0) = x_0$, $\beta(0) = \beta_0$ and $\overline{y}(0) = \overline{y}_0$ is the solution corresponding to $\lambda = 0$. The Jacobian of this system evaluated at $(x_0, \beta_0, \overline{y}_0; 0)$ is

$$J_0 = \left(\frac{\partial(A_1, A_2, A_3)}{\partial(x, \beta, \overline{y})}\right)_0 = \begin{vmatrix} c''(x_0) & -1 & 0 \\ 0 & \overline{y}_0 & \beta_0 \\ 1 & \frac{1}{n\beta_0^2} & 0 \end{vmatrix} = -\frac{c''(x_0)}{n\beta_0} - \beta_0 < 0.$$

Functions A_k, $k = 1, 2, 3$, are C^1 in some neighborhood of $(x_0, \beta_0, \overline{y}_0; 0)$, and hence the Implicit Function Theorem implies that there exist a $\bar{\lambda} > 0$ and unique C^1 functions $x(\lambda)$, $\beta(\lambda)$ and $\overline{y}(\lambda)$ solving the system of equations for any $\lambda \in [0, \bar{\lambda}]$.

Example. Suppose $n = 2$, $c(x) = -\Phi(x) = x^2$, and $\lambda < 1$. Then, (6) simplifies to

$$\beta = c'(x) + \lambda\phi(x) = 2(1-\lambda)x.$$

In addition, the equation for A_2 becomes

$$1 + \lambda\Phi(\overline{y}) = 1 - \lambda\overline{y}^2 = \beta\overline{y} \Rightarrow \overline{y} = \frac{2}{\beta + \sqrt{\beta^2 + 4\lambda}} = \frac{1}{(1-\lambda)x + \sqrt{(1-\lambda)^2x^2 + \lambda}}.$$

Applying these to the equation for A_3, we arrive at

$$\begin{aligned} x &= \int_0^{\overline{y}} ydG(y) = \int_0^{\overline{y}} yd\left(\beta y + \lambda y^2\right) = \frac{\beta}{2}\overline{y}^2 + \frac{2\lambda}{3}\overline{y}^3 = \frac{2\overline{y}}{3} - \frac{\beta(1-\beta\overline{y})}{6\lambda} \\ &= \frac{(1-\lambda)x + 2\sqrt{(1-\lambda)^2x^2 + \lambda}}{3\left((1-\lambda)x + \sqrt{(1-\lambda)^2x^2 + \lambda}\right)^2}. \end{aligned}$$

If $\lambda = 0$ then this equation reduces to $4x^2 = 1$, leading to $x_0 = 1/2$, which is consistent with x_g in our baseline model.

4.4 Risk taking in the presence of exogenous noise

Suppose that in addition to endogenous gambling the contestants face exogenous noise they cannot control. That is, player i's output is $Y_i = x_i + \varepsilon_i + \eta_i$, where ε_i is chosen as before, but η_i with zero mean and absolutely continuous distribution F_η is exogenously given. Let $[\underline{\eta}, \overline{\eta}]$ denote its support (possibly infinite). We will also allow for a general lower bound $\underline{y}$ for output. Player i's problem then becomes to maximize

$$\mathbb{P}\{Y_i > Y_j \text{ for all } j \neq i\} - c(x_i) \;\; \text{s.t. } Y_i \text{ is a mean-preserving spread of } x_i + \eta_i,\; Y_i \geqslant \underline{y} \text{ a.s.}$$

Based on Lemma 1, the only way gambling can work in a symmetric equilibrium is if $G(y)^{n-1}$ is linear in the support of Y_i, i.e., $G(y)^{n-1} = (y - \underline{y})/(\overline{y} - \underline{y})$. The equilibrium effort and $\overline{y}$ are then determined as in Proposition 6. Thus, the question is whether or not $G(y)$ given by Proposition 6 is a mean-preserving spread of $\underline{x}_g + \eta_i$. If the answer is yes, we have a gambling equilibrium as before.

Proposition 8 *Suppose $c'(0) = 0$, $\underline{\eta}$ is finite, and $\underline{y} \leqslant \underline{x}_g + \underline{\eta}$. Then the gambling equilibrium exists for n large enough.*

Proof. It is sufficient to show that for n large enough Y_i with distribution $G(y)$ from Proposition 6 is a mean-preserving spread of $\underline{x}_g + \eta_i$. The latter is distributed with $F_\eta(y - \underline{x}_g)$. It is, therefore, sufficient to prove that for n large enough

$$\int_{\underline{y}}^{y} G(t)dt \geqslant \int_{\underline{y}}^{y} F_\eta(t-\underline{x}_g)dt \text{ for all } y \geqslant \underline{y}.$$

Note that $\overline{y} \to \infty$ for $n \to \infty$ because $c'(0)=0$. We have

$$\begin{aligned}
\int_{\underline{y}}^{y} G(t)dt &= \int_{\underline{y}}^{y} \left(\frac{t-\underline{y}}{n(\underline{x}_g-\underline{y})}\right)^{\frac{1}{n-1}} dt = n(\underline{x}_g-\underline{y}) \int_0^{\frac{y-\underline{y}}{n(\underline{x}_g-\underline{y})}} z^{\frac{1}{n-1}} dz \\
&= (n-1)(\underline{x}_g-\underline{y}) \left(\frac{y-\underline{y}}{n(\underline{x}_g-\underline{y})}\right)^{\frac{n}{n-1}} \to y-\underline{y} \text{ as } n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\underline{y}}^{y} F_\eta(t-\underline{x}_g)dt &= tF_\eta(t-\underline{x}_g)|_{\underline{y}}^{y} - \int_{\underline{y}}^{y} tdF_\eta(t-\underline{x}_g) = yF_\eta(y-\underline{x}_g) - \int_{\underline{y}}^{y} tdF_\eta(t-\underline{x}_g) \\
&= yF_\eta(y-\underline{x}_g) - \mathbb{E}(\underline{x}_g+\eta|\underline{x}_g+\eta<y)F_\eta(y-\underline{x}_g) \\
&\leqslant yF_\eta(y-\underline{x}_g) - (\underline{x}_g+\underline{\eta})F_\eta(y-\underline{x}_g) \leqslant y-\underline{y}.
\end{aligned}$$

The inequality is strict for $y > \underline{x}_g + \underline{\eta}$ because the conditional expectation is then strictly greater than $\underline{x}_g + \underline{\eta} \geqslant \underline{y}$. The result follows. ■

5 Conclusions

In this paper, we study contests in which agents can strategically choose both effort and risk. This approach provides a natural connection between two distinct classes of models of contests—the all-pay contest (or pure-effort) model where performance is determined entirely by effort, and the noisy rank-order tournament model where performance is determined jointly by effort and (exogenous) noise. We endogenize the noise by allowing agents to choose arbitrary fair gambles around their effort, subject to a natural stop-loss constraint.

There are three main results. First, we provide a full characterization of the unique symmetric equilibrium. The equilibrium always exists, and features pure effort. This is a nice feature compared to the standard Lazear-Rosen model with exogenous noise where a pure strategy equilibrium only exists when noise is sufficiently dispersed, and under further constraints on the cost function.

Second, we show that the equilibrium effort in the contest with endogenous risk-taking exceeds the expected effort in the pure-effort contest with the number of agents is sufficiently large, and agents earn positive rents. Moreover, by modifying the stop-loss constraint on output, it is possible to extract the maximum feasible effort from the agents in the gambling contest.

Third, we show that effort in the gambling contest is invariant to prize allocations. This result is in contrast to both the pure-effort model and the Lazear-Rosen model. In the former, it is shown by Fang et al. (2020) that under convex costs it is optimal to share the prize equally among all agents except the one ranked last. In the latter, the optimal allocation of prizes depends on the distribution of noise (Drugov and Ryvkin, 2020).

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A Omitted Proofs

Proof of Proposition 1.

Suppose player i has chosen his effort x_i. Given x_i and $G(y)^{n-1}$, (**??**) reduces to

$$\max_{G_i\in\Delta(\mathbb{R}_+)} \int \max\left\{\frac{y}{\overline{y}}, 1\right\} dG_i(y) \text{ subject to } \int y dG_i(y) = x_i.$$

This is a familiar linear programming problem in the literature on Bayesian persuasion (Kamenica and Gentzkow, 2011; Aumann and Maschler, 1995) and strategic risk taking (e.g., Myerson, 1993; Fang and Noe, 2018; Seel and Strack, 2013), for which the method of concavification can be used to identify an optimal solution. Since the value function $G(y)^{n-1} = \min\{y/\overline{y}, 1\}$ is concave, it is straightforward that *one* optimal distribution is the degenerate one at x_i, regardless of the value of x_i. This implies that the player's maximized payoff is given as follows:

$$u(x_i) \equiv \max\left\{\frac{x_i}{\overline{y}}, 1\right\} - c(x_i).$$

This function is maximized at x^*,[17] implying that x^* is an individual player's optimal effort level. The individual optimality of G follows from a player's indifference between δ_{x^*} and G, that is,

$$\int \max\left\{\frac{y}{\overline{y}}, 1\right\} dG(y) = \int_0^{\overline{y}} \frac{y}{\overline{y}} dH(y) = \frac{\mathbb{E}[y]}{\overline{y}} = \frac{x^*}{\overline{y}} = \int \max\left\{\frac{y}{\overline{y}}, 1\right\} d\delta_{x^*}.$$

In this appendix, we prove that G such that $G(y)^{n-1} = \min\{y/\overline{y}, 1\}$ yields the unique symmetric equilibrium of our gambling model. Specifically, we show that G is the only distribution that satisfies a necessary condition for symmetric equilibrium.

Let H denote *a* symmetric equilibrium distribution. Then, H must be a solution to

$$\max_{H_i \in \Delta(\mathbb{R}_+)} \int H(y)^{n-1} dH_i(y) - c\left(\mathbb{E}_{H_i}[y]\right).$$

This implies that H must be also a solution to

$$\max_{H_i \in \Delta(\mathbb{R}_+)} \int H(y)^{n-1} dH_i(y) \text{ s.t. } \int y dH_i(y) = \mathbb{E}_H[y]. \tag{7}$$

For this second problem, the following result is well known.

Lemma 2 *A distribution H_i^* solves (7) if and only if there exists a linear function $\phi(y)$ such that*

$$\phi(y) \geqslant H(y)^{n-1} \textit{ for all } y \in \mathbb{R}_+ \textit{ and } \phi(y) = H(y)^{n-1} \textit{ whenever } y \in \operatorname{supp}(H_i^*).$$

Let $\underline{y}'$ and $\overline{y}'$ denote the lower and the upper bounds of $supp(H)$, respectively. We first show that $\underline{y}' = 0$. Suppose $\underline{y}' > 0$. Clearly, H cannot have an atom at $\underline{y}'$: if so, it is a profitable deviation for a player to marginally move the mass point above $\underline{y}'$. Now, consider the linear function $\phi(y)$ in Lemma 2. It should satisfy $\phi(y) \geqslant H(y)^{n-1} \geqslant 0$ for all y, and $\phi(\underline{y}') = 0$. The only linear function that satisfies these properties is $\phi(y) = 0$ for all y. This implies $H(y) = 0$ for all y, which clearly cannot be an equilibrium.

The fact that $\underline{y}' = 0$, $\overline{y}' \in \operatorname{supp}(H)$, and Lemma 2 together imply that $H(y)^{n-1} \leqslant \phi(y) = y/\overline{y}'$. Now, we show that $H(y)^{n-1} = \phi(y) = y/\overline{y}'$ for all $y \leqslant \overline{y}'$. Suppose there exist $y_1 \in [0, \overline{y}')$ and $y_2 (\in (y_1, \overline{y}']$ such that $(y_1, y_2) \cap \operatorname{supp}(H) = \varnothing$. Without loss of generality, assume that (y_1, y_2) is the largest such interval so that $y_1, y_2 \in \operatorname{supp}(H)$. This implies that H is flat over $[y_1, y_2)$ but has

[17] Given the particular shape of $\max\{x_i/\overline{y}, 1\}$ and strict monotonicity and convexity of $c(x_i)$, $u(x_i)$ is maximized either at x^* or at $\overline{y}$. The latter requires that $\lim_{y \to \overline{y}-} u'(y) = 1/\overline{y} - c'(\overline{y}) \geqslant 0$, which necessarily fails because

$$1 \geqslant \overline{y}c'(\overline{y}) = nx^* c'(nx^*) > nx^* c'(x^*) = 1.$$

a jump at y_2 (so that $H(y_2)^{n-1} = \phi(y_2)$). But, such an atom cannot be sustained in equilibrium, because a player can marginally move the mass point slightly above y_2.

The above result implies that $supp(H) = [0, \overline{y}']$ and $H(y)^{n-1} = \min\{y/\overline{y}', 1\}$. Given this, by the same arguments used in the main text, the equilibrium effort x' satisfies $1/\overline{y}' = c'(x')$ and

$$x' = \int y dH(y) = \int_0^{\overline{y}'} yd\left(\frac{y}{\overline{y}'}\right)^{\frac{1}{n-1}} = \frac{1}{n-1}\int_0^{\overline{y}'}\left(\frac{y}{\overline{y}'}\right)^{\frac{1}{n-1}} dy = \frac{\overline{y}'}{n}.$$

It is then straightforward that $H = G$. ■

Proof of Proposition 3 in the general case.

Consider the ratio x_e/x_g. Integrating by parts and splitting the integral, obtain:

$$\begin{aligned}\frac{x_e}{x_g} &= \frac{1}{x_g}\int_0^{c^{-1}(1)} xdc(x)^{\frac{1}{n-1}} = \frac{1}{x_g}\left[xc(x)^{\frac{1}{n-1}}\Big|_0^{c^{-1}(1)} - \int_0^{c^{-1}(1)} c(x)^{\frac{1}{n-1}}dx\right] \\ &= \frac{1}{x_g}\left[c^{-1}(1) - \int_0^{c^{-1}(1)} c(x)^{\frac{1}{n-1}}dx\right] = \frac{1}{x_g}\int_0^{c^{-1}(1)}\left[1 - c(x)^{\frac{1}{n-1}}\right]dx \\ &= \frac{1}{x_g}\int_0^{x_g}\left[1 - c(x)^{\frac{1}{n-1}}\right]dx + \frac{1}{x_g}\int_{x_g}^{c^{-1}(1)}\left[1 - c(x)^{\frac{1}{n-1}}\right]dx.\end{aligned}$$

From the mean-value theorem for definite integrals, there exists a $\xi_n \in (0, x_g)$ such that the first term in the last line is equal $1 - c(\xi_n)^{\frac{1}{n-1}}$, which converges to zero for $n \to \infty$. The second term can be bounded as

$$\int_{x_g}^{c^{-1}(1)} \frac{1 - c(x)^{\frac{1}{n-1}}}{x_g}dx = \int_{x_g}^{c^{-1}(1)} \frac{1 - c(x)^{\frac{1}{n-1}}}{\frac{1}{n-1}(n-1)x_g}dx < \int_{x_g}^{c^{-1}(1)} \frac{-\ln c(x)}{(n-1)x_g}dx,$$

where the inequality holds because, for each $x \in (x_g, c^{-1}(1))$, $s(\alpha) = 1 - c(x)^{\alpha}$ is a strictly concave function of α and hence $s(\alpha)/\alpha < s'(0) = -\ln c(x)$ for any $\alpha > 0$. Furthermore, using the definition of x_g,

$$\int_{x_g}^{c^{-1}(1)} \frac{-\ln c(x)}{(n-1)x_g}dx = \frac{n}{n-1}\int_{x_g}^{c^{-1}(1)} \frac{-\ln c(x)}{nx_g}dx = -\frac{n}{n-1}\int_{x_g}^{c^{-1}(1)} c'(x_g)\ln c(x)dx. \tag{8}$$

From the strict convexity of $c(x)$, the last integral can be bounded above as

$$-\int_{x_g}^{c^{-1}(1)} c'(x_g)\ln c(x)dx < -\int_{x_g}^{c^{-1}(1)} c'(x)\ln c(x)dx = -\int_{c(x_g)}^{1} \ln z\, dz < 1.$$

Here, the first inequality holds independent of n; moreover, the difference between the two terms is decreasing in x_g and hence increasing in n:

$$\frac{\partial}{\partial x_g}\left[-\int_{x_g}^{c^{-1}(1)}(c'(x)-c'(x_g))\ln c(x)dx\right]=\int_{x_g}^{c^{-1}(1)}c''(x_g)\ln c(x)dx<0.$$

Therefore, the limit of the last term in (8) as $n \to \infty$ is strictly less than one, which gives the result.

It can also be seen from (8) that, if $c'(0) = 0$ and $\int_0^{c^{-1}(1)} \ln c(x)dx < \infty$ (which holds in many cases), we have $x_e/x_g \to 0$ for $n \to \infty$.

■