Dissolving a Partnership Dynamically*

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Abstract

In financial disputes arising from divorce, inheritance, or the dissolution of a partnership, frequently the need arises to assign ownership of an indivisible item to one member of a group. This paper introduces and analyzes a dynamic auction for simply and efficiently allocating an item when participants are privately informed of their values. In the auction, the price rises continuously. A bidder who drops out of the auction, in return for surrendering his claim to the item, obtains compensation equal to the difference between the price at which he drops and the preceding drop price. When only one bidder remains, that bidder wins the item and pays the compensations of his rivals. We characterize the unique equilibrium with risk-neutral and CARA risk averse bidders. We show that dropout prices are decreasing as bidders become more risk averse. Each bidder's equilibrium payoff is at least $1/N$-th of his value for the item. Indeed, we show that each bidder's security payoff is $1/N$-th of his value. We introduce the notion of a perfect security strategy, we show that each bidder has a unique perfect security strategy, and that it coincides with the equilibrium bidding strategy as bidders becomes infinitely risk averse.

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1 Introduction

In financial disputes arising from divorce, inheritance, or the dissolution of a partnership, frequently the need arises to assign ownership of an indivisible item to one member of a group. This paper introduces and analyzes a dynamic auction for simply and efficiently resolving such disputes.

The canonical example of a division mechanism is divide and choose. In addition to helping children split pieces of cake, this procedure is widely used in a variety of other practical settings. A version of divide and choose called a "Texas Shoot-Out" is a commonly used exit mechanism found in two-person equal-share partnership contracts.[1] In this mechanism, the owner who wants to dissolve the partnership names a price and the other owner is compelled to either purchase his partner's share or sell his own share at the named price.

Divide and choose is simple and fair. Parents (lawyers) can explain the procedure to their children (clients) without difficulty. Moreover, whether a participant is the divider or the chooser, they can guarantee themselves at least half of their value for the object by following a simple security strategy. In a Texas Shoot-Out, for example, an owner who names a price that leaves him indifferent to whether his partner buys or sells is guaranteed to receive half of his value for the partnership. Likewise, his partner, by simply taking the best deal, either selling or buying at the proposed price, cannot leave with less than fifty percent of her value for the partnership.

Despite these properties, divide and choose has several flaws which limit its applicability and attractiveness. First, the procedure does not easily scale to more than two participants. Second, it does not treat the participants symmetrically: there is an advantage to being the divider when information is complete and to being the chooser when information is incomplete. Finally, when information is incomplete, then divide and choose is not efficient.[2]

[1] Brooks, Landeo, and Spier (2010) detail the popularity of this exit mechanism and examine why Texas Shoot-Outs are rarely triggered in real-world contracts.

[2] In a complete information environment, these issues have been well studied. Crawford

We present a dynamic auction which avoids the negative features of divide and choose while retaining its many attractive properties. In the auction, the price, starting from zero, rises continuously. Bidders may drop out at any point. A bidder who drops out surrenders his claim to the item and, in return, receives compensation from the (eventual) winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. That bidder wins the item and compensates the other bidders. Thus in an auction with N bidders, if $\{p_k\}_{k=1}^{N-1}$ is the sequence of dropout prices, then the compensation of the k-th bidder to drop is $p_k - p_{k-1}$, where $p_0 = 0$, and the winner's total payment is $p_{N-1} = \Sigma_{k=1}^{N-1}(p_k - p_{k-1})$.[3] Hereafter, we refer to this auction as the compensation auction.

In our setting, a strategy for a bidder is a sequence of bid functions, where the k-th bid function identifies the price at which the bidder drops out as a function of his value and the $k-1$ prior dropout prices. In the symmetric independent private values setting we provide necessary and sufficient conditions for a sequence of bid functions to be a symmetric Bayes Nash equilibrium in increasing and differentiable strategies. We characterize the unique such equilibrium when bidders are risk neutral and when they are CARA risk averse; in equilibrium the compensation auction efficiently dissolves partnerships. We show that equilibrium dropout prices are decreasing as bidders become more risk averse. Equilibrium is also interim proportional, i.e., each bidder's equilibrium expected payoff is equal to at least $1/N$-th of

(1979) shows that auctioning off the divider role in divide and choose can correct the asymmetry of the procedure, and Demange (1984) offers a procedure for N players that is fair and efficient. In an incomplete information environment, de Frutos and Kittsteiner (2008) show how bidding to be the chooser can restore efficiency to a Texas Shoot-Out.

[3]The auction can equivalantly be framed as follows: At the beginning of each round, compensation is set zero and then increased continuously until one of the participants agrees to take this compensation in return for giving up his claim to the item. This participant exits, and the process is repeated, until only one participant remains. The last participant is awarded the item and pays each of the others their individualized compensation.

his value for the item, and thus it is individual rational for each bidder to participate in the auction if $1/N$-th of his value is his disagreement payoff.

We show that when the price reaches a risk-neutral bidder's equilibrium dropout price (and at least three bidders remain in the auction) then the bidder, rather than dropping out, obtains the same payoff by remaining in the auction and "mimicking" the equilibrium behavior of the bidder with the next highest value. The analogous feature arises in Dutch multi-unit sequential auctions. We show that the sequence of compensations obtained by mimicking higher-value bidders forms a martingale.

In the actual application of any dissolution mechanism it is useful to be able provide participants with a simple strategy which is guaranteed to do "not too badly." A bidder's security payoff is the largest payoff the bidder can guarantee himself, regardless of the values and strategies of the other bidders, and a security strategy is one that guarantee's a bidder his security payoff. We show for the compensation auction that a bidder's security payoff is equal to $1/N$-th of his value and we identify simple security strategies that guarantee a bidder this amount. One of these strategies has the property that it continues to be a security strategy in the auction that remains after any number of bidders has dropped. We say such a strategy is a "perfect security strategy" and we show there is a unique such strategy. We show that the equilibrium bidding function of a CARA risk averse bidder converges to the perfect security strategy as bidders become infinitely risk averse.

Related Literature

In an independent private values setting, Cramton, Gibbons, and Klemperer (1987) identify necessary and sufficient conditions for a N-bidder partnership to be efficiently dissolvable when bidders are risk neutral, and they identify a static bidding game that dissolves it. They show that only equal partnerships are dissolvable as the number of bidders grows large. When bidders do have equal ownership shares, they show that partnerships are dissolvable by simple $k+1$ auctions.[4] A mechanism is *simple* in McAfee's (1992) sense if it can be described without reference to the players' utility functions or the distribution of their values. Loertscher and Wasser (2015) characterize the optimal dissolution mechanism for arbitrary initial ownerships, when the objective is to maximize a weighted sum of revenue and social surplus.

To our knowledge, McAfee (1992) is the only paper to study the dissolution of partnerships when the participants are risk averse. It characterizes the equilibrium bid functions of several simple mechanisms when there are $N = 2$ CARA risk averse bidders: the Winner's bid auction, the Loser's bid auction, and the Texas Shootout (which he calls the Cake Cutting Mechanism).[5] Morgan (2004) considers fairness in dissolving a two-person partnership in a common value framework. Athanassoglou, Brams, and Sethuraman (2008) consider the problem of dissolving a partnership when the objective of the bidders is to minimize maximum regret.

The present paper is the first to propose and analyze a dynamic procedure for dissolving a partnership with $N > 2$ bidders. Abundant experimental evidence suggests that dynamic mechanisms perform more reliably than static ones, e.g., English ascending bid auctions achieve efficient allocations far more reliably than second-price sealed-bid auctions, despite being strategically equivalent.[6] The prior literature has imposed the restriction that either

[4]In a $k+1$ auction, bids are simultaneous, the item is transferred to the highest bidder, and he pays each of the other bidders a price equal to

$$\frac{1}{N}\left[kb_s + (1-k)b_f\right],$$

where b_s is the the second highest bid, b_f is the highest bid, and $k \in [0,1]$. This mechanism is also studied in Guth and van Damme (1986). de Frutos (2000) studies the $k = 0$ and $k = 1$ versions of this auction when bidders' values are drawn from asymmetric distributions. A similar family of auctions is considered by Lengwiler and Wolfstetter (2005).

[5]In the Winner's Bid auction the high bidder wins and pays half his own bid to the loser, while in the Loser's Bid auction he pays half the losing bid to the loser. The Loser's Bid auction is strategically equivalent to the two-player version of our compensation auction.

[6]See Kagel (1995) for a discussion of several such studies in his well-known survey of

bidders be risk neutral or there only be two bidders. We dispense with both restrictions.[7]

We address the efficient allocation of an indivisible object. However, the dynamic auction we propose is inspired by the early cake cutting literature which concerned the division of a divisible item.[8] In the classical cake cutting problem, N individuals are interested in dividing a heterogeneous cake. Assume that the cake is rectangular and of unit width, where $t = 0$ and $t = 1$ correspond to the left and right edge, respectively. Dubins and Spanier (1961) describe one solution to this problem: A referee holds a knife at the left edge of the cake (i.e., $t = 0$) and slowly moves it rightward across the cake, keeping it parallel to the left edge. At any time, any of the participants can call out "cut." If the first participant calls cut at t_1 then he takes the piece to the left of the knife, i.e., $[0, t_1)$, and exits. The knife now continues moving rightward until a second participant calls cut at some t_2, and he receives $[t_1, t_2)$ and exits. This continues until the $N-1$-st participant calls cut and takes the piece $[t_{N-2}, t_{N-1}]$. The last participant receives the remainder $[t_{N-1}, 1]$.

A participant who calls "cut" whenever his value for the piece of cake to the left of the knife is $1/N$-th of his value for whole cake is easily verified to obtain a piece no smaller that $1/N$-th (in his own estimation), independent of when the other participants call cut. If pieces of cake are viewed as compensation, then the Dubins and Spanier procedure is similar to our auction: In each round, compensation (money or cake) is continuously increased until one participant agrees to take the compensation and give up his right to continue. The process continues until a single participant remains, who wins the cake or the item, and who compensates the other participants (with

auction experiments.

[7]See Moldovanu (2002) for a survey of the literature on dissolving a partnership.

[8]Steinhaus (1948), Dubins and Spanier (1961), and Kuhn (1967) are early examples. See Brams and Taylor (1996) or Robertson and Webb (1998) for a textbook treatment of the subject. Chen, Lai, Parkes, and Procaccia (2013) is a more recent contribution.

either money or compensatory pieces of the cake). The two procedures are not identical, and we focus on equilibrium behavior rather than fair division. Nonetheless, the parallel is useful and is exploited later when we develop new results on security payoffs and security strategies.

2 The Model

A single indivisible item is to be allocated to one of $N \geq 2$ bidders. The bidders' values for the item are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Let $X_1, \ldots, X_N$ be N independent draws from F, and let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$, and let $G_k^{(N)}$ denote the *c.d.f.* of $Z_k^{(N)}$, i.e., $G_k^{(N)}$ is the distribution of the k-th lowest of N draws. It is easy to verify that the conditional density of $Z_{k+1}^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_k^{(N)} = z_k$ is

$$g^{(N)}_{Z_{k+1}^{(N)}|Z_1^{(N)},\ldots,Z_k^{(N)}}(z_{k+1}|z_1, \ldots, z_k) = (N-k)f(z_{k+1})\frac{[1-F(z_{k+1})]^{N-(k+1)}}{[1-F(z_k)]^{N-k}}$$

if $0 \leq z_1 \leq \ldots \leq z_{k+1}$ and is zero otherwise.[9] As the conditional distribution of $Z_{k+1}^{(N)}$ given $Z_1^{(N)}, \ldots, Z_k^{(N)}$ depends only on $Z_k^{(N)}$, we simply denote it by $G_{k+1}^{(N)}(z_{k+1}|Z_k = z_k)$ rather than the more cumbersome $G^{(N)}_{Z_{k+1}^{(N)}|Z_1^{(N)},\ldots,Z_k^{(N)}}(z_{k+1}|Z_1^{(N)} = z_1, \ldots, Z_k = z_k^{(N)})$, and likewise we write $g_{k+1}^{(N)}(z_{k+1}|z_k)$ for the conditional density. Define

$$\lambda_k^N(z) \equiv g_{k+1}^{(N)}(z|z) = (N-k)\frac{f(z)}{1-F(z)},$$

[9] See Claim 1 of the Supplemental Appendix for the derivation of this density.

to be the instantaneous probability that one of $N-k$ bidders has a value of z conditional on the k-th lowest value being z.

In the auction, the price starts at 0 and rises continuously until $N-1$ of the bidders drop out. The remaining bidder wins the item. A bidder may drop out at any point as the price ascends, dropping out is irrevocable, and dropout prices are publicly observed. Let $p_0 = 0$ and suppose $p_1 \leq p_2 \leq \ldots \leq p_{N-1}$ is the sequence of $N-1$ dropout prices.[10] The winner pays compensation of $p_k - p_{k-1}$ to the k-th bidder to drop, for each $k \in \{1, \ldots, N-1\}$. We say that the k-th bidder has dropped at "round" k. Thus if a bidder whose value is x wins the auction, then his total payment is $p_{N-1} = \sum_{k=1}^{N-1}(p_k - p_{k-1})$ and his payoff is $u(x_i - p_{N-1})$. The payoff of the k-th bidder to drop is $u(p_k - p_{k-1})$. We assume that $u' > 0$ and $u'' \leq 0$.

A strategy is a list of $N-1$ functions $\beta = (\beta_1, \ldots, \beta_{N-1})$, where $\beta_k(x; p_1, \ldots, p_{k-1})$ gives the dropout price in the k-th round of a bidder whose value is x, when $k-1$ bidders have previously dropped out at prices $p_1 \leq p_2 \leq \ldots \leq p_{k-1}$. Since a strategy must call for a feasible dropout price, we require that $\beta_k(x; p_1, \ldots, p_{k-1}) \geq p_{k-1}$ for each k and $p_1, \ldots, p_{k-1}$. Sometimes we refer to a bidder's dropout price simply as his bid.

3 Equilibrium Bidding Strategies

We characterize symmetric equilibria in increasing and differentiable bidding strategies, using a simple cost-benefit heuristic.

ROUND N-1

Suppose that β is a symmetric Bayes Nash equilibrium in increasing and differentiable strategies. In the last round (i.e., round $N-1$), two bidders remain. Let $\mathbf{p}_{N-2} = (p_1, \ldots, p_{N-2})$ be the vector of dropout prices from

[10] In the event that several bidders drop at the same price, then one randomly selected bidder drops, the rest remain, and the auction resumes.

the prior $N-2$ rounds. Since bidding strategies are increasing, the dropout prices reveal the $N-2$ smallest values $\mathbf{z}_{N-2} = (z_1, \ldots, z_{N-2})$ of the bidders who dropped in prior rounds.

Consider a bidder with value x at the moment the bid reaches his equilibrium bid of $\beta_{N-1}(x; \mathbf{p}_{N-2})$. He knows that he has the second highest value, i.e., $Z_{N-1}^{(N)} = x$. We consider the benefit and cost of remaining in the auction until the bid reaches $\beta_{N-1}(x+\Delta; \mathbf{p}_{N-2})$. If his rival's value $z = Z_N^{(N)}$ exceeds $x+\Delta$, then raising his bid increases his compensation from $\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2}$ to $\beta_{N-1}(x+\Delta; \mathbf{p}_{N-2}) - p_{N-2}$. The probability of this event is[11]

$$1 - G_N^{(N)}(x+\Delta | Z_{N-1}^{(N)} = x).$$

Thus the marginal expected benefit of raising his bid is

$$\left[\begin{array}{c} u(\beta_{N-1}(x+\Delta; \mathbf{p}_{N-2}) - p_{N-2}) \\ -u(\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2}) \end{array} \right] \left(1 - G_N^{(N)}(x+\Delta | Z_{N-1}^{(N)} = x)\right),$$

which, for Δ small, is approximately equal to

$$u'(\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2})\beta'_{N-1}(x; \mathbf{p}_{N-2})\Delta. \tag{1}$$

If his rival's value z satisfies $x < z < x + \Delta$, then as a result of raising his bid he wins the auction and obtains compensation of $x - \beta_{N-1}(z; \mathbf{p}_{N-2})$, rather than receiving equilibrium compensation of $\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2}$. Thus the marginal expected cost of raising his bid, conditional on the dropout prices observed thus far, is

$$\left[\begin{array}{c} u(\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2}) \\ -u(x - \beta_{N-1}(z; \mathbf{p}_{N-2})) \end{array} \right] \left(G_N^{(N)}(x+\Delta | Z_{N-1}^{(N)} = x) - G_N^{(N)}(x | Z_{N-1}^{(N)} = x)\right)$$

[11] Recall that the conditional distribution $G_N^{(N)}(z|Z_{N-1}^{(N)} = x)$ depends only on $Z_{N-1}^{(N)} = x$ and not on the entire vector $\mathbf{z}_{N-2}$ of revealed values.

which, since $z \approx x$ for small Δ, is approximately

$$[u(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x;\mathbf{p}_{N-2}))]g_N^{(N)}(x|Z_{N-1}^{(N)} = x)\Delta.$$

Using the definition of $\lambda_{N-1}^{N}(x)$, we write this as

$$[u(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x;\mathbf{p}_{N-2}))]\lambda_{N-1}^{N}(x)\Delta. \tag{2}$$

At equilibrium, marginal benefit equals marginal cost as Δ vanishes. Thus equations (1) and (2) yield

$$\begin{aligned} & u'(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2})\beta'_{N-1}(x;\mathbf{p}_{N-2}) \\ = \quad & [u(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x;\mathbf{p}_{N-2}))]\lambda_{N-1}^{N}(x). \end{aligned} \tag{3}$$

In a Bayes Nash equilibrium the bidding strategy β_{N-1} must satisfy this differential equation.

Round $k < N - 1$

Consider a bidder with value x at the moment the bid reaches his round-k equilibrium bid of $\beta_k(x;\mathbf{p}_{k-1})$. This bidder knows that $Z_k^{(N)} = x$. Analogous to the case for round $N-1$, the marginal benefit of remaining in the auction until the bid reaches $\beta_k(x+\Delta;\mathbf{p}_{k-1})$ is approximately

$$u'(\beta_k(x;\mathbf{p}_{k-1}) - p_{k-1})\beta'_k(x;\mathbf{p}_{k-1})\Delta. \tag{4}$$

The marginal expected cost of dropping out at a higher bid is slightly different in round $k < N-1$ than in round $N-1$. If the value of a rival bidder $z = Z_{k+1}^{(N)}$ satisfies $x < z < x + \Delta$ then, as a result of raising his bid, this rival drops out in round k and sets the dropout price $\tilde{p}_k = \beta_k(z;\mathbf{p}_{k-1})$, and the auction moves to round $k+1$. In round $k+1$ it is optimal, as we establish later, for the bidder to bid as though his type were z, i.e., to drop at the price $\beta_{k+1}(z;\mathbf{p}_{k-1},\tilde{p}_k)$. Thus, he will be the next bidder to drop and he will receive compensation of $\beta_{k+1}(z;\mathbf{p}_{k-1},\tilde{p}_k)-\beta_k(z;\mathbf{p}_{k-1})$ rather than $\beta_k(x;\mathbf{p}_{k-1})-p_{k-1}$. The probability of this event is

$$G_{k+1}^{(N)}(x+\Delta|Z_k^{(N)}=x)-G_{k+1}^{(N)}(x|Z_k^{(N)}=x)\approx g_{k+1}^{(N)}(x|Z_k^{(N)}=x)\Delta=\lambda_k^N(x)\Delta.$$

Hence, for Δ small, the marginal expected cost of raising one's bid is approximately[12]

$$\left[\begin{array}{c} u(\beta_k(x;\mathbf{p}_{k-1})-p_{k-1}) \\ -u(\beta_{k+1}(x;\mathbf{p}_{k-1},\beta_k(x;\mathbf{p}_{k-1}))-\beta_k(x;\mathbf{p}_{k-1})) \end{array}\right]\lambda_k^N(x)\Delta. \tag{5}$$

At equilibrium, marginal benefit equals marginal cost as Δ vanishes. Equations (4) and (5) yield the differential equation

$$\begin{aligned} & u'(\beta_k(x;\mathbf{p}_{k-1})-p_{k-1})\beta_k'(x;\mathbf{p}_{k-1}) \\ = & \left[\begin{array}{c} u(\beta_k(x;\mathbf{p}_{k-1})-p_{k-1}) \\ -u(\beta_{k+1}(x;\mathbf{p}_{k-1},\beta_k(x;\mathbf{p}_{k-1}))-\beta_k(x;\mathbf{p}_{k-1})) \end{array}\right]\lambda_k^N(x). \end{aligned}$$

An equilibrium bidding strategy $\beta=(\beta_1,\ldots,\beta_{N-1})$ must satisfy the system of differential equations for $k=1,\ldots,N-1$.

Equilibrium Theorem

Proposition 1(i) identifies necessary conditions for β to be a symmetric equilibrium in strictly increasing and differentiable strategies. Proposition 1(ii) establishes that any solution to this system of differential equations is an equilibrium. The remainder of this section establishes existence and uniqueness of equilibrium in two important special cases – (i) risk neutral bidders and (ii) bidders with constant absolute risk aversion.

[12] Since $z\approx x$ for Δ small, we replace z by x in the terms $\beta_{k+1}(z;\mathbf{p}_{k-1},\tilde{p}_k)$ and $\beta_k(z;\mathbf{p}_{k-1})$.

Proposition 1: *(i) Any symmetric equilibrium β, in increasing and differentiable bidding strategies, satisfies the following system of differential equations:*

$$u'(\beta_{N-1}(x;\mathbf{p}_{N-2})-p_{N-2})\beta'_{N-1}(x;\mathbf{p}_{N-2}) = \\ [u(\beta_{N-1}(x;\mathbf{p}_{N-2})-p_{N-2})-u(x-\beta_{N-1}(x;\mathbf{p}_{N-2}))]\lambda^N_{N-1}(x) \tag{6}$$

and, for $k\in\{1,\ldots,N-2\}$, that

$$u'(\beta_k(x;\mathbf{p}_{k-1})-p_{k-1})\beta'_k(x;\mathbf{p}_{k-1}) \\ = \left[\begin{array}{c} u(\beta_k(x;\mathbf{p}_{k-1})-p_{k-1}) \\ -u(\beta_{k+1}(x;\mathbf{p}_{k-1},\beta_k(x;\mathbf{p}_{k-1}))-\beta_k(x;\mathbf{p}_{k-1})) \end{array}\right]\lambda^N_k(x). \tag{7}$$

(ii) If $\beta=(\beta_1,\ldots,\beta_{N-1})$ is a solution to the system of differential equations in (i), then it is an equilibrium.

Risk Neutral Bidders

Proposition 2 characterizes equilibrium when bidders are risk neutral. It shows that in round k a bidder whose value is x sets a drop price equal to a weighted average of the dropout price observed in round $k-1$ and the expectation of the second highest value conditional on x being between the k-th and the $k-1$-st lowest values.

Proposition 2: *Suppose that bidders are risk neutral. The unique symmetric equilibrium in increasing and differentiable strategies is given, for $k=1,\ldots,N-1$, by*

$$\beta^0_k(x;\mathbf{p}_{k-1}) = \frac{N-k}{N-k+1}p_{k-1}+\frac{1}{N-k+1}E\left[Z^{(N)}_{N-1}|Z^{(N)}_k > x > Z^{(N)}_{k-1}\right]. \tag{8}$$

According to Proposition 2, equilibrium dropout prices in round k are determined by a bidder's value and the round $k-1$ dropout price, but do not depend on dropout prices in rounds prior to $k-1$.

Let N, k, N', and k', be integers such that $N-k=N'-k'\geq 1$, but otherwise be arbitrary. It is straightforward to verify that[13]

$$E[Z^{(N')}_{N'-1}|Z^{(N')}_{k'}>x>Z^{(N')}_{k'-1}]=E[Z^{(N)}_{N-1}|Z^{(N)}_{k-1}>x>Z^{(N)}_{k-1}]\ \forall x\in[0,\bar{x}].$$

In other words, the expectation of the second highest of N' draws, conditional on x being between the k'-th and $k'-1$-st lowest draws, is the same as the expectation of the second highest of N draws, conditional on x being between the k-th and $k-1$-st lowest draws. Corollary 1 follows immediately from Proposition 2. In stating the corollary it is useful to write $\beta^0_{k,N}(x;\mathbf{p}_k)$ for the equilibrium bid function in round k of an auction with N bidders.

Corollary 1: *If $N'-k'=N-k$ and bidders are risk neutral, then the equilibrium bid function in round k' of an auction with N' bidders is the same as the equilibrium bid function in round k of an auction with N bidders. Equilibrium bids depend on only the number of rounds remaining in the auction and the last observed dropout price. In particular, $\beta^0_{k',N'}(x;\mathbf{p}_{k'})=\beta^0_{k,N}(x;\mathbf{p}_k)$* whenever $p_{k'}=p_k$.

Corollary 1 identifies an intuitive property of equilibrium, but it depends on the uniqueness of equilibrium. If there are multiple equilibria, then one might make a selection based on the number of bidders.

Example 1: Suppose $N=3$, bidders are risk neutral, and values are distributed $U[0,1]$. Equilibrium drop out prices in round 1 are given by

$$\beta^0_1(x)=\frac{1}{6}x+\frac{1}{6},$$

[13]See Claims 2 and 3 of the Supplemental Appendix.

and in round 2 are given by

$$\beta_2^0(x;p_1) = \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_1.$$

By Corollary 1, the equilibrium bid function in round 3 of an auction with 4 bidders is $\beta_3^0(x;\mathbf{p}_2) = \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_2$.

CARA Bidders

Proposition 3 characterizes equilibrium when bidders have constant absolute risk aversion (CARA), i.e., their utility functions are given by

$$u^\alpha(x) = \frac{1 - e^{-\alpha x}}{\alpha},$$

where $\alpha > 0$ is their index of risk aversion. Note that $\lim_{\alpha\to 0} u^\alpha(x) = x$, i.e., bidders are risk neutral in the limit as α approaches zero. Denote by β_k^α the equilibrium bid function in round k when bidders have CARA index of risk aversion α.

Proposition 3: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$. The unique symmetric equilibrium in increasing and differentiable strategies is given, for $k = 1, \ldots, N-1$, by*

$$\beta_k^\alpha(x;\mathbf{p}_{k-1}) = \frac{N-k}{N-k+1}p_{k-1} - \frac{N-k}{(N-k+1)\,\alpha}\ln\left(J_k^\alpha(x)\right), \tag{9}$$

where

$$J_{N-1}^\alpha(x) = E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]$$

and, for $k < N-1$, $J_k^\alpha(x)$ is defined recursively as

$$J_k^\alpha(x) = E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right].$$

Example 2: Suppose $N = 3$, bidders are CARA risk averse with index of risk aversion α, and values are distributed $U[0, 1]$. Equilibrium drop out prices in round 1 are given by

$$\beta_1^\alpha(x) = -\frac{2}{3\alpha} \ln \left(\frac{\int_x^1 \left(\frac{\int_z^1 e^{-\alpha t} 2(1-t)dt}{(1-z)^2} \right)^{\frac{1}{2}} 3(1-z)^2 dz}{(1-x)^3} \right),$$

and in round 2 are given by

$$\beta_2^\alpha(x; p_1) = \frac{1}{2} p_1 - \frac{1}{2\alpha} \ln \left(\frac{\int_x^1 e^{-\alpha z} 2(1-z)dz}{(1-x)^2} \right).$$

Figure 1 (below) shows the equilibrium bid functions for $\alpha = 10$. The round 2 bid function is shown under the assumption that the first bidder drops at a bid of .2, which reveals (in equilibrium) his value is $z_1 = (\beta_1^{10})^{-1}(.2) \approx .35154$. Since this value is the lower bound of the set of buyer types remaining in the auction, the figure shows $\beta_2^{10}(x; 1/5)$ for $x \geq z_1$.

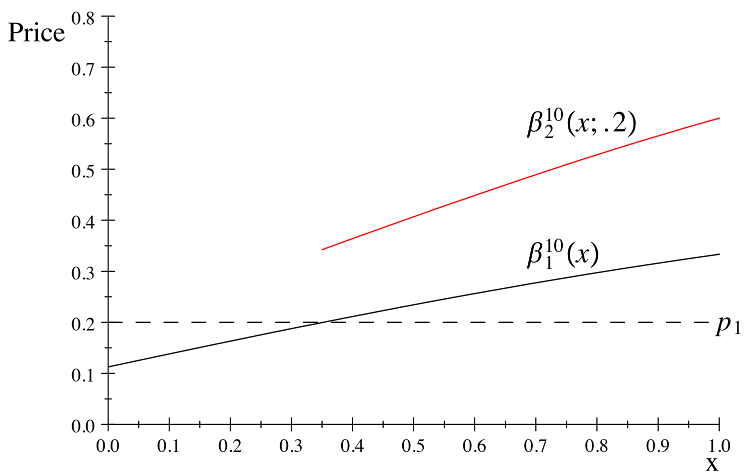

Figure 1: Equilibrium bids by round, for $N = 3, U[0, 1]$, and $CARA$ ($\alpha = 10$).

Proposition 4 establishes tight upper and lower bounds for the dropout prices of CARA risk averse bidders.

Proposition 4: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Then for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ we have that*

$$\beta_k^0(x; \mathbf{p}_{k-1}) > \beta_k^\alpha(x; \mathbf{p}_{k-1}) > \frac{x - p_{k-1}}{N-k+1} + p_{k-1} \text{ for } x < \bar{x},$$

i.e., CARA risk averse bidders demand less compensation than risk neutral bidders, but always demand compensation of at least $(x - p_{k-1})/(N-k+1)$.

Proposition 5 establishes the intuitive result that CARA bidders drop out at lower prices as they become more risk averse.

Proposition 5: *Suppose that bidders are CARA risk averse with index of risk aversion α. Dropout prices decrease as bidders become more risk averse, i.e., $\tilde{\alpha} > \alpha$ implies, for $k = 1, \ldots, N-1$, that*

$$\beta_k^\alpha(x; \mathbf{p}_{k-1}) > \beta_k^{\tilde{\alpha}}(x; \mathbf{p}_{k-1}) \; \forall k \in \{1, \ldots, N-1\}, \, \forall x \in [0, \bar{x}), \, \forall \mathbf{p}_{k-1},$$

except for bidders with the highest possible value $\bar{x}$, for whom the dropout price does not depend α.

Proposition 6 shows that as CARA bidders become infinitely risk averse, equilibrium bids approach the (linear) lower bound identified in Proposition 4.

Proposition 6: *Suppose that bidders are CARA risk averse with index of risk aversion α. Then for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ we have*

$$\lim_{\alpha \to \infty} \beta_k^\alpha(x; \mathbf{p}_{k-1}) = \frac{x - p_{k-1}}{N-k+1} + p_{k-1} \text{ for } x < \bar{x}.$$

Figure 2 below illustrates these results. It shows equilibrium bids in round 1 for $\alpha = 0$, 10, 100, and ∞ when $N = 3$ and values are distributed $U[0, 1]$. As α approaches infinity, $\lim_{\alpha\to\infty} \beta_1^{\alpha}(x) = x/3$. Later we shall see that $\beta_1^{\infty}(x) = x/3$ corresponds to a particular security strategy.

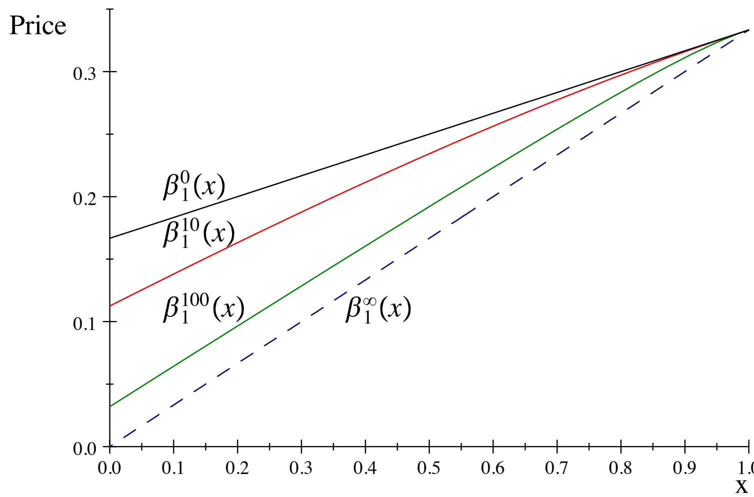

Figure 2: Round 1 equilibrium bids for $N = 3$, $U[0, 1]$, and $\alpha = 0, 10, 100,$ and ∞.

4 Properties of Equilibrium

In this section we examine the properties of equilibrium in the compensation auction. The auction is clearly ex-post efficient in any equilibrium in strictly increasing bidding strategies. Let $V^*(x)$ denote the (ex-ante) symmetric equilibrium expected utility of a bidder whose value is x.[14] We say that an auction is **interim proportional** if $V^*(x) \geq u(x/N)$ $\forall x \in [0, \bar{x}]$. In other words, each bidder in the auction obtains in expectation a utility equal to at least $1/N$-th of his value for the item. An auction is **ex-post proportional** if for each bidder and each of his possible values $x \in [0, \bar{x}]$, the *realized* payoff of the bidder is at least $u(x/N)$.

[14] Recall from Propositions 1 and 2 there is a unique symmetric equilibrium when bidders are either risk neutral or CARA risk averse.

Proposition 7: *The compensation auction is interim proportional.*

The intuition for Proposition 7 is straightforward: If a bidder with value x_i follows the strategy of dropping out whenever his compensation reaches x_i/N, then he *guarantees* himself a payoff of at least $u(x_i/N)$. In particular, regardless of the strategies and values of the other bidders, he either drops at some stage k and obtains compensation of exactly x_i/N or he wins the item at a price no more than $(N-1)x_i/N$. Since a bidder's equilibrium strategy must give him at least this payoff, the compensation auction is interim proportional.

In equilibrium a bidder acts to maximize his expected payoff. Thus, while a bidder can guarantee himself a payoff of at least $u(x_i/N)$ in the compensation auction, he may follow instead some other strategy which gives him a higher expected payoff but which possibly realizes a payoff ex-post of less than $u(x_i/N)$.

An auction that is interim proportional is "fair" in the sense that, prior to the auction, each bidder expects to obtain at least $1/N$-th of what he himself regards as the value of the item. If the auction is "ex-post" proportional, then even bidders who do not win the auction will regard the outcome as fair. Example 3 illustrates our results on interim proportionality, and shows that the compensation auction is not ex-post proportional.

Example 3: Suppose $N = 3$, bidders are risk neutral, and values are distributed $U[0, 1]$. In the compensation auction, a bidder with value zero has an expected payoff of $1/6$. Therefore, by the Revenue Equivalence Theorem, the expected payoff of a bidder with value x is $\frac{1}{6} + \frac{1}{3}x^3$. The compensation auction is interim proportional since $\frac{1}{6} + \frac{1}{3}x^3 \geq \frac{1}{3}x$ for $x \in [0, 1]$.

It is also ex-post proportional for the first two bidders to drop. If the value of the first bidder to drop is x_1 then his compensation is $\beta_1(x_1)$. Since $x_1 \leq 1$, then $\beta_1(x_1) = \frac{1}{6}x_1 + \frac{1}{6} \geq \frac{1}{3}x_1$. If the value of the second bidder to drop is x_2, then $p_1 \leq \frac{1}{3}$ implies his compensation is $\beta_2(x_2, p_1) - p_1 = \frac{1}{3}x_2 + \frac{1}{6} - \frac{1}{2}p_1 \geq \frac{1}{3}x_2$.

The auction need not be ex-post proportional for the winner. If all three bidders have values near zero, for example, then the first bidder drops at approximately $\frac{1}{6}$, the second bidder drops at approximately $\frac{1}{4}$, and thus the payoff of the winner is approximately $-\frac{1}{4}$. $\square$

Mimic Deviations

Suppose bidders are risk neutral and follow a symmetric equilibrium β. In this section we show that bidders' equilibrium strategies are not strict best responses. A bidder also obtains his equilibrium payoff by a deviation from β in which, rather than dropping at his equilibrium bid, he "mimics" the equilibrium behavior of a bidder with a higher value. Consider, in particular, a bidder for whom in round k the auction price has just reached his equilibrium dropout price. (This bidder therefore has the k-th lowest value $Z_k^{(N)}$.) In a one-round mimic deviation this bidder remains in the auction until the next bidder drops, he infers the bidder's value $Z_{k+1}^{(N)}$, and then in round $k+1$ he bids as if his own value were $Z_{k+1}^{(N)}$.[15] Likewise, a m**-round mimic deviation** is one in which the bidder allows his dropout price to pass, he remains in the auction and observes the next m lowest type bidders drop at rounds $k, \dots, k+m-1$, inferring their values $Z_{k+1}^{(N)}, \dots, Z_{k+m}^{(N)}$, and then in round $k+m$ he bids as though his own value is $Z_{k+m}^{(N)}$. Note that a $m > 0$ mimic deviation entails dropping out at a later round, rather than winning the auction.

For each $m \in \{0, \dots, N-k-1\}$, let C_{k+m} be the random variable which is the bidder's payoff from the m-round mimic deviation, where C_k is his compensation if he obeys β, evaluated at the moment he drops out. Our main result is that the sequence $\{C_{k+m}\}_{m=0}^{N-k-1}$ is a martingale.[16]

[15] This bidder will be the next bidder to drop since the values of the remaining bidders exceed $Z_{k+1}^{(N)}$.

[16] Martingales have also been studied in standard sequential first and second price sealed bid auctions by Weber (1983) and Milgrom and Weber (2000) which demonstrated that the sequence of sale prices forms a martingale. Mezzetti (2011) and Hu and Zou (2015)

Proposition 8: *For any $k < N-1$, the sequence $\{C_{k+m}\}_{m=0}^{N-k-1}$ of m round mimic compensations is a martingale, i.e., $E[C_{k+m+1}|C_{k+m},\ldots,C_k] = C_{k+m}$* for $m \in \{0,\ldots,N-k-1\}$.

The following corollary is an immediate consequence of the martingale property of mimic compensations.

Corollary: $E[C_{k+m}|Z_k^{(N)} = x_k,\ldots,Z_1^{(N)} = x_1] = C_k$ for all $m = 0,\ldots,N-1-k$.

A m-round mimic deviation does not influence which bidder wins the auction (so long as $m \leq N-1-k$) and hence has no effect on the surplus realized in the auction. Further, since the expected compensation of the deviating bidder is unchanged, the expected payoff of all the other bidders is unchanged as well.

Example 4: Mimic Martingales Suppose $N = 4$, bidders are risk neutral, and values are distributed $U[0,1]$. The equilibrium bidding functions are $\beta_1^0(x) = \frac{1}{10}x + \frac{3}{20}$, $\beta_2^0(x;p_1) = \frac{1}{6}x + \frac{1}{6} + \frac{2}{3}p_1$, and $\beta_3^0(x;p_1,p_2) = \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_2$. In what follows it is useful to observe that $E\left[Z_2^{(4)}|Z_1^{(4)} = \frac{1}{2}\right] = \frac{5}{8}$, $E\left[Z_3^{(4)}|Z_1^{(4)} = \frac{1}{2}\right] = \frac{3}{4}$, and $E\left[Z_4^{(4)}|Z_1^{(4)} = \frac{1}{2}\right] = \frac{7}{8}$.

Consider a bidder with type $x = 1/2$ when, in round 1, the bid has reached his equilibrium drop-out price of $\beta_1^0(1/2) = 1/5$. He knows he has the lowest type (i.e., $Z_1^{(4)} = 1/2$) and, if he obeys β^0, he obtains compensation of $1/5$. Suppose instead he follows a 1-round mimic deviation. He remains in the auction until the bidder with the next lowest value drops in round 1 at $p_1 = \beta_1^0(Z_2^{(4)})$. He infers $Z_2^{(4)}$ from the price p_1 and then in round 2 he bids as if his value were $Z_2^{(4)}$, i.e., he drops at the price $\beta_2^0(Z_2^{(4)};\beta_1^0(Z_2^{(4)}))$. The

study how price sequences depend on the bidders' risk attitudes.

compensation from this strategy is the random variable

$$\begin{aligned} C_2(Z_2^{(4)}) &= \beta_2^0(Z_2^{(4)}; \beta_1^0(Z_2^{(4)})) - \beta_1^0(Z_2^{(4)}) \\ &= \frac{1}{6}Z_2^{(4)} + \frac{1}{6} - \frac{1}{3}\left(\frac{1}{10}Z_2^{(4)} + \frac{3}{20}\right) \\ &= \frac{2}{15}Z_2^{(4)} + \frac{7}{60}. \end{aligned}$$

The bidder's expected payoff from the 1-round mimic deviation is

$$E\left[C_2(Z_2^{(4)})|Z_1^{(4)} = \frac{1}{2}\right] = \frac{2}{15}E\left[Z_2^{(4)}|Z_1^{(4)} = \frac{1}{2}\right] + \frac{7}{60} = \frac{2}{15}\left(\frac{5}{8}\right) + \frac{7}{60} = \frac{1}{5}.$$

In other words, conditional on the bid reaching his dropout price in round 1, the bidder obtains the same expected payoff from following the 1-round mimic strategy.

If he follows a 2-round mimic strategy, the bidder waits two rounds, inferring the second and third lowest values $Z_2^{(4)}$ and $Z_3^{(4)}$, respectively, from the drop prices in round 1 and 2, and he then bids $Z_3^{(4)}$ in round 3, i.e., he drops at the price $\beta_3^0(Z_3^{(4)}; \beta_2^0(Z_3^{(4)}, \beta_1^0(Z_2^{(4)})))$.[17] The compensation from this strategy is

$$\begin{aligned} C_3(Z_3^{(4)}, Z_2^{(4)}) &= \beta_3^0(Z_3^{(4)}; \beta_2^0(Z_3^{(4)}, \beta_1^0(Z_2^{(4)}))) - \beta_2^0(Z_3^{(4)}; \beta_1^0(Z_2^{(4)})) \\ &= \frac{1}{3}Z_3^{(4)} + \frac{1}{6} - \frac{1}{2}\left(\frac{1}{6}Z_3^{(4)} + \frac{1}{6} + \frac{2}{3}\left(\frac{1}{10}Z_2^{(4)} + \frac{3}{20}\right)\right) \\ &= \frac{1}{4}Z_3^{(4)} - \frac{1}{30}Z_2^{(4)} + \frac{1}{30} \end{aligned}$$

The bidder's expected payoff from the 2-round mimic deviation is likewise

$$E\left[C_3(Z_3^{(4)}, Z_2^{(4)})|Z_1^{(4)} = \frac{1}{2}\right] = \frac{1}{4}\left(\frac{3}{4}\right) - \frac{1}{30}\left(\frac{5}{8}\right) + \frac{1}{30} = \frac{1}{5}.$$

[17] We abuse notation here by supressing p_1 in $\beta_3^0(x; p_1, p_2)$ and instead writing $\beta_3^0(Z_3^{(4)}; \beta_2^0(Z_3^{(4)}, \beta_1^0(Z_2^{(4)})))$.

The bidder's payoff, however, is strictly lower if he simply remains in the auction until he wins. In this case he obtains his value minus the price at which the third bidder drops

$$\frac{1}{2}-\beta_3^0(Z_4^{(4)};\beta_2^0(Z_3^{(4)};\beta_1^0(Z_2^{(4)}))) = \frac{1}{2}-\left[\frac{1}{3}Z_4^{(4)}+\frac{1}{6}+\frac{1}{2}\left[\frac{1}{6}Z_3^{(4)}+\frac{1}{6}+\frac{2}{3}\left[\frac{1}{10}Z_2^{(4)}+\frac{3}{20}\right]\right]\right].$$

Expected compensation, conditional on $Z_1^{(4)} = 1/2$, is only $-7/40$.

5 Security Strategies

A significant practical obstacle to the actual implementation of any dissolution mechanism may be that the participants are uncertain of their equilibrium strategies or uncertain of whether the other participants will play their part of an equilibrium, and therefore uncertain of what payoff they are likely to obtain via the mechanism. In this section we provide advice to the bidders with the goal of guaranteeing that they do not do "too badly" regardless of the behavior of their rivals.

We identify security strategies and security payoffs in the compensation auction. A player's security payoff is the largest payoff that he can guarantee himself, regardless of the values and strategies of the other players, and a security strategy is a strategy which guarantees a player his security payoff. Write $v(x_i, x_{-i}, \beta^i, \beta^{-i})$ for the payoff to a bidder whose value is x_i and who follows the strategy β^i, where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $\beta^{-i} = (\beta^1, \ldots, \beta^{i-1}, \beta^{i+1}, \ldots, \beta^N)$ are the values and strategies of the remaining players.

Definition: *Bidder i's security payoff when his value is x_i* is the largest value $\bar{v}(x_i)$ for which he has a strategy $\bar{\beta}^i$ such that

$$v(x_i, x_{-i}, \bar{\beta}^i, \beta^{-i}) \geq \bar{v}(x_i)\ \forall x_{-i}, \beta^{-i}.$$

We say that $\bar{\beta}^i$ a *security strategy for bidder* i if for each $x_i \in [0, \bar{x}]$ the strategy guarantees him $\bar{v}(x_i)$.

Proposition 9 identifies bidder i's security payoff and a security strategy which attains it.

Proposition 9: *The strategy which calls for bidder* i *to drop out when his compensation reaches* x_i/N *is a security strategy and realizes the security payoff of* x_i/N*. More formally, the strategy* $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = x_i/N + p_{k-1}$ *for each* $k \in \{1, \ldots, N-1\}$*, and every* $x_i \in [0, \bar{x}]$ *and* $\mathbf{p}_{k-1}$ *such that* $0 \leq p_1 \leq \ldots \leq p_{k-1}$*, is a security strategy.*

The strategy given in Proposition 9 is simple in the sense that the compensation a bidder demands does not depend on the prior history of dropout prices – he drops as soon as the current bid exceeds the prior dropout price by x_i/N. A bidder, however, has many security strategies. Of particular interest is the one which calls for a bidder to drop in stage k when the bid exceeds the prior dropout price by $(x_i - p_{k-1})/(N-k+1)$. Proposition 10 establishes that this strategy is also security strategy.

Proposition 10: *Let* $\bar{\beta}^i$ *be such that* $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = (x_i - p_{k-1})/(N-k+1) + p_{k-1}$ *for each* $k \in \{1, \ldots, N-1\}$*, and every* $x_i \in [0, \bar{x}]$ *and* $\mathbf{p}_{k-1}$ *such that* $0 \leq p_1 \leq \ldots \leq p_{k-1} \leq x_i$*.*[18] *Then* $\bar{\beta}^i$ *is a security strategy.*

Proposition 11 generalizes Proposition 10 by identifying a class of security strategies.[19] It shows that any strategy in which the bidder demands compensation between $x_i/N + p_{k-1}$ and $(x_i - p_{k-1})/(N-k+1) + p_{k-1}$ is a security strategy.

[18] No restriction is placed on $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1})$ if $x_i < p_{k-1}$ since this contingency never arises if bidder i follows $\bar{\beta}^i$.

[19] We adopt the usual convention that $[a, b] = \{a\}$ if $a = b$, and $[a, b] = \emptyset$ if $a > b$.

Proposition 11: *Let $\bar{\beta}^i$ be such that $\bar{\beta}^i_k(x_i;\mathbf{p}_{k-1}) \in [\frac{x_i}{N}+p_{k-1}, \frac{x_i-p_{k-1}}{N-k+1}+p_{k-1}]$ for each $k \in \{1,\ldots,N-1\}$, and every $x_i \in [0,\bar{x}]$ and $\mathbf{p}_{k-1}$ such that $0 \leq p_1 \leq \ldots \leq p_{k-1} \leq \bar{x}$.*[20] *Then $\bar{\beta}^i$ is a security strategy.*

Perfect Security Strategies

A bidder's security payoff is the maximum payoff he can guarantee himself at the start of the auction. The main result in this subsection is to identify the unique strategy which maximizes the payoff a bidder guarantees himself following any sequence $p_1,\ldots,p_k$ of drop out prices.

To proceed, it is useful to introduce the notation of a subauction. In the subauction $\Gamma(n,p_0)$ there are $n \leq N$ bidders and the price ascends from p_0, where p_0 may be strictly positive. If $p_1 \leq \ldots \leq p_{n-1}$ is the sequence of dropout prices in $\Gamma(n,p_0)$, then the winner pays the difference $p_k - p_{k-1}$ to the k-th bidder to drop for $k = \{1,\ldots,n-1\}$, and also pays p_0 to a third party.

Our results to this point concern the auction $\Gamma(N,0)$. However, if at round k the sequence of dropout prices is $p_1,\ldots,p_{k-1}$, then the remaining bidders participate in $\Gamma(N-(k-1),p_{k-1})$, i.e., the subauction with $N-(k-1)$ bidders and the price ascending from p_{k-1}.

Proposition 10 identified a security strategy for $\Gamma(N,0)$. Proposition 12 is the analogue to Proposition 10 for $\Gamma(n,p_0)$. It identifies the bidders' security payoffs and a security strategy when the initial price p_0 need not be zero.

Proposition 12: *Let $p_0 \geq 0$. In the subauction $\Gamma(n,p_0)$ the strategy $\bar{\beta}^i$, given by*

$$\bar{\beta}^i_k(x_i;\mathbf{p}_{k-1}) = \begin{cases} (x_i - p_{k-1})/(n-k+1) + p_{k-1} & \text{if } x_i \geq p_{k-1} \\ \\ p_{k-1} & \text{if } x_i < p_{k-1} \end{cases}$$

[20] Observe that no restriction is placed on dropout prices for k, x_i, and $\mathbf{p}_{k-1}$ such that $[\frac{x_i}{N}+p_{k-1}, \frac{x_i-p_{k-1}}{N-k+1}+p_{k-1}]$ is empty.

for each $k \in \{1, \ldots, n-1\}$, and every $x_i \in [0, \bar{x}]$ and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$, is a security strategy. Furthermore, bidder i's security payoff when his value is x_i is $(x_i - p_0)/n$.

An implication of Proposition 12 is that a bidder's security payoff weakly increases from one round to the next when he follows the security strategy $\bar{\beta}^i$ identified in Proposition 10. To see this, consider a bidder whose value is x_i and who remains in the auction at round $k+1$ following drops at prices $p_1, \ldots, p_k$. By Proposition 12, his security payoff in the subauction $\Gamma(N-k, p_k)$ is

$$\frac{x_i - p_k}{N-k}.$$

Since the bidder did not drop in round k, then the bid at which a rival dropped must be less than his own round k bid, i.e.,

$$p_k \leq \bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = \frac{x_i - p_{k-1}}{N-k+1} + p_{k-1}.$$

Hence

$$\frac{x_i - p_k}{N-k} \geq \frac{x_i - (\frac{x_i - p_{k-1}}{N-k+1} + p_{k-1})}{N-k} = \frac{x_i - p_{k-1}}{N-(k-1)},$$

where the right hand side was the bidder's security payoff in round k in $\Gamma(N-(k-1), p_{k-1})$. Indeed, so long as bidder i is never indifferent between dropping or continuing, the inequalities above are strict and bidder i's security payoff strictly increases from one round to the next.

A security strategy is *perfect* if it continues to be a security strategy in the auction that remains following any sequence of drops. Formalizing this idea requires introducing the notion of the restriction of a strategy (for $\Gamma(N, 0)$) to a subauction. Let $\beta^i|_{\mathbf{p}_{k-1}}$ be the restriction of β^i to the auction $\Gamma(N-(k-1), p_{k-1})$ obtained after $k-1$ bidders in $\Gamma(N, 0)$ drop at prices

$(p_1, \ldots, p_{k-1})$, i.e., define

$$\begin{aligned} \beta_1^i|_{\mathbf{p}_{k-1}}(x_i) &\equiv \beta_k^i(x_i, \mathbf{p}_{k-1}), \\ \beta_2^i|_{\mathbf{p}_{k-1}}(x_i; p_k) &\equiv \beta_{k+1}^i(x_i; \mathbf{p}_{k-1}, p_k), \\ &\vdots \\ \beta_{N-k}^i|_{\mathbf{p}_{k-1}}(x_i; p_k, \ldots, p_{N-2}) &\equiv \beta_{N-1}^i(x_i; \mathbf{p}_{k-1}, p_k, \ldots, p_{N-2}). \end{aligned}$$

Formally, a perfect security strategy is defined as follows:

Definition: $\bar{\beta}^i$ is a **perfect security strategy** for bidder i if for each $x_i \in [0, \bar{x}]$, $k \in \{1, \ldots, N-1\}$, and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$, then $\bar{\beta}^i|_{\mathbf{p}_{k-1}}$ is a security strategy for bidder i in $\Gamma(N-(k-1), p_{k-1})$.

Proposition 13 shows that the security strategy identified in Proposition 10 is the unique perfect security strategy.

Proposition 13: *In the compensation auction $\Gamma(N, 0)$ the strategy $\bar{\beta}^i$, given by*

$$\bar{\beta}_k^i(x_i; \mathbf{p}_{k-1}) = \begin{cases} (x_i - p_{k-1})/(n-k+1) + p_{k-1} & \text{if } x_i \geq p_{k-1} \\ p_{k-1} & \text{if } x_i < p_{k-1} \end{cases}$$

for each $k \in \{1, \ldots, N-1\}$, and every $x_i \in [0, \bar{x}]$ and $\mathbf{p}_{k-1}$ such that $0 \leq p_1 \leq \ldots \leq p_{k-1}$, is the unique perfect security strategy.

6 Discussion

Compensation auctions can also be used to allocate an indivisible undesirable item (e.g., a waste dump or a nuclear power plant) or an indivisible costly task or chore (e.g., an administrative position). An allocation mechanism in such a setting must determine which of the N players is to accept the undesirable item or complete the chore and how the other players are going to compensate him. We consider the problem of allocating a chore.

The key to employing the compensation auction (which is defined for a "good") is to make the chore desirable. Suppose each bidder's cost of completing the chore is independently and identically distributed according to cumulative distribution function F with support $[0, \bar{c}]$. In order to make the chore desirable, each of the N bidders contributes $\bar{c}/N$ into a pot which will be awarded to the bidder assigned to complete the chore. Thus, if bidder i with cost c_i undertakes the chore, then he receives a total payoff of $v_i = \bar{c} - c_i \geq 0$. The compensation auction can be used to allocate the chore to a bidder and to determine the compensations (which can be viewed as rebates of $\bar{c}/N$) that the winner provides to the remaining bidders.

The auctions operates as before: The price, starting from zero, rises continuously and a bidder may drop out at any point. A bidder who drops out surrenders the opportunity to do the chore but, in return, receives compensation from the winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. Since the auction is interim proportional (see Proposition 7), then bidder i's equilibrium payoff is at least $v_i/N = (\bar{c} - c_i)/N$. Thus bidder i's payoff, net of his contribution $\bar{c}/N$, is at least

$$\frac{\bar{c} - c_i}{N} - \frac{\bar{c}}{N} = -\frac{c_i}{N}.$$

In other words, each bidder's payoff is equal to at least $1/N$-th of his cost of undertaking the chore. Each bidder i has a security strategy which guarantees that he incurs a cost no more than c_i/N (Proposition 9). Furthermore, since the auction is ex-post efficient, the chore is allocated to the bidder for whom the cost of completing the chore is smallest.

7 Appendix

Lemma 0 found in McAfee (1992) is not directly applicable to our paper since the payoff function may not be C^2 for all x and y. However, the following simple extension plays the same role and can be applied in our setting.

Lemma 0: *Suppose an agent of type x who reports y receives profits equal to*

$$\pi(x,y) = \begin{cases} \pi_H(x,y) & \text{if } y \geq x \\ \pi_L(x,y) & \text{if } y \leq x \end{cases} .$$

Further suppose that for all x we have

$$\frac{\partial}{\partial y}\pi(x,x) = \frac{\partial}{\partial y}\pi_H(x,x) = \frac{\partial}{\partial y}\pi_L(x,x) = 0 \tag{10}$$

and that

$$\begin{aligned} &\tfrac{\partial^2}{\partial x \partial y}\pi_H(x,y) \geq 0 \quad \text{for } y > x \\ &\tfrac{\partial^2}{\partial x \partial y}\pi_L(x,y) \geq 0 \quad \text{for } y < x. \end{aligned} \tag{11}$$

Then π is maximized over y at $y = x$.

Proof of Lemma 0: First, from (10), we have that $\frac{\partial}{\partial y}\pi(x,x) = \frac{\partial}{\partial y}\pi_H(x,x) = \frac{\partial}{\partial y}\pi_L(x,x) = 0$ for all x. Second, since (10) and (11) if $y > x$, then

$$\frac{\partial}{\partial y}\pi_H(x,y) \leq 0$$

and if $y < x$, then

$$\frac{\partial}{\partial y}\pi_L(x,y) \geq 0.$$

Hence, we have established that: (i) if $y < x$, then $\frac{\partial}{\partial y}\pi(x,y) = \frac{\partial}{\partial y}\pi_L(x,y) \geq 0$; (ii) if $y = x$, then $\frac{\partial}{\partial y}\pi(x,x) = 0$; and (iii), if $y > x$, then $\frac{\partial}{\partial y}\pi(x,y) = \frac{\partial}{\partial y}\pi_H(x,y) \leq 0$. Therefore π is maximized over y at $y = x$. $\square$

Proof of Proposition 1: Part (i) is established by the heuristic derivation. We prove (ii). Let $\beta = (\beta_1, \ldots, \beta_{N-1})$ be a solution to the system of differential equations. Since equilibrium is in increasing strategies, the sequence of dropout prices $(p_1, \ldots, p_{k-1})$ at round k reveals the $k-1$ lowest values $(z_1, \ldots, z_{k-1})$. In the proof it is convenient to write the round k equilibrium bid as a function of the prior dropout values rather than as a function of the prior dropout prices. In particular, we write $\beta_k(x|\mathbf{z}_{k-1})$ rather than $\beta_k(x; \mathbf{p}_{k-1})$.

For each $k < N$, let $\pi_k(y, x|\mathbf{z}_{k-1})$ be the expected payoff to a bidder with value x who in round k deviates from equilibrium and bids as though his value is y (i.e., he bids $\beta_k(y|\mathbf{z}_{k-1})$), when $\mathbf{z}_{k-1}$ is the profile of values of the $k-1$ bidders to drop so far and the remaining bidders follow β. Define

$$\Pi_k(x|\mathbf{z}_{k-1}) = \pi_k(x, x|\mathbf{z}_{k-1}).$$

It is clearly never optimal for a bidder to bid as though his type were less than z_{k-1}, i.e., bid less than $\beta_k(z_{k-1}|\mathbf{z}_{k-1})$, since bidding z_{k-1} yields a greater compensation.[21]

Consider the following two-part claim for round k:

(a) For each $\mathbf{z}_{k-1}$: if $x \geq z_{k-1}$ then $x \in \arg\max_y \pi_k(y, x|\mathbf{z}_{k-1})$, i.e., it is optimal for the bidder to follow β_k in round k; if $x < z_{k-1}$ then $z_{k-1} \in \arg\max_y \pi_k(y, x|\mathbf{z}_{k-1})$.

(b) For each $\mathbf{z}_{k-1}$ we have that

$$\frac{d\Pi_k(x|\mathbf{z}_{k-1})}{dx} \geq 0\,.$$

We prove by induction that the claim is true for each $k \in \{1, \ldots, N-1\}$.

We first show the claim is true for round $N-1$. Let $\mathbf{z}_{N-2}$ be arbitrary.

[21]Hereafter it is convenient to say the bidder bids y rather than saying he bids $\beta_k(y|\mathbf{z}_{k-1})$.

Consider an active bidder in the k-th round whose value is x but who bids as though it were $y \geq z_{N-2}$. Suppose that $x \geq z_{N-2}$. With a bid of $y \geq z_{N-2}$, the bidder wins and obtains $x - \beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2})$ if $y > z_{N-1}$, and he obtains compensation $\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2}$ if $y < z_{N-1}$, where $p_{N-2} = \beta_{N-2}(z_{N-2}|\mathbf{z}_{N-3})$. Hence

$$\begin{aligned}\pi_{N-1}(y,x|\mathbf{z}_{N-2}) = & \int\limits_{z_{N-2}}^{y} u(x - \beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1} \\ & + \int\limits_{y}^{\bar{x}} u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & [u(x - \beta_{N-1}(y|\mathbf{z}_{N-2})) - u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})]g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2}) \\ + \quad & u'(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})\beta'_{N-1}(y|\mathbf{z}_{N-2})(1 - G_{N-1}^{(N-1)}(y|Z_{N-2}^{(N-1)} = z_{N-2})). \end{aligned} \tag{12}$$

Since equation (3) holds for all x and, in particular, for $x = y$, we have that

$$\begin{aligned} & u'(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})\beta'_{N-1}(y|\mathbf{z}_{N-2}) \\ = \quad & \left[u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2}) - u(y - \beta_{N-1}(y|\mathbf{z}_{N-2}))\right] \lambda_{N-1}^{(N-1)}(y|z_{N-2}). \end{aligned} \tag{13}$$

Substituting (13) into (12) and simplifying yields

$$\frac{\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial y} = \left[u(x - \beta_{N-1}(y|\mathbf{z}_{N-2})) - u(y - \beta_{N-1}(y|\mathbf{z}_{N-2}))\right] g_{N-1}^{(N-1)}(y|z_{N-2}).$$

Clearly, $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y|_{y=x} = 0$. Moreover, for $y \geq z_{N-2}$ we have

$$\frac{\partial^2\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial y \partial x} = u'(x - \beta_{N-1}(y|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(y|z_{N-2}) \geq 0,$$

where the inequality holds since $u' > 0$ and $g_{N-1}^{(N-1)}(y|z_{N-2}) > 0$.

Suppose $x < z_{N-2}$. As already noted, it is never optimal to bid y less than z_{N-2}. Furthermore, $y \geq z_{N-2} > x$ implies that

$$[u(x - \beta_{N-1}(y|\mathbf{z}_{N-2})) - u(y - \beta_{N-1}(y|\mathbf{z}_{N-2}))]g_{N-1}^{(N-1)}(y|z_{N-2}) < 0.$$

Thus $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ for $y \geq z_{N-2}$ and therefore $z_{N-2} \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$. Hence (a) is true for $k = N - 1$ by Lemma 0 of McAfee.

To prove (b), note that

$$\begin{aligned} \frac{d\Pi_{N-1}(x|\mathbf{z}_{N-2})}{dx} &= \left.\frac{\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})}{\partial y}\right|_{y=x} + \left.\frac{\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})}{\partial x}\right|_{y=x} \\ &= \int_{z_{N-2}}^{x} u'(x - \beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1} \\ &\geq 0, \end{aligned}$$

where $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y|_{y=x} = 0$. Hence (b) holds for $k = N - 1$.

Assume the claim is true for rounds $k + 1$ through $N - 1$. We show that the claim is true for round k. Let $\mathbf{z}_{k-1}$ be arbitrary. Case (i): Consider an active bidder in the k-th round whose value is x but who bids as if it were $y \in [z_{k-1}, x]$. If $z_k \in [z_{k-1}, y]$ he continues to round $k + 1$ where, by the induction hypothesis, he optimally bids x and he obtains $\Pi_{k+1}(x|z_k, \mathbf{z}_{k-1})$. If $z_k \geq y$ he obtains compensation $u(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1})$ in round k, where $p_{k-1} = \beta_{k-1}(z_{k-1}|\mathbf{z}_{k-2})$. Hence his payoff is

$$\begin{aligned} \pi_k(y, x|\mathbf{z}_{k-1}) = & \int_{z_{k-1}}^{y} \Pi_{k+1}(x|z_k, \mathbf{z}_{k-1})g_k^{(N-1)}(z_k|z_{k-1})dz_k \\ & + \int_y^{\bar{x}} u(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1})\, g_k^{(N-1)}(z_k|z_{k-1})dz_k. \end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned} \frac{\partial\pi_k(y, x|\mathbf{z}_{k-1})}{\partial y} = & [\Pi_{k+1}(x|y, \mathbf{z}_{k-1}) - u(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1})]g_k^{(N-1)}(y|z_{k-1}) \\ & + u'(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1})\, \beta_k'(y|\mathbf{z}_{k-1})(1 - G_k^{(N-1)}(y|Z_{k-1}^{(N-1)} = z_{k-1})) \end{aligned}$$

From our heuristic derivation we have

$$\begin{aligned}&u'\left(\beta_k(y|\mathbf{z}_{k-1})-p_{k-1}\right)\beta_k'(y|\mathbf{z}_{k-1})\\ =\ &-\left[u(\beta_{k+1}(y|y,\mathbf{z}_{k-1})-\beta_k(y|\mathbf{z}_{k-1}))-u\left(\beta_k(y|\mathbf{z}_{k-1})-p_{k-1}\right)\right]\lambda_k^{(N-1)}(y|z_{k-1}).\end{aligned}$$

Substituting this expression into the prior equation, and using the definition of $\lambda_k^{(N-1)}(y|z_{k-1})$, yields

$$\frac{\partial\pi_k(y,x|\mathbf{z}_{k-1})}{\partial y}=[\Pi_{k+1}(x|y,\mathbf{z}_{k-1})-u(\beta_{k+1}(y|\beta_k(y|\mathbf{z}_{k-1}))-\beta_k(y|\mathbf{z}_{k-1}))]g_k^{(N-1)}(y|z_{k-1}).$$

When a bidder has the same value x as the last bidder to drop, then he is the next bidder to drop and he obtains compensation $u(\beta_{k+1}(x|x,\mathbf{z}_{k-1})-\beta_k(x|\mathbf{z}_{k-1}))$. Hence

$$\Pi_{k+1}(x|x,\mathbf{z}_{k-1})=u(\beta_{k+1}(x|x,\mathbf{z}_{k-1})-\beta_k(x|\mathbf{z}_{k-1})),$$

and therefore $\partial\pi_k(y,x|\mathbf{z}_{k-1})/\partial y|_{y=x}=0$.

For $y\in[z_{k-1},x]$ we have

$$\frac{\partial^2\pi_k(y,x|\mathbf{z}_{k-1})}{\partial y\partial x}=\frac{d}{dx}\Pi_{k+1}(x|y,\mathbf{z}_{k-1})g_k^{(N-1)}(y|z_{k-1})\geq 0,$$

where the inequality follows since (b) is true for round $k+1$ by the induction hypothesis.

Case (ii): Suppose the bidder bids as if his value is $y\geq x$. If $z_k\in[z_{k-1},x]$, then he continues to round $k+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{k+1}(x|z_k,\mathbf{z}_{k-1})$. If $z_k\in[x,y]$, then he continues to round $k+1$ and, by the induction hypothesis, he bids z_k and obtains compensation $\beta_{k+1}(z_k|z_k,\mathbf{z}_{k-1})-\beta_k(z_k|\mathbf{z}_{k-1})$. If $z_k>y$ then in round k he wins compensation $\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}$. Thus his payoff at round k is

$$\begin{aligned}\pi_k(y, x|\mathbf{z}_{k-1}) = \quad & \textstyle\int_{z_{k-1}}^{x} \Pi_{k+1}(x|z_k, \mathbf{z}_{k-1}) g_k^{(N-1)}(z_k|z_{k-1}) dz_k \\ & + \textstyle\int_x^y u(\beta_{k+1}(z_k|z_k, \mathbf{z}_{k-1}) - \beta_k(z_k|\mathbf{z}_{k-1})) g_k^{(N-1)}(z_k|z_{k-1}) dz_k, \\ & + \textstyle\int_y^{\bar{x}} u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(z_k|z_{k-1}) dz_k.\end{aligned}$$

Differentiating with to y yields

$$\begin{aligned}\frac{\partial \pi_k(y, x|\mathbf{z}_{k-1})}{\partial y} \quad = \quad & u(\beta_{k+1}(y|y, \mathbf{z}_{k-1}) - \beta_k(y|\mathbf{z}_{k-1})) g_k^{(N-1)}(y|z_{k-1}) \\ & - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(y|z_{k-1}) \\ & + u'\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(y|\mathbf{z}_{k-1})(1 - G_k^{(N-1)}(z_k|z_{k-1})) dz_k.\end{aligned}$$

From the heuristic derivation we have

$$\begin{aligned}& u'\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(y|\mathbf{z}_{k-1}) \\ = \quad & -\left[u(\beta_{k+1}(y|y, \mathbf{z}_{k-1}) - \beta_k(y|\mathbf{z}_{k-1})) - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right)\right] \lambda_k^{(N-1)}(y|z_{k-1}).\end{aligned}$$

Substituting this expression into the prior equation, and using the definition of $\lambda_k^{(N-1)}(y|z_{k-1})$, yields for $y \geq x$ that

$$\begin{aligned}\frac{\partial \pi_k(y, x|\mathbf{z}_{k-1})}{\partial y} \quad = \quad & u(\beta_{k+1}(y|y, \mathbf{z}_{k-1}) - \beta_k(y|\mathbf{z}_{k-1})) g_k^{(N-1)}(y|z_{k-1}) \\ & - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(y|z_{k-1}) \\ & - \left[u(\beta_{k+1}(y|y, \mathbf{z}_{k-1}) - \beta_k(y|\mathbf{z}_{k-1})) - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right)\right] g_k^{(N-1)}(y|z_{k-1}) \\ = \quad & 0.\end{aligned}$$

We have shown that $\partial \pi_k(y, x|\mathbf{z}_{k-1})/\partial y|_{y=x} = 0$ and $\partial \pi_k(y, x|\mathbf{z}_{k-1})/\partial y$ for $y \geq z_{k-1}$. If $x < z_{k-1}$ then clearly $z_{k-1} \in \arg\max_y \pi_k(y, x|\mathbf{z}_{k-1})$. Hence (a) is true for round k by Lemma 0 of McAfee (1992).

To establish (b) is true for round k, observe that

$$\begin{aligned}
\frac{d\Pi_k(x|\mathbf{z}_{k-1})}{dx} &= \left.\frac{\partial \pi_k(y,x|\mathbf{z}_{k-1})}{\partial y}\right|_{y=x} + \left.\frac{\partial \pi_k(y,x|\mathbf{z}_{k-1})}{\partial x}\right|_{y=x} \\
&= \int\limits_{z_{k-1}}^{x} \frac{d}{dx}\Pi_{k+1}(x|z_k,\mathbf{z}_{k-1})g_k^{(N-1)}(z_k|z_{k-1})dz_{N-1} \\
&\quad +\Pi_{k+1}(x|x,\mathbf{z}_{k-1})g_k^{(N-1)}(x|z_{k-1}) \\
&\geq 0,
\end{aligned}$$

since $d\Pi_{k+1}(x|z_k,\mathbf{z}_{k-1})/dx \geq 0$ by the induction hypothesis and $\Pi_{k+1}(x|x,\mathbf{z}_{k-1}) \geq 0$. $\square$

Proof of Proposition 2: The proof is symmetric to the proof of Proposition 3, and is therefore omitted. Alternatively, one can obtain the risk neutral bidding functions as limits of the CARA risk averse functions, i.e., as $\beta_k(x;\mathbf{p}_{k-1}) = \lim_{\alpha\to 0}\beta_k^{\alpha}(x;\mathbf{p}_{k-1})$. $\square$

Proof of Proposition 3: We first solve for the round $N-1$ bid function. When $u(x) = (1-e^{-\alpha x})/\alpha$, then (6) yields the differential equation

$$-\alpha e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})-p_{N-2}\right)}\frac{d\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})}{dx} = \left(e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})-p_{N-2}\right)} - e^{-\alpha x}\right)\lambda_{N-1}^{N}(x).$$

Multiply both sides by $2(1-F(x))^2$, this equation can be written as

$$\frac{d}{dx}\left(e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})-p_{N-2}\right)}(1-F(x))^2\right) = -2e^{-\alpha x}f(x)(1-F(x)).$$

By the Fundamental Theorem of Calculus

$$e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})-p_{N-2}\right)}(1-F(x))^2 = -\int_0^x e^{-\alpha z}2f(z)(1-F(z))dz + C,$$

where C is the constant of integration. Since the left hand side of this equation is zero when $x = \bar{x}$, then

$$C = \int_0^{\bar{x}} e^{-\alpha z} 2(1 - F(z)) f(z) dz.$$

and therefore the equation can be written as

$$e^{-\alpha\left(2\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2}\right)}(1 - F(x))^2 = \int_x^{\bar{x}} e^{-\alpha z} 2(1 - F(z)) f(z) dz.$$

Solving yields

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2} p_{N-2} - \frac{1}{2\alpha} \ln \left[\int_x^{\bar{x}} e^{-\alpha z} \frac{2(1 - F(z)) f(z)}{(1 - F(x))^2} dz \right],$$

which, by Claim 4 in the Supplemental Appendix, can be written as[22]

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2} p_{N-2} - \frac{1}{2\alpha} \ln \left(E\left[e^{-\alpha Z^{(N)}_{N-1}} | Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2} \right] \right).$$

Finally, by the definition of J^{α}_{N-1}, we can write

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2} p_{N-2} - \frac{1}{2\alpha} \ln \left(J^{\alpha}_{N-1}(x) \right).$$

Next, we solve for the round k bid function when $k < N - 1$. Assume that in round $k + 1$ bidders follow the bid function

$$\beta^{\alpha}_{k+1}(x;\mathbf{p}_k) = \frac{N - k - 1}{N - k} p_k - \frac{N - k - 1}{(N - k)\,\alpha} \ln \left(J^{\alpha}_{k+1}(x) \right).$$

Then

$$\beta^{\alpha}_{k+1}(x;\beta^{\alpha}_k(x;\mathbf{p}_{k-1}),\mathbf{p}_{k-1}) = \frac{N - k - 1}{N - k} \beta^{\alpha}_k(x;\mathbf{p}_{k-1}) - \frac{N - k - 1}{(N - k)\,\alpha} \ln \left(J^{\alpha}_{k+1}(x) \right).$$

[22] See Claim 2 of the Supplemental Appendix for the conditional density $g^{(N)}_{N-1}(z_{N-1}|Z^{(N)}_k > x > Z^{(N)}_{k-1})$.

and thus $\beta^{\alpha}_{k+1}(x;\beta^{\alpha}_k(x;\mathbf{p}_{k-1}),\mathbf{p}_{k-1}) - \beta^{\alpha}_k(x;\mathbf{p}_{k-1})$ equals

$$-\frac{N-k-1}{(N-k)\,\alpha}\ln\left(J^{\alpha}_{k+1}(x)\right) - \frac{1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1}).$$

For round $k < N$, by equation (7) we have

$$\begin{aligned}&-\alpha e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1})-p_{k-1}\right)}\tfrac{d\beta^{\alpha}_k(x;\mathbf{p}_{k-1})}{dx}\\ &= \left[e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1})-p_{k-1}\right)} - J^{\alpha}_{k+1}(x)^{\frac{N-k-1}{N-k}}\right](N-k)\tfrac{f(x)}{1-F(x)}.\end{aligned}$$

Multiplying both sides of this equation by $\frac{N-k+1}{N-k}\left(1-F(x)\right)^{N-k+1}$ yields

$$\begin{aligned}&\tfrac{d}{dx}\left(e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1})-p_{k-1}\right)}(1-F(x))^{N-k+1}\right)\\ &= -J^{\alpha}_{k+1}(x)^{\frac{N-k-1}{N-k}}\,(N-k+1)\,(1-F(x))^{N-k}f(x).\end{aligned}$$

Applying the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}&e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1})-p_{k-1}\right)}(1-F(x))^{N-k+1}\\ &= -\textstyle\int_0^x J^{\alpha}_{k+1}(z)^{\frac{N-k-1}{N-k}}\,(N-k+1)\,(1-F(z))^{N-k}f(z)dz + C,\end{aligned} \tag{14}$$

where C is an arbitrary constant.

Since the LHS of (14) is zero when $x = \bar{x}$, then

$$C = \int_0^x J^{\alpha}_{k+1}(z)^{\frac{N-k-1}{N-k}}\,(N-k+1)\,(1-F(z))^{N-k}f(z)dz.$$

Hence

$$\begin{aligned}&e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_k(x;\mathbf{p}_{k-1})-p_{k-1}\right)}(1-F(x))^{N-k+1}\\ &= \textstyle\int_x^{\bar{x}} J^{\alpha}_{k+1}(z)^{\frac{N-k-1}{N-k}}\,(N-k+1)\,(1-F(z))^{N-k}f(z)dz.\end{aligned}$$

Hence $\beta_k^\alpha(x;\mathbf{p}_{k-1})$ equals

$$\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(\int_x^{\bar{x}}\left(J_{k+1}^\alpha(z)\right)^{\frac{N-k-1}{N-k}}\frac{(N-k+1)\,(1-F(z))^{N-k}\,f(z)}{(1-F(x))^{N-k+1}}dz\right).$$

By Claim 4 of the Supplemental Appendix, we can write

$$\beta_k^\alpha(x;\mathbf{p}_{k-1})=\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}\right]\right),$$

and hence by the definition of J_k^α we have

$$\beta_k^\alpha(x;\mathbf{p}_{k-1})=\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(J_k^\alpha(x)\right),$$

which is the desired result. $\square$

Let

$$H_{k+1}^0(x)=\frac{1}{N-k}E[Z_{N-1}^{(N)}|Z_{k+1}^{(N)}>x>Z_k^{(N)}].$$

The conditional density for this expectation is given by

$$g_{N-1}^{(N)}(t|Z_{k+1}^{(N)}>x>Z_k^{(N)})=\frac{(N-k)\,(N-k-1)\,[F(t)-F(x)]^{N-k-2}[1-F(t)]f(t)}{(1-F(x))^{N-k}}.$$

The following lemma is useful in proving Proposition 4.

Lemma A:

$$\int_x^{\bar{x}}H_{k+1}^0(t)\frac{(N-k+1)[1-F(t)]^{N-k}f(t)}{[1-F(x)]^{N-k+1}}dt=\frac{1}{N-k}E\left[Z_{N-1}^{(N)}|Z_k^{(N)}>x>Z_{k-1}^{(N)}\right]$$

Proof: We have

$$\begin{aligned}
&\int_x^{\bar{x}} H_{k+1}^0(t) \frac{(N-k+1)[1-F(t)]^{N-k} f(t)}{[1-F(x)]^{N-k+1}} dt \\
= &\int_x^{\bar{x}} \left(\frac{1}{N-k} \int_t^{\bar{x}} q \frac{(N-k)(N-k-1)\,[F(q)-F(t)]^{N-k-2}\,[1-F(q)]f(q)}{[1-F(t)]^{N-k}} \right) dq \\
&\times \frac{(N-k+1)[1-F(t)]^{N-k} f(t)}{[1-F(x)]^{N-k+1}} dt \\
= &\int_x^{\bar{x}} \int_t^{\bar{x}} q \frac{(N-k+1)(N-k-1)\,[F(q)-F(t)]^{N-k-2}\,[1-F(q)]f(q)f(t)}{[1-F(x)]^{N-k+1}} dq dt.
\end{aligned}$$

Changing the order of integration, we can write this as

$$\int_x^{\bar{x}} \int_x^{q} q \frac{(N-k+1)(N-k-1)\,[F(q)-F(t)]^{N-k-2}\,[1-F(q)]f(q)f(t)}{[1-F(x)]^{N-k+1}} dt dq,$$

or

$$\int_x^{\bar{x}} q \frac{(N-k+1)(N-k-1)[1-F(q)]f(q)}{[1-F(x)]^{N-k+1}} \left(\int_x^q [F(q)-F(t)]^{N-k-2} f(t) dt \right) dq.$$

Since

$$\int_x^q [F(q)-F(t)]^{N-k-2} f(t) dt = - \frac{[F(q)-F(t)]^{N-k-1}}{N-k-1} \Bigg|_{t=x}^{t=q} = \frac{[F(q)-F(x)]^{N-k-1}}{N-k-1},$$

the above expression is equal to

$$\frac{1}{N-k} \int_x^{\bar{x}} q \frac{(N-k+1)(N-k)\,[F(q)-F(x)]^{N-k-1}\,[1-F(q)]f(q)}{[1-F(x)]^{N-k+1}} dq$$

which is just $\frac{1}{N-k} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)} \right]$. $\square$

Proof of Proposition 4: Part (i). We first show that for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ that $\beta_k^0(x; \mathbf{p}_{k-1}) \geq \beta_k^\alpha(x; \mathbf{p}_{k-1})$ for $x < \bar{x}$. The proof is by induction.

For $k = N-1$, since e^x is a convex function, then by Jensen's Inequality, for $x < \bar{x}$ we have

$$e^{E[-\alpha Z_{N-1}^{(N)}|Z_{N-1}^{(N)}>x>Z_{N-2}^{(N)}]} < E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}].$$

Noting that the RHS is $J_{N-1}^{\alpha}(x)$, taking the log of both sides, and then multiplying through by $-1/(2\alpha)$ yields

$$\frac{1}{2}E[Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] > -\frac{1}{2\alpha}\ln(J_{N-1}^{\alpha}(x))$$

Adding $\frac{1}{2}p_{N-2}$ to both sides yields the result $\beta_{N-1}^{0}(x;\mathbf{p}_{N-2}) > \beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})$ for $x < \bar{x}$.

For $k \leq N-1$, define

$$H_k^0(x) = \frac{1}{N-k+1}E[Z_{N-1}^{(N)}|Z_k^{(N)} > x > Z_{k-1}^{(N)}],$$

and

$$H_k^{\alpha}(x) = -\frac{N-k}{(N-k+1)\alpha}\ln(J_k^{\alpha}(x)) = -\frac{1}{\alpha}\ln\left(J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}}\right),$$

where $J_k^{\alpha}(x)$ is defined in Proposition 3. We have that

$$e^{-\alpha H_k^{\alpha}(x)} = J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}}.$$

We established above that $H_{N-1}^0(x) > H_{N-1}^{\alpha}(x)$, i.e.,

$$\frac{1}{2}E[Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] > -\frac{1}{2\alpha}\ln(E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]).$$

Assume for $k < N-2$ that $H_{k+1}^0(x) > H_{k+1}^{\alpha}(x)$ for $x < \bar{x}$. We show that $H_k^0(x) > H_k^{\alpha}(x)$ for $x < \bar{x}$. Since $-\alpha H_{k+1}^0(x) < -\alpha H_{k+1}^{\alpha}(x)$ and e^x is increasing, then

$$e^{-\alpha H_{k+1}^{0}(x)} < e^{-\alpha H_{k+1}^{\alpha}(x)} \text{ for } x < \bar{x},$$

or

$$e^{-\alpha H_{k+1}^{0}(x)} < J_{k+1}^{\alpha}(x)^{\frac{N-k-1}{N-k}} \text{ for } x < \bar{x},$$

Thus

$$E[e^{-\alpha H_{k+1}^{0}(Z_k^{(N)})} | Z_k^{(N)} > x > Z_{k-1}^{(N)}] < E[J_{k+1}^{\alpha}(Z_k^{(N)})^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}]. \tag{15}$$

The right hand side is $J_k^{\alpha}(x)$. Consider the left hand side. Since e^x is convex, then

$$e^{E[-\alpha H_{k+1}^{0}(Z_k^{(N)}) | Z_k^{(N)} > x > Z_{k-1}^{(N)}]} < E[e^{-\alpha H_{k+1}^{0}(Z_k^{(N)})} | Z_k^{(N)} > x > Z_{k-1}^{(N)}].$$

This inequality and (15) imply

$$e^{E[-\alpha H_{k+1}^{0}(Z_k^{(N)}) | Z_k^{(N)} > x > Z_{k-1}^{(N)}]} < J_k^{\alpha}(x).$$

Taking logs of both sides of this inequality yields

$$E[-\alpha H_{k+1}^{0}(Z_k^{(N)}) | Z_k^{(N)} > x > Z_{k-1}^{(N)}] < \ln\left(J_k^{\alpha}(x)\right).$$

Multiplying both sides by $-\frac{N-k}{(N-k+1)\alpha}$ yields

$$\int_x^{\bar{x}} H_{k+1}^{0}(z) \frac{(N-k)[1-F(z)]^{N-k} f(z)}{(1-F(x))^{N-k+1}} dz > -\frac{N-k}{(N-k+1)\alpha} \ln\left(J_k^{\alpha}(x)\right).$$

By Lemma A, the LHS can be written as

$$\frac{1}{N-k+1} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right].$$

Hence

$$\frac{1}{N-k+1}E\left[Z_{N-1}^{(N)}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right] > -\frac{N-k}{(N-k+1)\alpha}\ln\left(J_k^\alpha(x)\right).$$

Adding $\frac{N-k}{N-k+1}p_{k-1}$ to both sides yields $\beta_k^0(x;\mathbf{p}_{k-1}) > \beta_k^\alpha(x;\mathbf{p}_{k-1})$ for $x < \bar{x}$. This proves Part (i).

Part (ii). We now show that for each $k = 1,\ldots,N-1$ and $\mathbf{p}_{k-1}$ that

$$\beta_k^\alpha(x;\mathbf{p}_{k-1}) > \frac{1}{N-k+1}x + \frac{N-k}{N-k+1}p_{k-1} \text{ for } x < \bar{x}.$$

The proof is by induction.

Since $e^{-\alpha z} < e^{-\alpha x}$ for $z \in (x, \bar{x}]$ then

$$J_{N-1}^\alpha(x) = E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] < e^{-\alpha x}.$$

Taking logs of both sides and rearranging yields

$$-\frac{1}{2\alpha}\ln(J_{N-1}^\alpha(x)) > \frac{1}{2}x,$$

i.e., $H_{N-1}^\alpha(x) > x/2$ for $x < \bar{x}$. Adding $\frac{1}{2}p_{N-2}$ to both sides yields $\beta_{N-1}^\alpha(x;\mathbf{p}_{k-1}) > x/2 + p_{N-2}/2$ for $x < \bar{x}$.

Assume for $k < N-2$ that $H_{k+1}^\alpha(x) > 1/(N-k)$ for $x < \bar{x}$. We show that

$$H_k^\alpha(x) > \frac{1}{N-k+1}x \text{ for } x < \bar{x}.$$

Since $H_{k+1}^\alpha(x)$ is increasing, then for $z > x$ we have $H_{k+1}^\alpha(z) > H_{k+1}^\alpha(x) > x/(N-k)$ or $-\alpha H_{k+1}^\alpha(z) < -\alpha H_{k+1}^\alpha(x) < -\alpha x/(N-k)$ and thus

$$e^{-\alpha H_{k+1}^\alpha(z)} = J_{k+1}^\alpha(z)^{\frac{N-k-1}{N-k}} < e^{-\alpha H_{k+1}^\alpha(x)} < e^{-\alpha\frac{x}{N-k}}.$$

Hence

$$E[J^{\alpha}_{k+1}(Z^{(N)}_k)^{\frac{N-k-1}{N-k}}|Z^{(N)}_k > x > Z^{(N)}_k] < e^{-\alpha\frac{x}{N-k}}.$$

Taking logs of both sides yields

$$\ln(E[J^{\alpha}_{k+1}(Z^{(N)}_k)^{\frac{N-k-1}{N-k}}|Z^{(N)}_k > x > Z^{(N)}_k]) < -\alpha\frac{x}{N-k},$$

i.e.,

$$-\frac{N-k}{(N-k+1)\alpha}\ln(E[J^{\alpha}_{k+1}(Z^{(N)}_k)^{\frac{N-k-1}{N-k}}|Z^{(N)}_k > x > Z^{(N)}_k]) > \frac{x}{N-k+1}.$$

Hence $H^{\alpha}_k(x) > x/(N-k+1)$ for $x < \bar{x}$. Adding $\frac{N-k}{N-k+1}p_{k-1}$ to each side gives us

$$\beta^{\alpha}_k(x;\mathbf{p}_{k-1}) > \frac{x}{N-k+1} + \frac{N-k}{N-k+1}p_{k-1} \text{ for } x < \bar{x}. \ \square$$

Proof of Proposition 5: The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}&\left(E[e^{-\tilde{\alpha}Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-1}]\right)^{\frac{\alpha}{\tilde{\alpha}}}\\ &\geq E[\left(e^{-\tilde{\alpha}Z^{(N)}_{N-1}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-1}]\\ &= E[e^{-\alpha Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-1}].\end{aligned} \tag{16}$$

Next, after applying logs to both sides of (16), doing some algebraic manipulations, and adding $\frac{1}{2}p_{N-2}$ to both sides of (16) we have

$$\begin{aligned}&\tfrac{1}{2}p_{N-2} - \tfrac{1}{2\alpha}\ln E[e^{-\alpha Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-1}]\\ &\geq \tfrac{1}{2}p_{N-2} - \tfrac{1}{2\tilde{\alpha}}\ln E[e^{-\tilde{\alpha}Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-1}]\end{aligned} \tag{17}$$

Thus, we have $\beta^{\alpha}_{N-1}(x;p_{N-1}) \geq \beta^{\tilde{\alpha}}_{N-1}(x;p_{N-1})$.

(Induction Hypothesis): Suppose $\beta^{\alpha}_{k+1}(x;p_k) \geq \beta^{\tilde{\alpha}}_{k+1}(x;p_k)$. Let $H^{\alpha}_{k+1}(x)$ be the non-linear part of $\beta^{\alpha}_{k+1}(x;p_k)$ and let $H^{\tilde{\alpha}}_{k+1}(x)$ be the non-linear part of $\beta^{\tilde{\alpha}}_{k+1}(x;p_k)$.

We now consider the k-th round. As before, since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned} &\left(E[e^{-\tilde{\alpha}H^{\tilde{\alpha}}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\ &\geq E[e^{-\alpha H^{\tilde{\alpha}}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]. \end{aligned} \tag{18}$$

By the induction hypothesis we have that $H^{\alpha}_{k+1}(x) \geq H^{\tilde{\alpha}}_{k+1}(x)$ and therefore the RHS of (18) is greater than

$$E[e^{-\alpha H^{\alpha}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]. \tag{19}$$

Consequently, the LHS of (18) is greater than (19) or

$$\begin{aligned} &\left(E[e^{-\tilde{\alpha}H^{\tilde{\alpha}}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\ &\geq E[e^{-\alpha H^{\alpha}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]. \end{aligned} \tag{20}$$

Using simple manipulations of (20) we have

$$\begin{aligned} H^{\alpha}_k(x) &= -\tfrac{N-k}{(N-k+1)\alpha} \ln E[e^{-\alpha H^{\alpha}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}] \\ &\geq -\tfrac{N-k}{(N-k+1)\tilde{\alpha}} \ln E[e^{-\tilde{\alpha} H^{\tilde{\alpha}}_{k+1}(Z^{(N)}_{N-1})}|Z^{(N)}_k > x > Z^{(N)}_{k-1}] = H^{\tilde{\alpha}}_k(x) \end{aligned}$$

and therefore that $\beta^{\alpha}_k(x;p_{k-1}) \geq \beta^{\tilde{\alpha}}_k(x;p_{k-1})$. □

Proof of Proposition 6: The bidding function $\beta^{\alpha}_k(x;\mathbf{p}_{k-1})$ in an arbitrary round $k \leq N-1$ can be written as

$$\beta_k^\alpha(x;\mathbf{p}_{k-1}) = \frac{N-k}{N-k+1}p_{k-1} - \frac{1}{\alpha}\ln\left(J_k^\alpha(x)^{\frac{N-k}{N-k+1}}\right).$$

By the definition of $J_k^\alpha(x)$ we have

$$J_k^\alpha(x)^{\frac{N-k}{N-k+1}} = \left(E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\right]\right)^{\frac{N-k}{N-k+1}}.$$

Since $y^{\frac{N-k}{N-k+1}}$ is concave, then by Jensen's inequality

$$\begin{aligned}
J_k^\alpha(x)^{\frac{N-k}{N-k+1}} &\geq E\left[\left(\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}\right)^{\frac{N-k}{N-k+1}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\right] \\
&= E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k+1}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\right] \\
&= \int_x^{\bar{x}} J_{k+1}^\alpha(z_k)^{\frac{N-k-1}{N-k+1}}\frac{(N-k+1)(1-F(z_k))^{N-k}}{(1-F(x))^{N-k+1}}dF(z_k).
\end{aligned}$$

Likewise, since $y^{\frac{N-k-1}{N-k+1}}$ is concave, repeating the same argument yields

$$\begin{aligned}
J_{k+1}^\alpha(z_k)^{\frac{N-k-1}{N-k+1}} &\geq E\left[\left(J_{k+2}^\alpha(Z_{k+1}^{(N)})\right)^{\frac{N-k-2}{N-k+1}}|Z_{k+1}^{(N)} > x > Z_k^{(N)}]\right] \\
&= \int_{z_k}^{\bar{x}} J_{k+2}^\alpha(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{(N-k)(1-F(z_{k+1}))^{N-k-1}}{(1-F(z_k))^{N-k}}dF(z_{k+1}).
\end{aligned}$$

Substituting this expression into the prior one, and simplifying yields

$$J_k^\alpha(x)^{\frac{N-k}{N-k+1}} \geq \frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}}\int_{z_k}^{\bar{x}} J_{k+2}^\alpha(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{(1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}}dF(z_{k+1})dF(z_k).$$

Changing the order of integration, the right hand side is

$$\begin{aligned}
&\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}}\int_x^{z_{k+1}} J_{k+2}^{\alpha}(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{(1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}}dF(z_k)dF(z_{k+1})\\
=\;&\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}} J_{k+2}^{\alpha}(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{[F(z_{k+1})-F(x)](1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}}dF(z_{k+1}).
\end{aligned}$$

This last integral is just an expectation, taken with respect to the density of $Z_{k+1}^{(N)}$ conditional on $Z_k^{(N)} > x > Z_{k-1}^{(N)}$. Thus

$$J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}} \geq E[J_{k+2}^{\alpha}(Z_{k+1}^{(N)})^{\frac{N-k-2}{N-k+1}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}].$$

Continuing in this fashion, we obtain

$$J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}} \geq E[J_{N-1}^{\alpha}(Z_{N-2}^{(N)})^{\frac{1}{N-k+1}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]. \tag{21}$$

Since

$$f_{N-2}(z_{N-2}|Z_k^{(N)} > x > Z_{k-1}^{(N)}) = \frac{(N-k+1)!}{(N-k-2)!2!}\frac{[F(z_{N-2})-F(x)]^{N-k-2}[1-F(z_{N-2})]^2}{[1-F(x)]^{N-k+1}}f(z_{N-2}),$$

the right hand side of (21) can be written as

$$\int_x^{\bar{x}} J_{N-1}^{\alpha}(z_{N-2})^{\frac{1}{N-k+1}}\frac{(N-k+1)!}{(N-k-2)!2!}\frac{[F(z_{N-2})-F(x)]^{N-k-2}[1-F(z_{N-2})]^2}{[1-F(x)]^{N-k+1}}dF(z_{N-2}). \tag{22}$$

By a now-standard argument, we have

$$\begin{aligned}
J_{N-1}^{\alpha}(x)^{\frac{1}{N-k+1}} &= \left(E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]\right)^{\frac{1}{N-k+1}}\\
&\geq E[e^{-\frac{\alpha}{N-k+1}Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]\\
&= \int_x^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{2(1-F(z_{N-1}))}{[1-F(x)]^2}dF(z_{N-1}).
\end{aligned}$$

Replacing $J^{\alpha}_{N-1}(z_{N-2})^{\frac{1}{N-k+1}}$ in (22) with the right hand side of this expression with $x = z_{N-2}$ yields

$$\frac{(N-k+1)!}{(N-k-2)!}\int_x^{\bar{x}}\int_{z_{N-2}}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-2})-F(x)\right]^{N-k-2}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-1})dF(z_{N-2}).$$

Changing the order of integration, this expression can be written as

$$\frac{(N-k+1)!}{(N-k-2)!}\int_x^{\bar{x}}\int_{x}^{z_{N-1}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-2})-F(x)\right]^{N-k-2}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-2})dF(z_{N-1}).$$

Since

$$\begin{aligned}\int_x^{z_{N-1}}\left[F(z_{N-2})-F(x)\right]^{N-k-2}f(z_{N-2})dz_{N-2} &= \frac{1}{N-k-1}\left[F(z_{N-2})-F(x)\right]^{N-k-1}\Big|_x^{z_{N-1}}\\ &= \frac{1}{N-k-1}\left[F(z_{N-1})-F(x)\right]^{N-k-1},\end{aligned}$$

the expression further simplifies to

$$\begin{aligned}&\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-1})-F(x)\right]^{N-k-1}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-1})\\ =\; &E[e^{-\frac{\alpha}{N-k+1}Z^{(N)}_{N-1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]).\end{aligned}$$

Thus we have established that

$$\frac{1}{\alpha}\ln(J^{\alpha}_k(x)^{\frac{N-k}{N-k+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha}{N-k+1}Z^{(N)}_{N-1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}])\right).$$

The round k equilibrium bidding function therefore is bounded above by

$$\beta^{\alpha}_k(x;\mathbf{p}_{k-1}) \leq \frac{N-k}{N-k+1}p_{k-1} - \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha}{N-k+1}Z^{(N)}_{N-1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}])\right).$$

We show that $\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln(E[e^{-\frac{\alpha}{N-k+1}Z^{(N)}_{N-1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]) = \frac{x}{N-k+1}$, i.e.,

$$\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(\int\limits_{x}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}t} h(t)dt\right) = \frac{x}{N-k+1},$$

where

$$h(t) = \frac{(N-k+1)!}{(N-k-1)!}\frac{[1-F(t)]\,[F(t)-F(x)]^{N-k-1}}{(1-F(x))^{N-k+1}} f(t).$$

Then $\lim_{\alpha\to\infty}\beta_k^\alpha(x;\mathbf{p}_{k-1}) \leq \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}$. By Proposition 4, $\beta_k^\alpha(x;\mathbf{p}_{k-1}) \geq \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}$ for $\alpha > 0$. Hence we have

$$\lim_{\alpha\to\infty}\beta_k^\alpha(x;\mathbf{p}_{k-1}) = \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}.$$

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\lim_{\alpha\to\infty}\frac{1}{N-k+1}\frac{\int\limits_{x}^{\bar{x}} te^{-\frac{\alpha}{N-k+1}t}h(t)dt}{\int\limits_{x}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}t}h(t)dt}.$$

We show that

$$\lim_{\alpha\to\infty}\frac{\int\limits_{x}^{\bar{x}} te^{-\alpha t}h(t)dt}{\int\limits_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} = x.$$

Clearly

$$\frac{\int\limits_{x}^{\bar{x}} te^{-\alpha t}h(t)dt}{\int\limits_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} \geq \frac{x\int\limits_{x}^{\bar{x}} e^{-\alpha t}h(t)dt}{\int\limits_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} = x.$$

Also, for any $\Delta > 0$ small

$$\begin{aligned}
\lim_{\alpha\to\infty} \frac{\int_x^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_x^{\bar{x}} e^{-\alpha t}h(t)dt} &\leq \lim_{\alpha\to\infty} \frac{\int_x^{x+\Delta} te^{-\alpha t}h(t)dt + \int_{x+\Delta}^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_x^{x+\Delta} e^{-\alpha t}h(t)dt} \\
&\leq \lim_{\alpha\to\infty} \frac{(x+\Delta)\int_x^{x+\Delta} e^{-\alpha t}h(t)dt + e^{-\alpha(x+\Delta)}\int_{x+\Delta}^{\bar{x}} \bar{x}h(t)dt}{\int_x^{x+\Delta} e^{-\alpha t}h(t)dt} \\
&= x+\Delta+\lim_{\alpha\to\infty} \frac{\int_{x+\Delta}^{\bar{x}} \bar{x}h(t)dt}{\int_x^{x+\Delta} e^{\alpha(x+\Delta-t)}h(t)dt}.
\end{aligned}$$

Since $h(t) > 0$ for $t \in [x, x+\Delta]$, then $\lim_{\alpha\to\infty} \int_x^{x+\Delta} e^{\alpha(x+\Delta-t)}h(t)dt = \infty$ for any $\Delta > 0$. Hence we have shown that for any $\Delta > 0$ we have

$$x \leq \lim_{\alpha\to\infty} \frac{\int_x^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_x^{\bar{x}} e^{-\alpha t}h(t)dt} \leq x+\Delta.$$

Thus

$$\lim_{\alpha\to\infty} \frac{\int_x^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_x^{\bar{x}} e^{-\alpha t}h(t)dt} = x. \ \square$$

Proof of Proposition 7: Consider the strategy $\bar{\beta}$ which calls for bidder i to drop out when his compensation reaches x_i/N. More formally, suppose bidder i's strategy is $\bar{\beta}_i^k(x_i; \mathbf{p}_{k-1}) = x_i/N + p_{k-1}$ for each $k \in \{1, \ldots, N-1\}$, and every $x_i \in [0, \bar{x}]$ and $(p_1, \ldots, p_{k-1})$ such that $0 \leq p_1 \leq \ldots \leq p_{k-1}$. We show that guarantee's bidder i a payoff of at least $u(x_i/N)$. Thus in any equilibrium bidder i's payoff is at least $u(x_i/N)$.

Suppose that bidder i has value x_i and follows $\bar{\beta}_i$. Let x_{-i} and β_{-i} be arbitrary, and let $p_1, \ldots, p_{N-1}$ be the sequence of dropout prices that result. The sequence is uniquely determined unless there is a tie at some stage. If there is a tie then, depending on which bidder drops, one of several different prices sequences $(p_1, p_2, \ldots, p_{N-1})$ may result. In this case, let $(p_1, \ldots, p_{N-1})$ be an arbitrary such sequence.

Either bidder i drops out at some stage k or all the other bidders drop out first. In the former case, i's compensation is $x_i/N + p_{k-1} - p_{k-1} = x_i/N$ and he obtains utility $u(x_i/N)$. If all the other bidders drop out before bidder i, then it must be the case that $p_1 \leq x_i/N$, $p_2 - p_1 \leq x_i/N, \ldots, p_{N-1} - p_{N-2} \leq x_i/N$ since otherwise, if $p_k - p_{k-1} > x_i/N$ then bidder i would have dropped out at stage k. Hence $p_1 + (p_2 - p_1) + \ldots + (p_{N-1} - p_{N-2}) = p_{N-1} \leq (N-1)x_i/N$. Bidder i wins the item and obtains utility $u(x_i - p_{N-1}) \geq u(x_i/N)$. □

Proof of Proposition 8: Consider a bidder who has the k-th lowest value. Let m be such that $k + m < N - 1$. If the bidder follows the m round mimic strategy, then at round $k+m$ the bidder (i) infers $Z_1^{(N)} = x_1, \ldots, Z_{k-1}^{(N)} = x_{k-1}$, (ii) he knows his own value $Z_k^{(N)} = x_k$ is the k-th lowest and, if $m > 0$, then (iii) he infers $Z_{k+1}^{(N)} = x_{k+1}, \ldots, Z_{k+m}^{(N)} = x_{k+m}$.[23] Under the mimic strategy, he bids in round $k + m$ as though his type is x_{k+m} and drops at price $\tilde{p}_{k+m} = \beta_{k+m}(x_{k+m}; \tilde{\mathbf{p}}_{k+m-1})$, where $\tilde{\mathbf{p}}_{k+m-1} = (\tilde{p}_1, \ldots, \tilde{p}_{k+m-1})$ is given by

$$\tilde{p}_1 = \beta_1(x_1) \text{ and } \tilde{p}_j = \beta_j(x_j; \tilde{p}_1 \ldots, \tilde{p}_{j-1}) \text{ for } j \in \{2, \ldots, k-1\}$$

and $\tilde{p}_k = \beta_k(x_k; \tilde{p}_1 \ldots, \tilde{p}_{k-1})$ if $m = 0$ and

$$\tilde{p}_j = \beta_j(x_{j+1}; \tilde{p}_1 \ldots, \tilde{p}_{j-1}) \text{ for } j \in \{k, \ldots, k+m-1\}$$

[23] $m = 0$ corresponds to following his equilibrium strategy.

otherwise. He obtains compensation

$$
\begin{aligned}
c_{k+m} &= \beta_{k+m}(x_{k+m}; \tilde{\mathbf{p}}_{k+m-1}) - \tilde{p}_{k+m-1} \\
&= \frac{1}{N-(k+m)+1} \left\{ E\left[Z_{N-1}^{(N)} | Z_{k+m}^{(N)} > x_{k+m} > Z_{k+m-1}^{(N)} \right] - \tilde{p}_{k+m-1} \right\}.
\end{aligned}
$$

We show that the bidder obtains the same expected compensation if, instead of dropping at round $k+m$ at price $\tilde{p}_{k+m}$, he follows the $m+1$ round mimic strategy.[24] In that case, he observes the rival with the next lowest value drop in round $k+m$ and infers his rivals' type to be $Q = Z_{k+m+1}^{(N)}$. In round $k+m+1$ he bids as though his own type is $Q = Z_{k+m+1}^{(N)}$ and therefore he is the next bidder to drop since all bidders of type Q or lower have already dropped. He obtains compensation

$$
C_{k+m+1}(Q) = \beta_{k+m+1}(Q; \tilde{\mathbf{p}}_{k+m-1}, \tilde{P}_{k+m}(Q)) - \tilde{P}_{k+m}(Q),
$$

where $\tilde{P}_{k+m}(Q) = \beta_{k+m}(Q; \tilde{\mathbf{p}}_{k+m-1})$ is the price in round $k+m$.

Using the equilibrium bidding function in Proposition 2, if q is the realized value of Q, then in round $k+m+1$ the bidder obtains compensation

$$
c_{k+m+1}(q) = \frac{1}{N-(k+m)} \left\{ E\left[Z_{N-1}^{(N)} | Z_{k+m+1}^{(N)} > q > Z_{k+m}^{(N)} \right] - \beta_{k+m}(q; \tilde{\mathbf{p}}_{k+m-1}) \right\},
$$

where

$$
\beta_{k+m}(q; \tilde{\mathbf{p}}_{k+m-1}) = \frac{1}{N-(k+m)+1} E\left[Z_{N-1}^{(N)} | Z_{k+m}^{(N)} > q > Z_{k+m-1}^{(N)} \right] + \frac{N-(k+m)}{N-(k+m)+1} \tilde{p}_{k+m-1}.
$$

[24] Since $k+m < N-1$ then $k+m+1 \leq N-1$ and the bidder drops out rather than winning.

Defining

$$\begin{aligned} D(q) \quad &\equiv \quad \frac{N-(k+m)+1}{N-(k+m)} E\left[Z_{N-1}^{(N)} | Z_{k+m+1}^{(N)} > q > Z_{k+m}^{(N)}\right] \\ & \quad - \frac{1}{N-(k+m)} E\left[Z_{N-1}^{(N)} | Z_{k+m}^{(N)} > q > Z_{k+m-1}^{(N)}\right], \end{aligned}$$

we can write

$$c_{k+m+1}(q) = \frac{1}{N-(k+m)+1}\{D(q) - \tilde{p}_{k+m-1}\}.$$

The term $E[Z_{N-1}^{(N)} | Z_{k+m+1}^{(N)} > q > Z_{k+m}^{(N)}]$ is

$$\int_q^{\bar{x}} t \frac{(N-(k+m))(N-(k+m)-1) f(t)[1-F(t)][F(t)-F(q)]^{N-(k+m)-2}}{[1-F(q)]^{N-(k+m)}} dt$$

and the term $E[Z_{N-1}^{(N)} | Z_{k+m}^{(N)} > q > Z_{k+m-1}^{(N)}]$ is

$$\int_q^{\bar{x}} t \frac{(N-(k+m)+1)(N-(k+m)) f(t)[1-F(t)][F(t)-F(q)]^{N-(k+m)-1}}{[1-F(q)]^{N-(k+m)+1}} dt.$$

Thus

$$\begin{aligned} D(q) \quad &= \quad \int_q^{\bar{x}} t \frac{(N-(k+m)+1)\,(N-(k+m)-1) f(t)[1-F(t)][F(t)-F(q)]^{N-(k+m)-2}}{[1-F(q)]^{N-(k+m)}} dt \\ & \quad - \int_q^{\bar{x}} t \frac{(N-(k+m)+1) f(t)[1-F(t)][F(t)-F(q)]^{N-(k+m)-1}}{[1-F(q)]^{N-(k+m)+1}} dt, \end{aligned}$$

which can be written as

$$\begin{aligned} D(q) \quad &= \quad \int_q^{\bar{x}} t\,(N-(k+m)+1)\, f(t)[1-F(t)] \\ & \quad \times \frac{[F(t)-F(q)]^{N-(k+m)-2}\,((N-(k+m)-1)[1-F(q)] - [F(t)-F(q)])}{[1-F(q)]^{N-(k+m)+1}} dt. \end{aligned}$$

Hence $E[D(Q)|Z_{k+m}^{(N)} = x_{k+m}]$ equals

$$\int_{x_{k+m}}^{\bar{x}} \int_{q}^{\bar{x}} t\,(N-(k+m)+1)\, f(t)[1-F(t)]$$
$$\times \frac{[F(t)-F(q)]^{N-(k+m)-2}\,((N-(k+m)-1)[1-F(q)]-[F(t)-F(q)])}{[1-F(q)]^{N-(k+m)+1}}$$
$$\times g_{k+m+1}^{(N)}(q|Z_{k+m}^{(N)} = x_{k+m})dtdq,$$

where

$$g_{k+m+1}^{(N)}(q|Z_{k+m}^{(N)} = x_{k+m}) = \frac{(N-(k+m))f(q)[1-F(q)]^{N-(k+m)-1}}{[1-F(x_{k+m})]^{N-(k+m)}}.$$

Changing the order of integration, this can be rewritten as

$$\int_{x_{k+m}}^{\bar{x}} \int_{x_{k+m}}^{t} [t\,(N-(k+m)+1)\, f(t)[1-F(t))]]$$
$$\times \frac{[F(t)-F(q)]^{N-(k+m)-2}\,((N-(k+m)-1)[1-F(q)]-[F(t)-F(q)])}{[1-F(q)]^{N-(k+m)+1}}$$
$$\times \frac{(N-(k+m))f(q)[1-F(q)]^{N-(k+m)-1}}{[1-F(x_{k+m})]^{N-(k+m)}} dqdt,$$

Simplifying further yields

$$\int_{x_{k+m}}^{\bar{x}} \frac{t\,(N-(k+m)+1)\,(N-(k+m))f(t)[1-F(t))]}{[1-F(x_{k+m})]^{N-(k+m)}}$$
$$\times \int_{x_{k+m}}^{t} \frac{[F(t)-F(q)]^{N-(k+m)-2} f(q)\,((N-(k+m)-1)[1-F(q)]-[F(t)-F(q)])}{[1-F(q)]^2} dqdt.$$

The inner integral

$$\int_{x_{k+m}}^{t} \frac{[F(t)-F(q)]^{N-(k+m)-2} f(q)\,((N-(k+m)-1)[1-F(q)]-[F(t)-F(q)])}{[1-F(q)]^2} dq$$

reduces to

$$\left[-\frac{[F(t)-F(q)]^{N-(k+m)-1}}{1-F(q)}\right]_{q=x_{k+m}}^{q=t} = \frac{[F(t)-F(x_{k+m})]^{N-(k+m)-1}}{1-F(x_{k+m})}.$$

Thus $E[D(Q)|Z_{k+m}^{(N)}=x_{k+m}]$ equals

$$\int_{x_{k+m}}^{\bar{x}} \frac{[t\,(N-(k+m)+1)\,(N-(k+m))f(t)[1-F(t)]]\,[F(t)-F(x_{k+m})]^{N-(k+m)-1}}{[1-F(x_{k+m})]^{N-(k+m)+1}},$$

i.e.,

$$E[D(Q)|Z_{k+m}^{(N)}=x_{k+m}] = E\left[Z_{N-1}^{(N)}|Z_{k+m}^{(N)}>x_{k+m}>Z_{k+m-1}^{(N)}\right].$$

Hence

$$\begin{aligned} E[C_{k+m+1}(Q)|Z_{k+m}^{(N)}=x_{k+m}] &= \tfrac{1}{N-(k+m)+1}\left\{E\left[Z_{N-1}^{(N)}|Z_{k+m}^{(N)}>x_{k+m}>Z_{k+m-1}^{(N)}\right]-\tilde{p}_{k+m-1}\right\} \\ &= c_{k+m}. \end{aligned}$$

This establishes that the sequence of mimic compensations $\{C_{k+m}\}_{m=0}^{N-1-k}$ is a martingale. $\square$

Proof of Proposition 9: We need to establish two facts: (i) $\bar{\beta}^i$ guarantees bidder i a payoff of at least x_i/N, and (ii) there is no strategy which guarantees player i a payoff above x_i/N. This establishes that $\bar{v}_i(x_i)=x_i/N$ is player i's security payoff and $\bar{\beta}^i$ is a security strategy.

Part (i). Suppose that bidder i has value x_i and follows $\bar{\beta}^i$ given in the proposition. Let x_{-i} and β^{-i} be arbitrary, and let $p_1,\ldots,p_{N-1}$ be the sequence of dropout prices that result. The sequence is uniquely determined unless there is a tie at some stage. If there is a tie then, depending on which bidder drops, one of several different prices sequences may result. In this case, let $(p_1,\ldots,p_{N-1})$ be an arbitrary such sequence.

Either bidder i drops out at some stage k, or all the other bidders drop out first. In the former case, i's payoff is $x_i/N+p_{k-1}-p_{k-1}=x_i/N$. Suppose that all the other bidders drop out before bidder i. Then it must be the case that $p_1 \leq x_i/N$, $p_2-p_1 \leq x_i/N, \ldots, p_{N-1}-p_{N-2} \leq x_i/N$ since otherwise, if $p_k-p_{k-1} > x_i/N$ for some k, then bidder i would have dropped out at round k. Hence $p_1+(p_2-p_1)+\ldots+(p_{N-1}-p_{N-2}) \leq (N-1)x_i/N$ and thus bidder i's payoff is at least $x_i-(N-1)x_i/N = x_i/N$.

Part (ii). Suppose to the contrary that for some $\hat{x}_i \in [0,\bar{x}]$ that there is a strategy $\hat{\beta}^i$ for bidder i such that

$$v(\hat{x}_i, x_{-i}, \hat{\beta}^i, \beta^{-i}) > \bar{v}(\hat{x}_i) = \frac{\hat{x}_i}{N} \; \forall x_{-i}, \beta^{-i}.$$

Since the inequality holds for all x_{-i} and β^{-i}, then it holds in particular for $\hat{x}_{-i}=(\hat{x}_i,\ldots,\hat{x}_i)$ and $\hat{\beta}^{-i}=(\hat{\beta}^i,\ldots,\hat{\beta}^i)$, i.e., $v(\hat{x}_i,\hat{x}_{-i},\hat{\beta}^i,\hat{\beta}^{-i}) > \hat{x}_i/N$. When every bidder has the same value $\hat{x}_i$ and follows the same strategy $\hat{\beta}^i$, then by symmetry every bidder has the same expected payoff, which is at least $\bar{v}(\hat{x}_i)$. Summing across the N bidders, the total payoff is at least $N\bar{v}(\hat{x}_i)$, which is greater than $\hat{x}_i$. This is a contradiction since the total gain to allocating the item, i.e., the sum of the bidders' payoffs, is $\hat{x}_i$ when every bidder's value is $\hat{x}_i$. $\square$

Proof of Proposition 10: Suppose that bidder i has value x_i and follows $\bar{\beta}^i$. Let x_{-i} and β^{-i} be arbitrary, and let $p_1,\ldots,p_{N-1}$ be the sequence of dropout prices that results. We show that bidder i's payoff is at least his security payoff of x_i/N. In the proof below, take $n=N$ and $p_0=0$.

Suppose that bidder i is not among the first $\hat{k}-1$ bidders to drop. We show for $k \in \{1,\ldots,\hat{k}-1\}$ that (i) $p_k-p_0 \leq k(x_i-p_0)/n$ and (ii) $p_k-p_{k-1} \leq (x_i-p_{k-1})/(n-k+1)$. Assume $x_i > p_0$. If bidder i is not the first to drop, then

$$\bar{\beta}^i_1(x_i;p_0) = \frac{x_i-p_0}{n-1+1}+p_0 \geq p_1,$$

i.e.,

$$p_1 - p_0 \leq \frac{x_i - p_0}{n}.$$

Hence (i) and (ii) hold for $k = 1$.

Assume that (i) and (ii) hold for some $k' < \hat{k} - 1$. We show they hold for $k' + 1$. By the induction hypothesis, $p_{k'} - p_0 \leq k'(x_i - p_0)/n$ and hence $k' < n$ and $x_i > p_0$ implies $p_{k'} - p_0 \leq x_i - p_0$, i.e., $p_{k'} \leq x_i$. Since bidder i did not drop at $k' + 1 \leq \hat{k} - 1$, then

$$\bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) = \frac{x_i - p_{k'}}{n - (k'+1) + 1} + p_{k'} \geq p_{k'+1},$$

which establishes (ii) for $k = k' + 1$. Rearranging, we obtain

$$p_{k'+1} - p_0 \leq \frac{x_i + (N - k' - 1)p_{k'}}{N - k'} - p_0 \leq \frac{x_i + (n - k' - 1)(\frac{k'(x_i - p_0)}{n} + p_0)}{n - k'} - p_0 = \frac{k' + 1}{n}(x_i - p_0),$$

where the second inequality holds by the induction hypothesis. Hence (i) holds for $k = k' + 1$.

If bidder i drops in round $\hat{k}$, then his payoff is $(x_i - p_{\hat{k}-1})/(n - \hat{k} + 1)$. Since $p_{\hat{k}-1} \leq (\hat{k} - 1)(x_i - p_0)/n + p_0$ then

$$\frac{x_i - p_{\hat{k}-1}}{n - \hat{k} + 1} \geq \frac{x_i - (\frac{\hat{k}-1}{n}(x_i - p_0) + p_0)}{n - \hat{k} + 1} = \frac{x_i - p_0}{n}.$$

If bidder i is not among the first $N-1$ bidders to drop, then $p_{N-1} - p_0 \leq (n-1)(x_i - p_0)/n$. He wins the auction and his payoff is $x_i - (\frac{n-1}{n}(x_i - p_0) + p_0) = -\frac{1}{n}(p_0 - x_i)$

$$x_i - p_{N-1} \geq x_i - (\frac{n-1}{n}(x_i - p_0) + p_0) = \frac{x_i - p_0}{n}.$$

Hence $\bar{\beta}^i$ guarantee's bidder i his security payoff of $(x_i - p_0)/n$ and is therefore a security strategy. $\square$

Proof of Proposition 11: Suppose that bidder i has value x_i and follows $\bar{\beta}^i$. Let x_{-i} and β^{-i} be arbitrary, and let $p_1, \ldots, p_{N-1}$ be the sequence of dropout prices that results.

Suppose bidder i has not dropped at round $\hat{k} \geq 1$. We show that $p_k \leq k/N$ for each $k \in \{1, \ldots, \hat{k}\}$. Since bidder i did not drop at round 1 then $p_1 \leq \bar{\beta}^i_1(x_i) = x_i/N$. Suppose that $p_k \leq k/N$ for some $k' < \hat{k}$. We show that $p_{k'+1} \leq (k'+1)/N$. Since $p_{k'} \leq k'/N$, then $[\frac{x_i}{N} + p_{k'}, \frac{x_i - p_{k'}}{N-k'} + p_{k'}]$ is non-empty, and hence $\bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) \in [\frac{x_i}{N} + p_{k'}, \frac{x_i - p_{k'}}{N-k'} + p_{k'}]$. Since bidder i did not drop at round $k'+1$, then

$$p_{k'+1} \leq \bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) \leq \frac{x_i - p_{k'}}{N-k} + p_{k'} = \frac{x_i + p_{k'}(N-k'-1)}{N-k'}.$$

Furthermore, $p_{k'} \leq k'/N$ implies

$$p_{k'+1} \leq \frac{x_i + \frac{k'}{N} x_i (N-k'-1)}{N-k'} = \frac{(k'+1)x_i}{N}.$$

By induction, $p_k \leq k/N$ for each $k \in \{1, \ldots, \hat{k}\}$.

Since $\bar{\beta}^i_1(x_i) = x_i/N$, if bidder i dropped at round 1 his payoff was x_i/N. If bidder i dropped at round $k > 1$ then $p_{k-1} \leq (k-1)/N$ (since he did not drop at round $k-1$) and hence his payoff is

$$\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) - p_{k-1} \geq \frac{x_i}{N} + p_{k-1} - p_{k-1} = \frac{x_i}{N}.$$

If bidder i wins the auction (i.e., he did not drop at round $N-1$) then $p_{N-1} \leq (N-1)x_i/N$ and his payoff is

$$x_i - p_{N-1} \geq x_i - \frac{N-1}{N} x_i = \frac{x_i}{N}.$$

Thus $\bar{\beta}^i$ is a security strategy for bidder i. $\square$

Proof of Proposition 12: If $x_i \geq p_0$, the proof of Proposition 10 goes through since it holds for general n and p_0.

If $x_i < p_0$, then bidder i's payoff is negative if he wins the auction. We first show that $\bar{\beta}^i$ guarantees bidder i a payoff of a least $(x_i - p_0)/n$. Since $\bar{\beta}^i_1$ calls for bidder i to drop immediately, his payoff is zero unless he wins the auction. The later occurs only if all $n-1$ other bidders drop immediately and ties are broken in bidder i's favor. In this case, bidder i's payoff is $x_i - p_0$. Since this occurs with at most probability $1/n$, his expected payoff is at least $(x_i - p_0)/n$.

To see that there is no strategy which guarantees bidder i a payoff above $(x_i - p_0)/n$, simply note that for any strategy he follows, if all of his rivals follow the same strategy and have the same values, then by symmetry each bidder wins with probability $1/n$ and bidder i's payoff is $(x_i - p_0)/n$. $\square$

Proof of Proposition 13: Suppose that $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) < (x_i - p_{k-1})/(N - k + 1) + p_{k-1}$ for some k, x_i and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$. We show that $\bar{\beta}^i$ is not a perfect security strategy. In particular, we show that $\bar{\beta}^i|_{\mathbf{p}_{k-1}}(x_i) < \bar{v}_{N-(k-1),p_{k-1}}(x_i)$ for some x_{-i} and β^{-i}.

From Proposition 10, the security payoff to player i in $\Gamma(N-(k-1), p_{k-1})$ is $\bar{v}_{N-(k-1),p_{k-1}}(x_i) = (x_i - p_{k-1})/(N-(k-1))$. Let x_{-i} and β^{-i} be such that the bids of the other $N-k$ bidders in round 1 of $\Gamma(N-(k-1), p_{k-1})$ are greater than $\bar{\beta}^i_k|_{\mathbf{p}_{k-1}}(x_i)$. Then bidder i drops in round 1 and his payoff is

$$\bar{\beta}^i_k|_{\mathbf{p}_{k-1}}(x_i) - p_{k-1} < \frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1} - p_{k-1} = \bar{v}_{N-(k-1),p_{k-1}}(x_i).$$

Hence $\bar{\beta}_i$ is not a perfect security strategy.

Suppose that $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) > (x_i - p_{k-1})/(N - k + 1) + p_{k-1}$ for some k, x_i and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$. Let x_{-i} and β^{-i} be such that (i) one of the other $N-k$ bidders in $\Gamma(N-(k-1), p_{k-1})$ has a dropout price

$\hat{p}_k$ satisfying

$$\bar{\beta}^i_k|_{\mathbf{p}_{k-1}}(x_i) > \hat{p}_k > \frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1},$$

and (ii) the remaining bidders' dropout prices are above $\bar{\beta}^i_k|_{\mathbf{p}_{k-1}}(x_i)$. Then bidder 1 does not drop out in round 1 of $\Gamma(N-(k-1), p_{k-1})$, but enters the subgame $\Gamma(N-k, \hat{p}_k)$. From Proposition 10 the largest payoff he can guarantee himself in this subgame is $\bar{v}_{N-k,\hat{p}_k}(x_i) = (x_i - \hat{p}_k)/(N-k)$. We have that

$$\frac{x_i - \hat{p}_k}{N-k} < \frac{x_i - \left[\frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1}\right]}{N-k} = \frac{x_i - p_{k-1}}{N-(k-1)} < \bar{v}_{N-(k-1),p_{k-1}}(x_i).$$

Hence $\bar{\beta}_i$ is not a perfect security strategy. $\square$

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