The optimal allocation of prizes in tournaments of heterogeneous agents*

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March 25, 2015

Abstract

Tournaments are widely used in organizations, explicitly or implicitly, to reward the best-performing employees, e.g., through promotion or bonuses, and/or to punish the worst-performing employees, e.g., through firing or unfavorable job assignments. We explore the impact of the allocation of prizes on the effectiveness of tournament incentive schemes. We show that while multiple prize allocation rules are equivalent when agents are symmetric in their ability, the equivalence is broken in the presence of heterogeneity. Under a wide range of conditions, *loser prize tournaments*, i.e., tournaments that award a low prize to relatively few bottom performers, are optimal for the firm. The reason is that low-ability agents are discouraged less in such tournaments, and hence can be compensated less to meet their participation constraints.

JEL classification codes: M52, J33, J24

Keywords: tournament, allocation of prizes, contract, heterogeneous agents

*This paper is a substantially reworked and improved version of an earlier working paper titled "To reward the best or to punish the worst? A comparison of two tournament mechanisms with heterogeneous agents," in which we analyzed a special case of the general optimal prize allocation problem solved here. We thank Matthias Sutter, and the participants of the FSU Quantitative Economics Workshop and the 2013 Conference on Tournaments, Contests and Relative Performance Evaluation for valuable comments.

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1 Introduction

Tournaments, or incentive schemes based on relative performance evaluation, are one of the mainstays in a manager's toolkit of motivational devices. In the workplace, employees may compete with one another to receive a reward, for example in the form of a promotion or bonus (see, e.g., Lazear and Rosen 1981; Bull et al. 1988; Orrison et al. 2004). A sometimes overlooked, but equally important type of workplace tournaments is the competition among co-workers to avoid being punished. Employee termination is the most severe but not the only form of punishment. For example, a manager may take an employee off attractive special projects, assign him or her to a more onerous job, or refuse to give an employee an otherwise expected bonus or promotion.

Given the wide use of rank-based rewards and punishments in organizations,[1] it is important to understand what combination of rewards and punishments is optimal for the firm. This question can be formulated quite generally as a *prize allocation* problem. In this paper, we explore the effect of the allocation of prizes on the effectiveness of tournament contracts. Our model builds on the seminal theory of Lazear and Rosen (1981). Workers perform by choosing effort, which is not observable by the manager. Each worker's performance depends positively on effort but also includes a random component ("noise"). The manager can only observe the ranking of workers by their performance levels and has to design a *tournament contract* that awards fixed prizes based on the workers' ranks. In the baseline case of homogeneous risk-neutral workers, multiple distributions of prizes are efficiency and profit-equivalent (Lazear and Rosen 1981), i.e., the predicted work effort and firm profits are the same under those incentive schemes. We depart from this symmetric setting and consider heterogeneous workers. We focus on the case of relatively weak heterogeneity because, first, it is analytically tractable, and, second, it is the most relevant for applications due to endogenous labor market sorting and efficiency considerations.[2]

We show that in the presence of weak heterogeneity the multiplicity of optimal prize allocations is broken in favor of a unique optimal tournament contract. The optimal contract, to the first order in the level of heterogeneity, is a *j-tournament* awarding two

[1] A recent Wall Street Journal article ("'Rank and Yank' Retains Vocal Fans," January 31, 2012, available at `http://online.wsj.com/article/SB10001424052970203363504577186970064375222.html`) states that 60% of Fortune 500 companies currently use some kind of a ranking system for incentive provision. Jack Welch, the former CEO of General Electric, regularly terminated the lowest 10% of the GE employees on the work performance scale.

[2] In fact, it is well-established that tournament incentive schemes become increasingly inefficient as the degree of worker heterogeneity rises, due to the discouragement of low-ability workers (see, e.g., O'Keeffe et al. 1984). Thus, tournament contracts are most likely to be used in relatively homogeneous groups.

distinct prizes:[3] a higher prize is awarded to the agents ranked 1 through j, and a lower prize to the remaining agents. Moreover, in a wide range of cases the optimal contract awards a low prize to relatively few workers ($j > n/2$). This result is a consequence of the finding that lower-ability workers are discouraged more in tournaments that focus on rewarding top performers than in tournaments punishing low performers as it is more important for them to avoid losing in the latter. Hence, lower compensation overall is needed in tournaments with punishment to satisfy the workers' participation constraints.

We show that tournament contracts are nearly efficient in weakly heterogeneous groups, in the sense that the inefficiency is a second-order effect with respect to the level of heterogeneity. There is, however, a first-order (negative) effect of heterogeneity on the firm's profit, and this effect depends critically on the allocation of prizes. The optimal allocation of prizes is, in turn, determined by the shape of the distribution of noise.

We restrict attention to tournament contracts satisfying anonymity, i.e., the principle that two workers cannot be compensated differently for the same output.[4] Such schemes are preferable from a managerial perspective because they do not involve worker discrimination, do not violate procedural equity, and are less demanding in terms of the information the principal needs to possess. As we show, in order to implement an anonymous tournament contract for weakly heterogeneous workers, the principal only needs to know average ability and the ability (but not the identity) of the least productive worker. Finally, we show that as long as workers' heterogeneity is not too strong, the inefficiency of anonymous contracts is negligible.

To the best of our knowledge this is the first paper that presents a general, yet tractable, theory of optimal prize allocation for heterogeneous workers in the Lazear and Rosen (1981) framework. In the analysis, we sacrifice precision for generality and use the linear approximation. This technique is reliable as long as the degree of workers' heterogeneity is not too strong and has proved fruitful in other settings (see, e.g., Fibich and Gavious 2003; Fibich et al. 2004, 2006; Ryvkin 2007, 2009). We show with an example of an otherwise intractable model that the linear approximation agrees with a high-precision numerical solution very well in a wide range of parameters.

The rest of the paper is organized as follows. Section 2 reviews the relevant theoretical

[3]We borrow the term "j-tournament" from Akerlof and Holden (2012). Importantly, Akerlof and Holden (2012) focus on homogeneous agents and analyze two-prize tournaments in an *ad hoc* fashion, whereas we show that they emerge as a unique optimal mechanism for weakly heterogeneous agents.

[4]It was shown previously that the inefficiencies arising in the traditional tournament contracts in the presence of heterogeneity can be removed by extending the class of possible contracts to those violating the principle of anonymity. Examples of such solutions include ability-specific piece rates (Lazear and Rosen 1981), handicaps (O'Keeffe et al. 1984), and ability-specific prizes (Gürtler and Kräkel 2010).

and empirical literature on the problem of prize allocation in tournaments. In Section 3, we describe the model and briefly characterize symmetric optimal contracts that are well-known and serve as the point of departure for further analysis. In Section 4, we present the main results and a numerical illustration. Section 5 concludes with a summary and discussion of our findings and their implications.

2 Review of the relevant literature

There is an extensive literature on tournaments in organizations (for a review of the earlier literature see, e.g., McLaughlin 1988, Lazear 1995, Prendergast 1999; for a more recent review see, e.g., DeVaro 2006, Konrad 2009). Most of this literature focuses on tournaments that reward the best-performing employees.[5] Punishment incentive schemes were initially mentioned by Mirrlees (1975) and later re-examined by Nalebuff and Stiglitz (1983), who note the equivalence of multiple prize allocation schemes in the symmetric case.[6]

Most of the existing theoretical literature on optimal prize allocation in tournaments focuses on two classes of models – perfectly discriminating contests and Tullock (1980) contests (for a detailed review, see Sisak 2009). Moldovanu and Sela (2001) study perfectly discriminating contests that are essentially all-pay auctions with private and possibly nonlinear bidding costs. They find that the optimal allocation of prizes that maximizes total effort depends on the curvature of the effort cost function: one top prize is optimal for linear or concave costs, while multiple prizes can be optimal for convex costs. Moldovanu et al. (2012) explore optimal prize structures in the same framework but explicitly allow for punishment (prizes below the agents' outside option) which may or may not be costly to the employer. They identify the relationship between the distribution of ability in the population and the prize structure and show that, in some cases, punishment can be optimal even if it is costly. Baye et al. (1996), Barut and Kovenock (1998) and Clark and Riis (1998), among others, study all-pay auctions of complete information

[5]Throughout this discussion, we focus on the standard static principal-agent models of tournaments in the tradition of Lazear and Rosen (1981). There is also an extensive literature on dynamic tournaments involving sequential elimination of employees (see, e.g., Rosen 1986, O'Flaherty and Siow 1995, Gradstein and Konrad 1999, Ryvkin and Ortmann 2008, Casas-Arce and Martinez-Jerez 2009, Fu and Lu 2009). Although elimination can be thought of as a form of punishment, it is typically not discussed as such. Instead, these models focus on the incentives of the remaining (promoted) agents.

[6]The equivalence of optimal tournament contracts with various configurations of prizes in the symmetric case was mentioned already by Lazear and Rosen (1981). Nalebuff and Stiglitz (1983) discuss the equilibrium existence and note that in the presence of punishment the agents' payoff functions remain concave as the number of agents n increases, whereas the pure strategy equilibrium disappears as n increases in the case of reward. Thus, punishment prize structures tend to reduce nonconvexities in the principal-agent problem.

and also find that multiple prizes can be optimal for some configurations of types. In the Tullock (1980) framework, it was found that heterogeneity (Baik 1994, Szymanski and Valetti 2005) can lead to the optimality of the second prize. Schweinzer and Segev (2012) show that multiple prizes can be optimal also in symmetric Tullock contests with a nested winner determination structure. Liu et al. (2013) study optimal prize allocation in tournaments as a general mechanism design problem under incomplete information and show that punishments arise as part of a second-best solution.

Gürtler and Kräkel (2012) study rank-order "dismissal tournaments" of two heterogeneous workers, one of whom is terminated as a result. Their primary focus is on the selection efficiency of the termination mechanism, defined as the probability that the high-ability worker is retained. Gürtler and Kräkel (2012) show that, if the low-ability worker has a relatively low outside option, potential termination incentivizes her more than the high-ability worker. This leads to the possibility that, in some instances, the high-ability worker contributes less effort and is more likely to be terminated. Kräkel (2012) uses a similar argument to discuss adverse selection in a sequential elimination setting.

Kräkel (2000) discusses reward and punishment tournaments in which workers may face "relative deprivation," a behavioral term in the payoff function making a worker minimize the distance between her income and the average income of a richer reference group. One of the results is that in the absence of relative deprivation, for symmetric workers, reward tournaments are more effective than punishment tournaments from the organizer's perspective. An important difference between our approach and that of Kräkel (2000) is that he does not calculate optimal contracts, and the result is driven by the assumption that the high and low prizes are the same in both tournament schemes and thus the punishment tournament always costs more to the organizer.

The paper that is related most closely to ours is by Akerlof and Holden (2012) (henceforth, AH12) who study optimal prize structures using the Lazear and Rosen (1981) framework.[7] They consider homogeneous agents and focus on the role of the shape of agents' utility function (risk aversion and prudence) in determining the optimal prize structure. They find that nontrivial profiles of prizes rewarding top performers and punishing bottom performers can be optimal depending on parameters. Our paper can be viewed as complementary to AH12 as we use a model with risk-neutral agents but focus on the effect of agents' heterogeneity in ability. In the extended working paper version of AH12, Akerlof and Holden (2007) provide some results for heterogeneous agents. First, they discuss a model in which agents learn their abilities after they choose effort levels; thus,

[7]Krishna and Morgan (1998) pose essentially the same question but restrict attention to tournaments of up to four agents.

agents are symmetric *ex ante* but heterogeneous *ex post*. The resulting equilibrium is symmetric and has properties similar to the equilibrium with *ex ante* symmetric agents. Second, Akerlof and Holden (2007) discuss some special cases of models with *ex ante* heterogeneous agents, restricting attention to tournaments with only two types of agents and an equal number of agents of each type ($n/2$ high ability agents and $n/2$ low ability agents); they also restrict the shape of the effort cost function to quadratic (in the case of additive heterogeneity) or power law (in the case of multiplicative heterogeneity). For additive heterogeneity, Akerlof and Holden (2007) show that the tournament that pays a low prize w_2 to the lowest-ranked agent and a high prize w_1 to the remaining $n-1$ agents (the "strict loser-prize tournament") induces a higher level of effort than the tournament that pays prize w_1 to the highest-ranked agent and prize w_2 to the remaining $n-1$ agents (the "strict winner-prize tournament"). This result is of limited practical value, however, because total compensation is clearly higher in the former tournament than in the latter. For multiplicative heterogeneity, Akerlof and Holden (2007) show that when heterogeneity is sufficiently large, the strict winner-prize scheme is preferred to the strict loser-prize scheme. In contrast to Akerlof and Holden (2007), our model does not restrict the number of player types, nor does it impose any parametric restrictions on the cost function of effort. Additionally, we keep various prize structures comparable by calculating optimal contracts in all cases.

Because of the difficulties in observing effort and prize valuations with field data, some of the initial empirical tests of tournament theory were conducted using laboratory experiments (see a recent review by Dechenaux et al. 2012). One of the first is by Bull et al. (1987) who showed that, on average, rank-order tournaments generated behavior similar to piece-rate pay schemes, albeit with a higher variance in behavior. With this result established, subsequent papers delved into more nuanced topics such as affirmative action (Schotter and Weigelt 1992), tournament size and prize structure (Harbring and Irlenbusch 2003; Orrison et al. 2004, Chen et al. 2011), sabotage (Harbring and Irlenbusch 2008; Falk et al. 2008; Carpenter et al. 2010, Harbring and Irlenbusch 2011), selection (Camerer and Lovallo 1999; Eriksson et al. 2009; Cason et al. 2010), dynamic tournaments (Sheremeta 2010), and gender effects (Gneezy et al. 2003), just to name a few.

The empirical literature using field data has looked at both sports tournaments (Ehrenberg and Bognanno 1990; Becker and Huselid 1992; Fernie and Metcalf 1999; Lynch 2005; Brown 2012) and corporate tournaments (Main et al. 1993; Eriksson 1999; Bognanno 2001; Conyon et al. 2001; DeVaro 2006). For the most part, the examined field evidence is in line with the theories' directional predictions with respect to effort when examining the spread between the winner and the loser prizes, the size of the tournament and the number of prizes available. The empirical analysis of punishment tournaments has mainly focused on the causes or effects of employee termination (see, e.g., a recent meta-analysis of implications of downsizing by Datta et al. 2010). Warner et al. (1988), and Gibbons and Murphy (1990) provide evidence that rank-order termination tournaments are used in the upper levels of management by showing that relative stock performance can be used in explaining CEO dismissals. In the financial sector, Chevalier and Ellison (1999) find a U-shaped relationship between relative fund performance and a fund manager's termination risk. Qiu (2003) builds upon this analysis and shows that a fund manager's risk attitudes are dependent upon their fund's relative performance rank.

3 The model

3.1 Model setup

Consider a tournament of $n \geq 2$ risk-neutral agents indexed by $i = 1, \ldots, n$. Each agent participates in the tournament by exerting effort $e_i \geq 0$ that costs her $c_i g(e_i)$. Here, $c_i > 0$ is the agent's cost parameter (higher c_i implies lower ability), and $g(\cdot)$ is a strictly convex and strictly increasing function, with $g(0) = 0$. All agents have the same outside option payoff ω.[8]

Following Lazear and Rosen (1981), we model agent i's output as $y_i = e_i + u_i$, where u_i is a zero-mean random shock. Shocks $u_1, \ldots, u_n$ are independent across individuals and drawn from the same distribution with support $[u_l, u_h]$, probability density function (pdf) $f(u)$ and cumulative density function (cdf) $F(u)$.[9]

The agents' output levels are ranked, and the agent ranked r receives prize V_r, with $V_1 \geq V_2 \geq \ldots \geq V_n$, where at least two prizes are distinct. Let $p^{(i,r)}(\mathbf{e})$ denote the probability, as a function of the vector of effort levels $\mathbf{e} = (e_1, \ldots, e_n)$, that agent i's output is ranked r in the tournament. Agent i's expected payoff then can be written as

$$\pi_i(\mathbf{e}) = \sum_{r=1}^{n} p^{(i,r)}(\mathbf{e}) V_r - c_i g(e_i).$$

[8] Outside option ω is the expected payoff of an agent if she does not participate in the tournament. It can represent unemployment insurance benefits, earnings in a different firm or sector, or income from self-employment. The assumption that ω is homogeneous across agents is warranted provided their abilities are part of job-specific human capital and thus not transferrable outside the firm. If it is not the case, agents' outside options can be correlated with abilities (see, e.g., Kräkel 2012). We discuss implications of such a correlation in Section 4.4.

[9] Under risk-neutrality, the results do not change if shocks u_i contain an additive common shock component, i.e., $u_i = \rho + \epsilon_i$ where ρ is the common shock and ϵ_i are zero-mean i.i.d.

For a given configuration of prizes, suppose an equilibrium in pure strategies exists and let $\mathbf{e}^* = (e_1^*, \ldots, e_n^*)$ denote the vector of equilibrium effort levels. There is a risk-neutral principal, whose objective function is the expected profit defined as the difference between aggregate effort and total prize payments, $\Pi = \sum_i e_i - \sum_r V_r$.[10] The principal chooses a tournament contract $(V_1, \ldots, V_n)$. Given the principal's objective, the optimal contract $(V_1^*, \ldots, V_n^*)$ solves

$$\max_{V_1,\ldots,V_n} \sum_i e_i^* - \sum_r V_r$$

subject to the participation constraints, $\pi_i(\mathbf{e}^*) \geq \omega$, $i = 1, \ldots, n$, and the incentive compatibility constraints ensuring that $\mathbf{e}^*$ is an equilibrium under the optimal contract.

3.2 Symmetric optimal contracts

The results of this section are well-known in the literature. We provide them here for completeness because they serve as the point of departure for the analysis that follows. Assume that all agents have the same ability, $c_1 = \ldots = c_n = \bar{c}$. In this section, we briefly characterize the symmetric equilibrium assuming it exists. The existence conditions are discussed in detail by AH12 for a more general setting with risk-averse agents.

Let $\bar{e}$ denote the symmetric equilibrium effort level. For a given configuration of prizes, $\bar{e}$ solves the symmetrized first-order condition

$$\sum_r \beta_r V_r = \bar{c} g'(\bar{e}), \tag{1}$$

where $\beta_r = p_1^{(1,r)}(\bar{e}, \ldots, \bar{e})$ is the derivative of an agent's probability to be ranked r with respect to the agent's own effort evaluated at the symmetric equilibrium point. The expression for β_r is provided in AH12:

$$\beta_r = \binom{n-1}{r-1} \int F(t)^{n-r-1}[1-F(t)]^{r-2}[n-r-(n-1)F(t)]f(t)^2 dt. \tag{2}$$

Coefficients β_r, referred to by AH12 as "weights," play a critical role in determining the optimal distribution of prizes for symmetric risk-averse agents. As we show below, however, a different set of coefficients enters the stage for heterogeneous agents.

Weights β_r are determined entirely by the distribution of noise F. The following additional properties of β_r are provided by AH12: (i) For any distribution F, $\sum_r \beta_r = 0$,

[10]The results below are also valid for a more general model with $\Pi = Q(\sum_i e_i) - \sum_r V_r$, where $Q(\cdot)$ is a smooth, strictly increasing and concave function. The results below correspond to normalization $Q'(n\bar{e}^s) = 1$, which can be adopted without loss of generality.

$\beta_1 \geq 0$, and $\beta_n \leq 0$; (ii) If F is symmetric, i.e., $f(t) = f(-t)$, then $\beta_r = -\beta_{n-r+1}$ for all r; (iii) If F is a uniform distribution on the interval $[-b, b]$, then $\beta_1 = -\beta_n = 1/(2b)$ and $\beta_r = 0$ for $1 < r < n$.

A critical issue that arises in the analysis below, and is also discussed by AH12, is whether weights β_r are monotonically decreasing in r. Although this appears to be the case for some prominent distributions (such as the uniform and the normal distributions), the monotonicity of β_r is not a universal property. Specifically, as mentioned by AH12, nonmonotonicities in the weights tend to arise when F is multimodal. In what follows, we will be making the assumptions of monotonicity of β_r and/or symmetry of F whenever necessary.

In the symmetric equilibrium, the probability of any agent winning the tournament is $1/n$; therefore, the equilibrium payoff of an agent is $\bar{\pi} = (1/n)\sum_r V_r - \bar{c}g(\bar{e})$. To calculate the optimal contract, write the principal's profit as $\bar{\Pi} = n[\bar{e} - (1/n)\sum_r V_r]$. Effort is costly, and compensation is independent of effort; therefore, the participation constraint binds, $\bar{\pi} = \omega$. This gives $\bar{\Pi} = n[\bar{e} - \bar{c}g(\bar{e}) - \omega]$. The principal will choose an optimal contract $(\bar{V}_1, \ldots, \bar{V}_n)$ such that the equilibrium effort $\bar{e}$ maximizes $\bar{\Pi}$. This gives the following system of equations:

$$\sum_r \beta_r V_r = \bar{c}g'(\bar{e}), \quad \sum_r V_r = n[\omega + \bar{c}g(\bar{e})], \quad \bar{c}g'(\bar{e}) = 1. \tag{3}$$

Let $\bar{e}^s$ denote the solution of the equation $\bar{c}g'(\bar{e}) = 1$. Then any configuration of prizes $(\bar{V}_1, \ldots, \bar{V}_n)$ that solves the system of equations

$$\sum_r \beta_r V_r = 1, \quad \sum_r V_r = n[\omega + \bar{c}g(\bar{e}^s)]. \tag{4}$$

will implement an optimal contract. The firm's optimal profit is $\bar{\Pi}^s = n[\bar{e}^s - \bar{c}g(\bar{e}^s) - \omega]$. The resulting contracts are socially optimal, in the sense that they maximize total surplus $n[\bar{e} - \bar{c}g(\bar{e})]$.

As seen from Eqs. (4), an optimal contract is only determined up to $n-2$ arbitrary prizes. Thus, two distinct prizes are sufficient to generate an optimal contract. The multiplicity of optimal contracts, the discussion of which goes back to Lazear and Rosen (1981), is a consequence of the symmetry (and risk-neutrality) of agents. As we show below, the multiplicity of optimal contracts will be broken when agents are heterogeneous, and in a wide range of scenarios a unique optimal contract will emerge.

4 Optimal contracts with weakly heterogeneous agents

4.1 Equilibrium with weakly heterogeneous agents

We now turn to tournaments of heterogeneous agents. While the case of arbitrary heterogeneity is analytically intractable, a lot can be said about the impact of relatively weak heterogeneity. From a practical viewpoint, weak heterogeneity means that agents' abilities are not very different from some average level. This is a reasonable assumption to make in most cases, as employees whose abilities are substantially different from group average are unlikely to be part of a tournament in the first place, due to the well-documented adverse effects of agent disparity on tournament efficiency (e.g., Lazear and Rosen 1981, O'Keeffe et al. 1984, Müller and Schotter 2010). Moreover, in many cases natural job market sorting will lead to attrition of employees whose ability is too far from the firm's average.

Let $\bar{c} = n^{-1} \sum_i c_i$ denote the average cost parameter. Introduce relative abilities (or, for brevity, abilities) a_i defined as negative relative deviations of cost parameters from the average: $c_i = \bar{c}(1 - a_i)$. By construction, $a_i < 1$, $\sum_i a_i = 0$, and higher a_i implies lower cost of effort, i.e., a higher ability. Moreover, $a_i > 0$ ($a_i < 0$) implies ability above (below) average.

Assume agents are weakly heterogeneous, in the sense that $\mu \equiv \max_i |a_i| \ll 1$. Thus, it is assumed that relative deviations of cost parameters c_i from the average cost parameter $\bar{c}$ are "small."

In what follows, we will assume that, for a given configuration of prizes $(V_1, \ldots, V_n)$, the pure strategy equilibrium with weakly heterogeneous agents exists and is governed by the corresponding system of first-order conditions:

$$\sum_r p_i^{(i,r)}(\mathbf{e}) V_r = c_i g'(e_i), \quad i = 1, \ldots, n. \tag{5}$$

This is a reasonable assumption to make provided the symmetric equilibrium exists and the agents' payoffs are smooth functions of parameters in the neighborhood of the symmetric equilibrium point.[11] In this case, we can look for the equilibrium effort levels in the form $e_i = \bar{e}(1 + x_i)$, where the relative deviations of effort from the symmetric equilibrium

[11]In line with other studies of tournament contracts, we focus on the symmetric equilibrium as the point of departure, even though it may not be the only possible equilibrium, because it is the most "natural" equilibrium for symmetric agents. The approximate equilibrium with weakly heterogeneous agents we identify is unique in the neighborhood of the symmetric equilibrium by construction, as it is given by the solution to a system of linear equations with full rank.

level, x_i, are also "small," $|x_i| \ll 1$. In the linear approximation, x_i can be found approximately, with accuracy $O(\mu^2)$, by expanding the first-order conditions (5) around the symmetric equilibrium point to the first order in μ. The result is given by the following proposition (all proofs are provided in the Appendix).

Proposition 1 *For a given configuration of prizes $(V_1, \ldots, V_n)$, in the linear approximation,*
(a) the equilibrium effort of agent i is $e_i^* = \bar{e}(1 + x_i)$, with

$$x_i = \xi(\bar{e})a_i + O(\mu^2), \quad \xi(\bar{e}) = \frac{\bar{c}g'(\bar{e})}{\bar{e}[\bar{c}g''(\bar{e}) - \sum_r \lambda_r V_r]}; \tag{6}$$

(b) the equilibrium payoff of agent i is

$$\pi_i = \frac{1}{n}\sum_r V_r - \bar{c}g(\bar{e}) + \eta(\bar{e})a_i + O(\mu^2), \quad \eta(\bar{e}) = \frac{\bar{c}^2 g'(\bar{e})^2}{(n-1)[\bar{c}g''(\bar{e}) - \sum_r \lambda_r V_r]} + \bar{c}g(\bar{e}). \tag{7}$$

Here,

$$\lambda_r = \Delta_r \tag{8}$$
$$+\frac{n(n-2)![(r-1)(r-2)M_r - 2(r-1)(n-r)M_{r+1} + (n-r)(n-r-1)M_{r+2}]}{2(n-r)!(r-1)!},$$

$$M_k = \int F(t)^{n-k}[1-F(t)]^{k-3} f(t)^3 dt, \tag{9}$$

$$\Delta_r = \begin{cases} f(u_h)^2 - f(u_l)^2, & n=2,\ r=1 \\ f(u_l)^2 - f(u_h)^2, & n=2,\ r=2 \\ \frac{nf(u_h)^2}{2}, & n \geq 3,\ r=1 \\ -\frac{n}{2}[f(u_l)^2 I_{n=3} + f(u_h)^2], & n \geq 3,\ r=2 \\ -\frac{n}{2}[f(u_l)^2 + f(u_h)^2 I_{n=3}], & n \geq 3,\ r=n-1 \\ \frac{nf(u_l)^2}{2}, & n \geq 3,\ r=n \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$I_{n=3}$ is the indicator equal 1 if $n = 3$ and zero otherwise.

In what follows, we will assume that the denominator in the expression for $\xi(\bar{e})$, Eq. (6), is positive, i.e., higher ability agents exert higher effort, as would be expected in a "well-behaved" equilibrium.[12]

[12]This is, of course, a consequence of our specification of the cost of effort in which effort and ability are complementary.

Proposition 1 shows that the deviations of agents' efforts and payoffs from the symmetric equilibrium levels are determined, in the linear approximation, by coefficients λ_r. As we show below, these coefficients, together with β_r, also determine the prize structure of optimal contracts.

Note that for $n = 2$, $\lambda_r = \Delta_r$; moreover, if F is symmetric, $\lambda_r = 0$. The following corollary follows directly from Eq. (8) and describes the properties of coefficients λ_r for $n \geq 3$.

Corollary 1 *For $n \geq 3$, coefficients λ_r have the following properties:*

(i) For any distribution F, $\sum_r \lambda_r = 0$, $\lambda_1 \geq 0$ and $\lambda_n \geq 0$;

(ii) If F is symmetric, $\lambda_r = \lambda_{n-r+1}$ for all r;

(iii) If F is a uniform distribution on the interval $[-b, b]$ then $\lambda_r = \Delta_r$, i.e., $\lambda_1 = \lambda_n = n/8b^2$; $\lambda_2 = \lambda_{n-1} = -n(I_{n=3} + 1)/8b^2$; and $\lambda_r = 0$ for $2 < r < n - 1$.

4.2 Optimal contracts

It follows from Proposition 1, Eq. (6), that the aggregate deviation of agents' effort from the symmetric equilibrium level is zero in the linear approximation, $\sum_i x_i = O(\mu^2)$; therefore, in the linear approximation the aggregate effort of agents is the same as in the symmetric tournament and, thus, the principal's objective function is $\Pi = n\bar{e} - \sum_r V_r + O(\mu^2)$. The first-order correction to the principal's profit arises due to the participation constraint that now will be binding only for the lowest-ability agent. Let agents be ordered, without loss of generality, so that $c_1 \leq c_2 \leq \ldots \leq c_n$. Then the participation constraint will be $\pi_n = \omega$, where π_n is the equilibrium payoff of agent n given by Eq. (7). This gives the principal's objective function $\Pi = n[\bar{e} - \bar{c}g(\bar{e}) - \omega + \eta(\bar{e})a_n] + O(\mu^2)$. The principal will choose the optimal contract $(V_1^*, \ldots, V_n^*)$ such that the equilibrium effort $\bar{e}$ maximizes Π and satisfies the participation constraint $(1/n)\sum_r V_r - \bar{c}g(\bar{e}) + \eta(\bar{e})a_n = \omega$.

A *j-tournament*, as defined by AH12, is a tournament prize structure that awards two distinct prizes, a prize W_1 to the agents ranked 1 through j and a prize W_2 to the agents ranked $j+1$ through n, with $W_1 > W_2$. It turns out that, in the linear approximation, the optimal tournament prize structure in the tournament of weakly heterogeneous agents is that of a j-tournament. The results are summarized in the following proposition.

Proposition 2 *In the tournament of weakly heterogeneous agents, in the linear approximation:*

(a) The optimal contract is a j-tournament, with $V_1^* = \ldots = V_j^* = W_1$, $V_{j+1}^* = \ldots = V_n^* = W_2$, and

$$j \in \arg \min_{1 \leq r \leq n-1} \frac{\Lambda_r}{B_r}.$$

Here,

$$B_r = \frac{(n-1)!}{(n-r-1)!(r-1)!} \int F(t)^{n-r-1}[1-F(t)]^{r-1} f(t)^2 dt, \tag{11}$$
$$\Lambda_r = \frac{n(n-2)!}{2(n-r-1)!(r-1)!}[(n-r-1)M_{r+2} - (r-1)M_{r+1}] + \sum_{k=1}^{r} \Delta_k.$$

(b) The optimal prizes are

$$W_1 = \omega + \bar{c}g(\bar{e}^s) + \frac{n-j}{nB_j} + \left[\tau - \eta(\bar{e}^s) + \frac{(n-j)\bar{c}g''(\bar{e}^s)\tau}{nB_j}\right] a_n + O(\mu^2), \tag{12}$$
$$W_2 = \omega + \bar{c}g(\bar{e}^s) - \frac{j}{nB_j} + \left[\tau - \eta(\bar{e}^s) - \frac{j\bar{c}g''(\bar{e}^s)\tau}{nB_j}\right] a_n + O(\mu^2), \quad \tau = \frac{\eta'(\bar{e}^s)}{\bar{c}g''(\bar{e}^s)}.$$

(c) The firm's optimal profit is

$$\Pi^* = \bar{\Pi}^s + n\eta(\bar{e}^s)a_n + O(\mu^2), \tag{13}$$

The intuition for part (c) of Proposition 2 is as follows. In the linear approximation, the optimal symmetric equilibrium effort $\bar{e}$ can be sought in the form $\bar{e} = \bar{e}^s + \tau a_n + O(\mu^2)$, where $\bar{e}^s$ is the optimal effort level for contracts with homogeneous agents, and τ is a constant to be determined. However, because the firm's profit is $\Pi = n[\bar{e} - \bar{c}g(\bar{e}) - \omega + \eta(\bar{e})a_n] + O(\mu^2)$, the correction τa_n will have no first-order effect on the profit through $\bar{e}$ due to the envelope theorem. Thus, the first-order effect of agents' heterogeneity on the firm's profit is simply $n\eta(\bar{e}^s)a_n$.

The resulting optimal contract is still nearly efficient (the inefficiency is of order $O(\mu^2)$).[13] Thus, to the first order in μ, heterogeneity leads to a *redistribution* of surplus from the principal to the agents, but not to a reduction in surplus. Indeed, by construction, $a_n < 0$; therefore, in the heterogeneous case the principal's profit is reduced, in the linear approximation, by $n\eta(\bar{e}^s)|a_n|$, as compared to the symmetric case.

Proposition 2 is the central result of this paper. It shows that the multiplicity of optimal contracts with symmetric agents is broken in the presence of weak heterogeneity. Two distinct prizes are still sufficient to implement an optimal contract, in the linear approximation, but the structure of the contract is determined critically by the distribution of noise through coefficients β_r and λ_r. Unfortunately, not much can be said about the properties of these coefficients for general distributions F, and thus the optimal j in the

[13] To see this, consider total surplus $S = \sum_i [e_i^* - c_i g(e_i^*)]$. In the linear approximation, with e_i^* determined by Eq. (6), we have $S = n[\bar{e} - \bar{c}g(\bar{e})] + O(\mu^2)$. With $\bar{e} = \bar{e}^s + \tau a_n + O(\mu^2)$, this gives $S = n[\bar{e}^s - \bar{c}g(\bar{e}^s)] + O(\mu^2)$, i.e., the same surplus as in the symmetric case, in the linear approximation.

j-tournament prize structure can potentially be (almost) anywhere. In the remainder of this section, we will explore how certain restrictions imposed on F lead to restrictions on the location of j.

Following AH12, we will refer to j-tournaments with $j \leq n/2$ as "winner-prize tournaments" because they award the high prize to relatively few top performers; and to j-tournaments with $j \geq n/2$ as "loser-prize tournaments" because they award the low prize to relatively few bottom performers. We will also use the terms "strict winner-prize tournament" and "strict loser-prize tournament" to refer to the extreme versions of the two tournaments with $j = 1$ and $j = n - 1$, respectively. In what follows, we show that for a wide class of distributions F the optimal tournament prize structure with weakly heterogeneous agents is that of a loser-prize tournament.

Figure 1 shows β_r and λ_r as functions of r for the normal distribution of noise with $n = 20$. As seen from Figure 1, both coefficients exhibit the predicted symmetry.

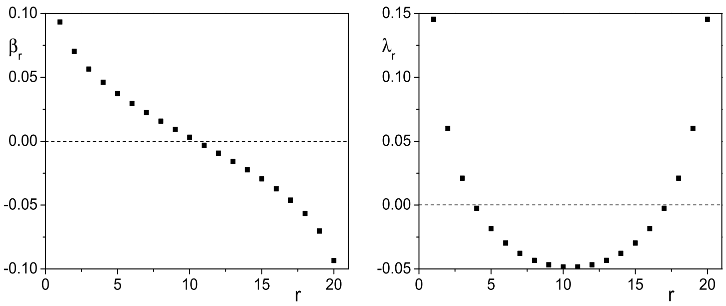

Figure 1: Coefficients β_r (left) and λ_r (right) as functions of r for $n = 20$ and the normal distribution of noise F with zero mean and unit variance.

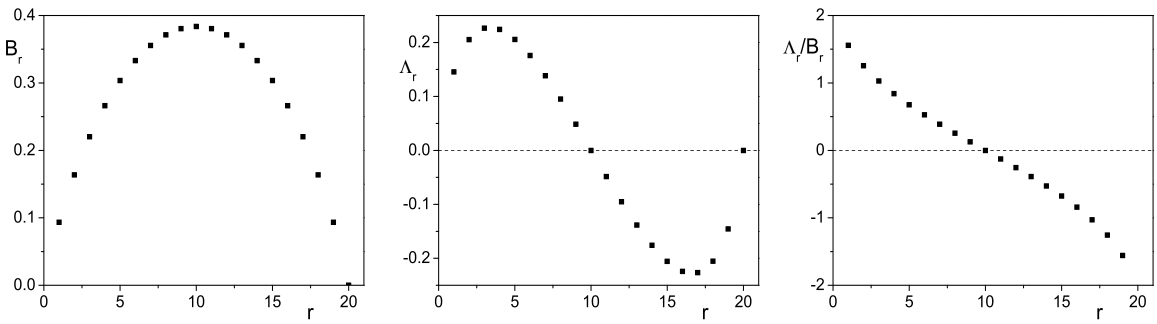

Figure 2: Coefficients B_r (left), Λ_r (center), and their ratio Λ_r/B_r (right) as functions of r for $n = 20$ and the normal distribution of noise F with zero mean and unit variance.

Moreover, β_r is monotonically decreasing in r, while λ_r is U-shaped. These shapes are quite generic and hold for a variety of single-peaked symmetric distributions. They have consequences for the dependence of the cumulative coefficients, B_r and Λ_r, on r, as shown in Figure 2.

Figure 2 shows the dependence of B_r, Λ_r and their ratio, Λ_r/B_r, on r for the same distribution of noise as in Figure 1. Recall that B_r is positive for any distribution F (cf. Eq. (11)) and will have the inverted-U shape as in Figure 2 (left) if β_r is decreasing in r. The maximum of B_r will be reached at the point where β_r crosses zero. It will be in the middle if β_r is symmetric (i.e., if distribution F is symmetric). Similarly, recall that $\sum_r \lambda_r = 0$ and λ_1 is positive for any distribution F; therefore, if λ_r is U-shaped as in Figure 1, Λ_r will be positive and will have a maximum at a relatively low r, then it will cross into the negative domain and will have a minimum for a relatively high r, as in Figure 2 (center). It will be symmetric around the middle if F is symmetric.

The ratio Λ_r/B_r appears to be monotonically decreasing in r when F is the normal distribution (Figure 2, left), and reaches its minimum for $r = n-1$. Thus, when F is the normal distribution, the optimal contract is the strict loser-prize tournament awarding prize W_1 to the agents ranked 1 though $n-1$ and prize W_2 to the agent ranked last. It is easy to see that the same is true when F is a uniform distribution. A more general result is given by the following proposition.

Proposition 3 *(a) Suppose the distribution of noise is symmetric and λ_r is U-shaped. Then the optimal tournament contract for weakly heterogeneous agents, in the linear approximation, is a loser-prize tournament.*
(b) For any distribution F under no circumstances is the strict winner-prize tournament optimal for $n \geq 3$.

To see why Proposition 3 is true, consider the shapes of B_r and Λ_r (Fig. 2). It is clear that the minimum of Λ_r/B_r will be reached when $\Lambda_r < 0$; therefore the optimal j cannot be equal to 1 for any F, and has to be greater than $n/2$ when F is symmetric.

Our results imply that, when agents are weakly heterogeneous, firms that use tournaments focusing on punishing the worst-performing workers will perform better.

4.3 A numerical illustration

In this section, we provide a numerical illustration of the results summarized in Propositions 1 and 2. The goal of this section is to demonstrate that the linear approximation approach used in Sections 4.1 and 4.2 produces results that are very close to high-precision numerical solutions in a wide range of parameters.

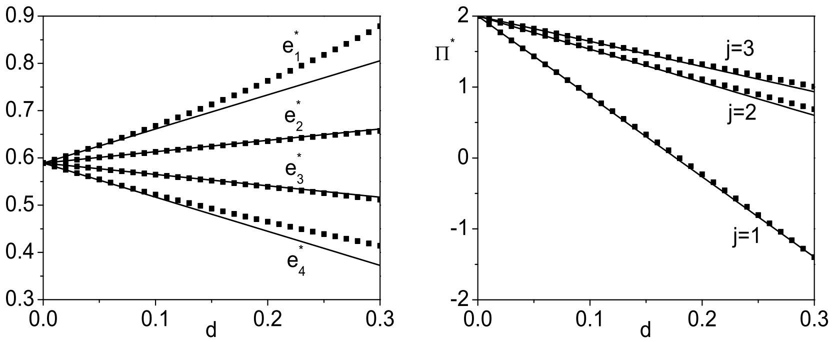

Figure 3: *Left*: Equilibrium efforts e_i^* as functions of heterogeneity parameter d. The solid lines show the linear approximation given by Eq. (6). The squares show the results of a high-precision numerical solution of the system of Eqs. (5). *Right*: The firm's profit Π^* as a function of heterogeneity parameter d for j-tournaments with $j = 1, 2, 3$. The solid lines show the linear approximation given by Eq. (13). The squares show the results of a high-precision numerical computation of the profit in the equilibrium generated by the corresponding j-tournament.

For illustration, consider a tournament of $n = 4$ agents with the cost of effort $g(e) = e^2/2$, the standard normal distribution of noise F, and the outside option $\omega = 0$. The average cost parameter $\bar{c} = 1$, and the agents' relative abilities are $a_1 = d$, $a_2 = d/3$, $a_3 = -d/3$ and $a_4 = -d$. Here, $d \geq 0$ is the heterogeneity parameter, with $d = 0$ corresponding to the homogeneous case. The weak heterogeneity approximation requires that d be small compared to unity. For practical purposes, $d \leq 0.1$ would typically be considered as "small" in applied mathematics. As we show below, the linear approximation in this example works remarkably well at least for $d \leq 0.2$, which corresponds to a nearly 40% variation in ability between the highest and the lowest ability agents.

We start with an illustration of the linearized equilibrium characterized in Proposition 1. Let the prizes be $V_1 = 2$, $V_2 = 1$, $V_3 = 0$ and $V_4 = 0$. This configuration of prizes is not optimal, but we use it here to demonstrate that the linear approximation works well for various configurations of prizes, not necessarily restricted to two-prize optimal contracts described in Proposition 2. The left panel in Figure 3 shows the dependence of equilibrium effort levels e_i^* on the heterogeneity parameter d for each of the four agents. The solid lines in the left panel show the linear approximation $e_i^* = \bar{e}(1 + x_i)$, with $\bar{e} = 0.589$ and x_i given by Eq. (6).[14] The squares show the results of a high-precision numerical solution of the system of Eqs. (5). As seen from the figure, the agreement for $d \leq 0.1$ is excellent and remains reasonably good for d at least up to 0.2. As expected, the equilibrium efforts are

[14]Recall that $\bar{e}$ is the solution of the equation $\sum_r \beta_r V_r = \bar{c}g'(e)$. In our example with $n = 4$ and the standard normal distribution F, Eq. (2) gives $\beta_1 = 0.257$, $\beta_2 = 0.0743$, $\beta_3 = -0.0743$ and $\beta_4 = -0.257$.

ranked in the same way as relative abilities, with more able agents exerting higher effort. As d increases, variation in effort between agents becomes substantial, and it is captured remarkably well by the linear approximation.

We now turn to an illustration of Proposition 2. A complete numerical computation of optimal contracts is prohibitively complex because it requires optimization of the firm's profit Π as a function of prizes $V_1, \ldots, V_4$, with the exact equilibrium computed at each step of the optimization process. We, therefore, present hybrid computational results. Since we already know, from the illustration above, that the equilibrium is evaluated very well by the linear approximation as long as d is not too large, we use the optimal prizes W_1^* and W_2^* computed in the linear approximation, Eq. (12), and calculate the exact profit of the firm, Π^*, for every j-tournament (with $j = 1, 2, 3$) generated by those prizes. The results are presented in the right panel of Figure 3 that shows the firm's optimal profit, Π^*, as a function of d for the three j-tournaments. The linear approximation, Eq. (13), is shown by the solid lines, while the squares show the results of a high-precision computation of Π^*. As seen from the figure, the optimal j-tournament is the strict loser-prize tournament with $j = 3$, as predicted by Proposition 2.[15] As expected, the firm's optimal profit decreases with heterogeneity. The agreement between the numerically computed profit and the linear approximation is excellent.

4.4 Heterogeneous outside options

One possible extension of the analysis presented above is to explore the effect of heterogeneity in the agents' outside options. Given that the agents' heterogeneity in ability is weak, it is reasonable to assume that their outside option payoffs ω_i are also close to the average value ω. Let $\omega_i = \omega + \kappa_i$, where $\sum_i \kappa_i = 0$ and $|\kappa_i/\omega| \ll 1$.

The outside options will affect the principal's problem through the participation constraints that will now take the form $\pi_i \geq \omega_i$. Thus, the binding participation constraint will not necessarily be that of the lowest ability agent, but of agent $k \in \arg\min_{1\leq i\leq n}(\pi_i - \omega_i)$. Depending on who that agent is, the optimal allocation of prizes can be the same or quite different from what we describe above. Specifically, nothing will change if $a_k < 0$, but if $a_k > 0$, the optimal j-tournament will have a j that *maximizes*, as opposed to minimizes, the ratio Λ_r/B_r. Thus, if the configuration of outside options is such that the participation constraint is binding for one of the high-ability agents, optimal contracts may shift in the direction of winner-prize tournaments, i.e., those that award a high prize to relatively few top performers, because now it is the top performers whose incentives

[15] Proposition 2 predicts that the optimal j is given by the $r \in \{1, 2, 3\}$ that minimizes Λ_r/B_r. In our example, $\Lambda_1/B_1 = 0.714$, $\Lambda_2/B_2 = 0$ and $\Lambda_3/B_3 = -0.714$.

are critical.

Kräkel (2012) assumes that the outside option is positively correlated with ability. We can explore the effect of such correlation by letting $\kappa_i = \kappa a_i$, where $\kappa > 0$ is some coefficient. Recall that agent i's payoff, in the linear approximation, is $\pi_i = \bar{\pi} + \eta(\bar{e})a_i$, where $\bar{\pi}$ is the payoff in the symmetric equilibrium. This gives $\pi_i - \omega_i = \bar{\pi} - \omega + (\eta(\bar{e}) - \kappa)a_i$. Thus, if κ is small compared to $\eta(\bar{e})$, there will be no effect on optimal contracts. If κ is large compared to η, optimal contracts will be reversed (i.e., focusing on reward instead of punishment). The nontrivial case is when κ is close to η in magnitude. Recall that η can be manipulated through the structure of prizes; at the same time, the optimal prize structure will depend on the sign of $\eta - \kappa$.

We conclude that the presence of heterogeneity in outside options does not change the basic j-tournament structure of optimal contracts; moreover, it does not change the optimality of punishment as long as the variation in outside options is small compared to variation in ability. Nontrivial reversals of optimal contracts in the direction of rewards may occur, however, if the variation in outside options is relatively strong.

5 Discussion and conclusions

The question of what works better – the carrot or the stick – is probably as old as life itself. In this paper, we address this question in a narrow sense: if a firm uses relative performance evaluation-based incentives, for example to decide on bonuses, promotion or firing, which prize structure is most effective? We use a standard principal-agent model of tournaments that yields the same levels of aggregate effort and firm's profit for various types of contracts involving rewards and/or punishments when workers are homogeneous in ability. For heterogeneous workers, however, the equivalence of multiple prize allocations is broken. We show that it is never optimal to just reward the best performer, and that under a wide range of conditions optimal contracts are those emphasizing punishment of relatively few worst-performing employees. The result follows from the effect different prize allocations have on the degree of discouragement of low-ability workers.

We also show that the efficiency of anonymous tournament contracts (i.e., contracts in which prizes can only be conditioned on the ranking of output but not on the individual worker's ability) is robust to heterogeneity as long as heterogeneity is not too strong. The inefficiency of such contracts is a second-order effect in the level of heterogeneity, while the differences in firms' profits across different prize allocations are of the first-order.

The broad interpretation of our result is in line with Bruton et al. (1996) in their conclusion that downsizing may be an important part of a healthy organization if done strategically. Caution should be exercised though as other consequences could offset the gains in productivity (Repenning 2000). Of course, termination is not the only form of punishment. Importantly, we show that because optimal contracts are determined mostly by the incentives of low-ability workers, it is the low-performing workers who should be distinguished and motivated most by the optimal mechanism. Our results predict that the firms using contracts that focus on punishing low-performing workers will do better.

Our results complement those of AH12 and Moldovanu et al. (2012), who show that tournament contracts involving punishment can be optimal, respectively, for homogeneous agents in the presence of risk aversion and in an all-pay auction setting under incomplete information about agents' abilities. We show that the agents' *ex ante* heterogeneity is an independent factor driving the effectiveness of punishment contracts.

We emphasize the key differences between our paper and AH12. From the outset, their focus is on homogeneous agents with risk-aversion while we look at heterogeneous risk-neutral agents. Thus, AH12 study a symmetric setting where the effect of prize allocation on efficiency is driven by the curvature of agents' utility function; whereas we study a (weakly) heterogeneous setting in which prize allocation affects efficiency through the discouragement of low-ability agents. Also, AH12 introduce and analyze j-tournaments in an *ad hoc* fashion, while we identify them as unique optimal prize structures under weak heterogeneity.[16] We further emphasize that although in the extended working paper version of AH12, Akerlof and Holden (2007) provide some results for heterogeneous agents, they only analyze a few restrictive special cases. We view our results as more general and, to an extent, complementary to Akerlof and Holden (2007) who find that in some special cases sufficiently strong multiplicative heterogeneity leads to winner-prize tournaments being preferred to loser-prize tournaments. We show that when heterogeneity is weak loser-prize tournaments are preferred to winner-prize tournaments under very general conditions.

Our analysis has several limitations dropping which can be of interest in terms of possible extensions. First, we assume that workers are risk-neutral. More complex incentive schemes involving more than two distinct prizes can be optimal under risk-aversion (AH12). Further steps in this direction include considering workers with heterogeneous risk attitudes and/or with preferences departing from the expected utility theory. Second, we restricted attention to the case of relatively weak heterogeneity. Although the impact of heterogeneity on aggregate effort is a second-order effect compared to the first-order effect of heterogeneity on the optimal aggregate compensation, it can become large and

[16] We stress again that we do not restrict attention to two-prize tournaments from the outset. In an unrestricted set of prize profiles, a two-prize scheme emerges as the unique optimal prize scheme for weakly heterogeneous agents.

surpass the latter in magnitude when heterogeneity becomes strong. As discussed in the Introduction, this effect is likely to be mitigated by endogenous sorting of employees; nevertheless, it may be of interest to explore the interplay between possible nonlinear gains from strong heterogeneity in terms of aggregate effort and losses in terms of aggregate compensation. Third, we follow the tradition of Lazear and Rosen (1981) and effectively collapse the dynamic nature of employment into one decision-making period. A richer model can study explicitly the multi-period principal-agent interaction and the role of reward and punishment (including termination) in the optimal provision of incentives in a dynamic setting.

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A Proofs of propositions

A.1 Proof of Proposition 1

We first prove the following two lemmas.

Lemma 1 *The expression for λ_r is given by Eq. (8).*

Proof. By definition, $\lambda_r = p_{11}^{(1,r)} - p_{12}^{(1,r)}$. Suppose agent 1 exerts effort e_1 and all agents $j \geq 2$ exert effort $\bar{e}$. The probability of player 1 being ranked r can be written as

$$p^{(1,r)}(e_1, \bar{e}, \ldots, \bar{e}) = \binom{n-1}{r-1} \int F(t + e_1 - \bar{e})^{n-r}[1 - F(t + e_1 - \bar{e})]^{r-1} f(t)dt. \qquad (14)$$

Then $p_{11}^{(1,r)}$ can be found by differentiating Eq. (14) twice with respect to e_1 and then setting $e_1 = \bar{e}$. For convenience, we will use the following notation for the integrals arising in this calculation:

$$M_k = \int F(t)^{n-k}[1 - F(t)]^{k-3} f(t)^3 dt.$$

Equation (14) then gives

$$p_{11}^{(1,r)} = \binom{n-1}{r-1} \frac{\partial}{\partial e_1} \Big[\int F(t + e_1 - \bar{e})^{n-r-1}[1 - F(t + e_1 - \bar{e})]^{r-2}$$

$$\times [(n-r)(1 - F(t + e_1 - \bar{e})) - (r-1)F(t + e_1 - \bar{e})] f(t + e_1 - \bar{e}) f(t) dt \Big]_{e_1 = \bar{e}}$$

$$= \binom{n-1}{r-1} [(n-r-1)[(n-r)M_{r+2} - (r-1)M_{r+1}] - (r-2)[(n-r)M_{r+1} - (r-1)M_r]$$

$$-(n-1)M_{r+1}+\int F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t))-(r-1)F(t)]f'(t)f(t)dt\Bigg].$$

Suppose now that agents 1 and 2 exert efforts e_1 and e_2, respectively, and all agents $j \geq 3$ exert effort $\bar{e}$. The probability of player 1 being ranked r can be written as

$$\begin{aligned}
&p^{(1,r)}(e_1, e_2, \bar{e}, \ldots, \bar{e}) \\
&= \binom{n-2}{r-1} \int F(t+e_1-\bar{e})^{n-r-1} F(t+e_1-e_2)[1-F(t+e_1-\bar{e})]^{r-1} f(t)dt \\
&+\binom{n-2}{r-2} \int F(t+e_1-\bar{e})^{n-r}[1-F(t+e_1-\bar{e})]^{r-2}[1-F(t+e_1-e_2)]f(t)dt.
\end{aligned} \tag{15}$$

Here, the first term is the probability that $y_1 > y_2$ and y_1 is ranked r among the remaining $n-1$ agents; and the second term is the probability that $y_1 < y_2$ and y_1 is ranked $r-1$ among the remaining $n-1$ agents. The expression for $p_{12}^{(1,r)}$ can be found by differentiating Eq. (15) with respect to e_1 and e_2 and then setting $e_1 = e_2 = \bar{e}$. This gives

$$\begin{aligned}
p_{12}^{(1,r)} = &\binom{n-2}{r-1} \frac{\partial}{\partial e_2} \Bigg[(n-r-1) \int F(t)^{n-r-2} F(t+\bar{e}-e_2)[1-F(t)]^{r-1} f(t)^2 dt \\
&+ \int F(t)^{n-r-1}[1-F(t)]^{r-1} f(t+\bar{e}-e_2) f(t)dt \\
&-(r-1) \int F(t)^{n-r-1} F(t+\bar{e}-e_2)[1-F(t)]^{r-2} f(t)^2 dt\Bigg]_{e_2=\bar{e}} \\
&+\binom{n-2}{r-2} \frac{\partial}{\partial e_2} \Bigg[(n-r) \int F(t)^{n-r-1}[1-F(t)]^{r-2}[1-F(t+\bar{e}-e_2)] f(t)^2 dt \\
&-(r-2) \int F(t)^{n-r}[1-F(t)]^{r-3}[1-F(t+\bar{e}-e_2)] f(t)^2 dt \\
&- \int F(t)^{n-r}[1-F(t)]^{r-2} f(t+\bar{e}-e_2) f(t)dt\Bigg]_{e_2=\bar{e}} \\
= &\binom{n-2}{r-1} \left[-(n-r-1)M_{r+2} + (r-1)M_{r+1} - \int F(t)^{n-r-1}[1-F(t)]^{r-1} f'(t)f(t)dt\right] \\
&+\binom{n-2}{r-2} \left[(n-r)M_{r+1} - (r-2)M_r + \int F(t)^{n-r}[1-F(t)]^{r-2} f'(t)f(t)dt\right].
\end{aligned}$$

In addition to the various M_k terms, the expressions for $p_{11}^{(1,r)}$ and $p_{12}^{(1,r)}$ contain the integrals involving $f'(t)$. These integrals can be dealt with through integration by parts.

Collecting the integrals from both expressions in $p_{11}^{(1,r)} - p_{12}^{(1,r)}$, obtain

$$\int \left[\binom{n-1}{r-1} F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t)) - (r-1)F(t)] \right.$$

$$\left. + \binom{n-2}{r-1} F(t)^{n-r-1}[1-F(t)]^{r-1} - \binom{n-2}{r-2} F(t)^{n-r}[1-F(t)]^{r-2} \right] f'(t)f(t)dt$$

$$= n \int \left[\binom{n-2}{r-1} F(t)^{n-r-1}[1-F(t)]^{r-1} - \binom{n-2}{r-2} F(t)^{n-r}[1-F(t)]^{r-2} \right] f'(t)f(t)dt$$

$$= \Delta_r - \frac{n}{2}\binom{n-2}{r-1}[(n-r-1)M_{r+2} - (r-1)M_{r+1}] + \frac{n}{2}\binom{n-2}{r-2}[(n-r)M_{r+1} - (r-2)M_r].$$

Here, Δ_r is the part determined by the boundary values of the distribution of noise:

$$\Delta_r = \frac{n}{2}\binom{n-2}{r-1} F(t)^{n-r-1}[1-F(t)]^{r-1} f(t)^2 \Big|_{u_l}^{u_h} - \frac{n}{2}\binom{n-2}{r-2} F(t)^{n-r}[1-F(t)]^{r-2} f(t)^2 \Big|_{u_l}^{u_h}.$$

It is easy to see that Δ_r is equal to zero except for some special values of n and r. Specifically, for $n = 2$, we have

$$\Delta_1 = -\Delta_2 = f(u_h)^2 - f(u_l)^2,$$

while for $n \geq 3$,

$$\Delta_1 = \frac{nf(u_h)^2}{2}, \quad \Delta_2 = -\frac{n}{2}(f(u_l)^2 I_{n=3} + f(u_h)^2),$$

$$\Delta_{n-1} = -\frac{n}{2}(f(u_l)^2 + f(u_h)^2 I_{n=3}), \quad \Delta_n = \frac{nf(u_l)^2}{2}.$$

In all other cases, $\Delta_r = 0$.

Going back to the expression for $p_{11}^{(1,r)} - p_{12}^{(1,r)}$, we can now collect all the remaining terms:

$$p_{11}^{(1,r)} - p_{12}^{(1,r)} = \Delta_r + \binom{n-1}{r-1}(n-r-1)(n-r)M_{r+2} - \binom{n-1}{r-1}(n-r-1)(r-1)M_{r+1}$$

$$-\binom{n-1}{r-1}(n-r)(r-2)M_{r+1} + \binom{n-1}{r-1}(r-1)(r-2)M_r - \binom{n-1}{r-1}(n-1)M_{r+1}$$

$$-\frac{n}{2}\binom{n-2}{r-1}(n-r-1)M_{r+2} + \frac{n}{2}\binom{n-2}{r-1}(r-1)M_{r+1} + \binom{n-2}{r-1}(n-r-1)M_{r+2}$$

$$-\binom{n-2}{r-1}(r-1)M_{r+1} - \binom{n-2}{r-2}(n-r)M_{r+1} + \binom{n-2}{r-1}(r-2)M_r$$

$$+\frac{n}{2}\binom{n-2}{r-2}(n-r)M_{r+1}-\frac{n}{2}\binom{n-2}{r-2}(r-2)M_r.$$

Collecting the terms with M_r, M_{r+1} and M_{r+2} and using the properties of binomial coefficients, finally obtain Eq. (8).
Q.E.D.

Lemma 2

$$p_2^{(1,r)}=-\frac{\beta_r}{n-1}.$$

Proof. The expression for $p_2^{(1,r)}$ can be obtained by differentiating Eq. (15) with respect to e_2 and setting $e_1=e_2=\bar{e}$. This gives

$$p_2^{(1,r)}=-\binom{n-2}{r-1}\int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2dt+\binom{n-2}{r-2}\int F(t)^{n-r}[1-F(t)]^{r-2}f(t)^2dt$$

$$=\frac{(n-2)!}{(r-2)!(n-r-1)!}\int F(t)^{n-r-1}[1-F(t)]^{r-2}\left(-\frac{1-F(t)}{r-1}+\frac{F(t)}{n-r}\right)f(t)^2dt$$

$$=\frac{(n-2)!}{(r-1)!(n-r)!}\int F(t)^{n-r-1}[1-F(t)]^{r-2}[-(n-r)(1-F(t))+(r-1)F(t)]f(t)^2dt$$

$$=-\frac{\beta_r}{n-1}.$$

Q.E.D.

We now go back to the proof of Proposition 1. For part (a), start by plugging the representations $e_i=\bar{e}(1+x_i)$ and $c_i=\bar{c}(1-a_i)$ into Eq. (5):

$$\sum_r p_i^{(i,r)}(\bar{e}(1+x_1),\ldots,\bar{e}(1+x_n))V_r=\bar{c}(1-a_i)g'(\bar{e}(1+x_i)).$$

The next step is to expand both sides of the equation in Taylor series to the first order in μ treating x_i and a_i as small corrections linear in μ. The left-hand side becomes

$$\sum_r(p_i^{(i,r)}+p_{ii}^{(i,r)}\bar{e}x_i+\sum_{j\neq i}p_{ij}^{(i,r)}\bar{e}x_j)V_r+O(\mu^2).$$

Here and below, all the derivatives of $p^{(i,r)}$ are evaluated at the symmetric equilibrium point $(\bar{e},\ldots,\bar{e})$. Note that, by symmetry, $p_i^{(i,r)}=p_1^{(1,r)}\equiv\beta_r$ for all i and, likewise, $p_{ii}^{(i,r)}=p_{11}^{(1,r)}$ for all i and $p_{ij}^{(i,r)}=p_{12}^{(1,r)}$ for all $i\neq j$. Introducing $X=\sum_i x_i$, finally obtain for the left-hand side of (5),

$$\sum_r(\beta_r+\lambda_r\bar{e}x_i+p_{12}^{(1,r)}\bar{e}X)V_r+O(\mu^2).$$

Here, $\lambda_r \equiv p_{11}^{(1,r)} - p_{12}^{(1,r)}$.

Similarly expanding the right-hand side of (5), obtain

$$c_i g'(e_i) = \bar{c}(1-a_i)g'(\bar{e}(1+x_i)) = \bar{c}[g'(\bar{e}) + g''(\bar{e})\bar{e}x_i - g'(\bar{e})a_i] + O(\mu^2).$$

Equating the two expressions and using Eq. (1), obtain

$$\sum_r (\lambda_r \bar{e}x_i + p_{12}^{(1,r)}\bar{e}X)V_r = \bar{c}\bar{e}g''(\bar{e})x_i - \bar{c}g'(\bar{e})a_i + O(\mu^2).$$

Summing this expression over i and using the restriction $\sum_i a_i = 0$ gives $X = O(\mu^2)$, which, together with Eq. (1), produces Eq. (6).

For part (b), write agent i's equilibrium payoff, $\pi_i = \sum_r p^{(i,r)}V_r - c_i g(e_i)$, using the representations $c_i = \bar{c}(1-a_i)$ and $e_i = \bar{e}(1+x_i)$:

$$\pi_i = \sum_r p^{(i,r)}(\bar{e}(1+x_1), \dots, \bar{e}(1+x_n))V_r - \bar{c}(1-a_i)g(\bar{e}(1+x_i)).$$

Expanding this expression to the first order in μ, obtain

$$\begin{aligned}\pi_i &= \sum_r \left(\frac{1}{n} + p_i^{(i,r)}\bar{e}x_i + \sum_{j\neq i} p_j^{(i,r)}\bar{e}x_j\right)V_r - \bar{c}(g(\bar{e}) + g'(\bar{e})\bar{e}x_i - a_i g(\bar{e})) + O(\mu^2) \\ &= \frac{1}{n}\sum_r V_r - \bar{c}g(\bar{e}) + \bar{c}g(\bar{e})a_i - \bar{e}\sum_r p_2^{(1,r)}V_r x_i + O(\mu^2).\end{aligned}$$

Using Lemma 2, this can be written as

$$\pi_i = \frac{1}{n}\sum_r V_r - \bar{c}g(\bar{e}) + \bar{c}g(\bar{e})a_i + \frac{\bar{e}\sum_r \beta_r V_r x_i}{n-1} + O(\mu^2).$$

which, together with Eqs. (1) and (6), gives the result.

A.2 Proof of Proposition 2

The principal's profit is $\Pi = n[\bar{e} - \bar{c}g(\bar{e}) - \omega + \eta(\bar{e})a_n] + O(\mu^2)$. Let $\bar{e} = \bar{e}^s + \tau a_n$. In the linear approximation, the optimal profit Π can be evaluated at $\bar{e} = \bar{e} = \bar{e}^s$, due to the envelope theorem; hence part (c) of the proposition.

For part (a), note that $\eta(\bar{e}^s)$ is evaluated at the parameters of the symmetric optimal contract, which gives (cf. Eq. (7) and the fact that $\bar{c}g'(\bar{e}^s) = 1$)

$$\eta(\bar{e}^s) = \frac{1}{(n-1)[\bar{c}g''(\bar{e}) - \sum_r \lambda_r \bar{V}_r]} + \bar{c}g(\bar{e}^s).$$

By construction the relative ability of agent n is negative, $a_n < 0$, therefore the principal will choose the prize structure $(\bar{V}_1, \ldots, \bar{V}_n)$ that minimizes the loss term in the profit, $n\eta(\bar{e}^s)|a_n|$, i.e., minimizes $\eta(\bar{e}^s)$. This leads to the following principal's problem:

$$\min_{\bar{V}_1,\ldots,\bar{V}_n} \sum_r \lambda_r \bar{V}_r \quad \text{s.t.} \sum_r \beta_r \bar{V}_r = 1, \quad \sum_r \bar{V}_r = n[\omega + \bar{c}g(\bar{e}^s)].$$

Let $D_r = \bar{V}_r - \bar{V}_{r+1}$ for $r = 1, \ldots, n-1$ denote the differences between adjacent prizes. By construction, $D_r \geq 0$. Prizes $\bar{V}_r$ can then be written as $\bar{V}_r = \sum_{j=r}^{n-1} D_j + \bar{V}_n$. This gives

$$\sum_r \beta_r \bar{V}_r = \beta_1(D_1 + \ldots + D_{n-1} + \bar{V}_n) + \beta_2(D_2 + \ldots + D_{n-1} + \bar{V}_n) + \ldots + \beta_n \bar{V}_n$$

$$= \beta_1 D_1 + (\beta_1 + \beta_2)D_2 + \ldots + (\beta_1 + \ldots + \beta_n)\bar{V}_n = \sum_{r=1}^{n-1} B_r D_r,$$

where $B_r = \sum_{j=1}^{r} \beta_r$; we also used the fact that $B_n = 0$. Similarly,

$$\sum_r \lambda_r \bar{V}_r = \sum_{r=1}^{n-1} \Lambda_r D_r, \quad \sum_r \bar{V}_r = \sum_{r=1}^{n-1} r D_r + n\bar{V}_n.$$

Here $\Lambda_r = \sum_{j=1}^{r} \lambda_r$, with $\Lambda_n = 0$.

The principal's problem can be written in terms of the variables D_r as

$$\min_{D_1,\ldots,D_{n-1}\geq 0} \sum_{r=1}^{n-1} \Lambda_r D_r \quad \text{s.t.} \sum_{r=1}^{n-1} B_r D_r = 1. \tag{16}$$

Note that the second constraint is no longer relevant for the minimization problem and only serves to determine the lowest prize: $\bar{V}_n = \omega + \bar{c}g(\bar{e}^s) - (1/n)\sum_{r=1}^{n-1} r D_r$.

The following lemma shows that the cumulative coefficients B_r and Λ_r are indeed given by Eqs. (11).

Lemma 3 $B_r = \sum_{k=1}^{r} \beta_k$ *and* $\Lambda_r = \sum_{k=1}^{r} \lambda_k$ *are given by Eqs. (11).*

Proof. It easy to see that for $r = 1$ both formulas are correct. It is, therefore, sufficient to show that $\beta_r = B_r - B_{r-1}$ and $\lambda_r = \Lambda_r - \Lambda_{r-1}$, with B_r and Λ_r given by Eqs. (11). We have

$$B_r - B_{r-1} = \frac{(n-1)!}{(n-r-1)!(r-1)!}\int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2dt$$

$$-\frac{(n-1)!}{(n-r)!(r-2)!}\int F(t)^{n-r}[1-F(t)]^{r-2}f(t)^2dt$$

$$= \frac{(n-1)!}{(n-r)!(r-1)!}\int F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t))-(r-1)F(t)]f(t)^2dt = \beta_r;$$

$$\Lambda_r - \Lambda_{r-1} = \frac{n(n-2)!}{2(n-r-1)!(r-1)!}[(n-r-1)M_{r+2}-(r-1)M_{r+1}]$$

$$-\frac{n(n-2)!}{2(n-r)!(r-2)!}[(n-r)M_{r+1}-(r-2)M_r]+\Delta_r$$

$$= \frac{n(n-2)!}{2(n-r)!(r-1)!}[(n-r)(n-r-1)M_{r+2}-2(n-r)(r-1)M_{r+1}+(r-1)(r-2)M_r]+\Delta_r = \lambda_r.$$

Q.E.D.

It follows from (11) that coefficients B_r are positive for all $r < n$. Thus, the constraints of problem (16) define a convex polygon whose vertices $k = 1, \ldots, n-1$ have $D_k = 1/B_k$ and $D_r = 0$ for all $r \neq k$. The objective function is linear, therefore the minimum will be reached at one of the vertices.[17] Specifically, an optimal vertex is $j \in \arg\min_{1\le r\le n-1} \Lambda_r/B_r$.

Thus, the optimal prize structure is such that $D_j = 1/B_j$ for some j and $D_r = 0$ for $r \neq j$. The nth prize, therefore, is $\bar{V}_n = \omega + \bar{c}g(\bar{e}^s) - j/nB_j$. This leads to the following optimal configuration of symmetric prizes: $\bar{V}_1 = \ldots = \bar{V}_j = \bar{V}_n + 1/B_j$ and $\bar{V}_{j+1} = \ldots = \bar{V}_n$.

Now that the optimal structure of symmetric prizes is determined, we are in a position to find the optimal prizes W_1 and W_2 and the optimal average effort $\bar{e}$ for weakly heterogeneous agents (part (b)). The first-order condition for the principal's profit maximization is

$$\frac{\partial \Pi}{\partial \bar{e}} = n[1 - \bar{c}g'(\bar{e}) + \eta'(\bar{e})a_n] = 0.$$

Plugging in the representation $\bar{e} = \bar{e}^s + \tau a_n$ and expanding this equation to the first order in μ, obtain

$$1 - \bar{c}g'(\bar{e}^s) - \bar{c}g''(\bar{e}^s)\tau a_n + \eta'(\bar{e}^s)a_n = 0,$$

which implies $\tau = \eta'(\bar{e}^s)/\bar{c}g''(\bar{e}^s)$.

The optimal prizes W_1 and W_2 satisfy the equations $\sum_r \beta_r V_r = \bar{c}g'(\bar{e})$ and $\pi_n = \omega$.

[17] It is possible to have multiple minima when $\Lambda_k/B_k = \Lambda_l/B_l$ for some $k \neq l$. Such solutions are nongeneric; besides, any of the optimal vertices can be used as a solution anyway.

With the j-tournament prize structure and $\bar{e} = \bar{e}^s + \tau a_n$, these become, in the linear approximation,

$$B_j(W_1 - W_2) = 1 + \bar{c}g''(\bar{e}^s)\tau a_n,$$

$$\frac{jW_1}{n} + \frac{(n-j)W_2}{n} - \bar{c}g(\bar{e}^s) - \tau a_n + \eta(\bar{e}^s)a_n = \omega.$$

Solving this system of equations, obtain the result.