

A Simple Market-Like Allocation Mechanism for Public Goods

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Abstract

We argue that since allocation mechanisms will not always be in equilibrium, their out-of-equilibrium properties must be taken into account along with their properties in equilibrium. For economies with public goods, we define a simple market-like mechanism in which the strong Nash equilibria yield the Lindahl allocations and prices. The mechanism satisfies critical out-of-equilibrium desiderata that previously-introduced mechanisms fail to satisfy, and always (weakly) yields Pareto improvements, whether in equilibrium or not. The mechanism requires participants to communicate prices and quantities, and turns these into outcomes according to a natural and intuitive outcome function. Our approach first exploits the equivalence, when there are only two participants, between the private-good and public-good allocation problems to obtain a two-person public-good mechanism, and then we generalize the public-good mechanism to an arbitrary number of participants. The results and the intuition behind them are illustrated in the familiar Edgeworth Box and Kölm Triangle diagrams.

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There is a substantial literature analyzing the design of institutions, or “mechanisms,” for achieving efficient allocations in the presence of public goods. The pioneering paper by Groves and Ledyard (1977) introduced the first public goods mechanism with Pareto efficient Nash equilibria. Subsequently, Hurwicz (1979) and Walker (1981), building on the ideas in Groves & Ledyard, defined mechanisms that attain Lindahl allocations — allocations that are individually rational as well as Pareto efficient. Subsequent theoretical research has focused on developing mechanisms with additional desirable properties, or mechanisms that can be applied to economies with other kinds of externalities.¹

A number of the mechanisms developed in this theoretical research have been the subject of experimental studies.² The experimental results have been mixed at best. The mechanisms have variously failed to converge to equilibrium, or have exhibited slow convergence, or while out of equilibrium have suffered failures of individual rationality, failures of collective feasibility, or severely inefficient outcomes. These results are serious red flags for practical implementation. They suggest that in the case of public goods there remains a gap between implementation in theory and implementation in practice, and that perhaps a different approach might be fruitful.

The failures when out of equilibrium are especially troubling. Even for mechanisms that have good stability properties, we can’t realistically expect to be in equilibrium very often (if ever!). And the failures when out of equilibrium are often unacceptable: outcomes that make some (or all) the participants far worse off than they would have been had they been simply left alone, or outcomes that are not well-defined, because they are not feasible for some individuals or for the economy as a whole. This suggests that a focus on just the equilibrium properties of mechanisms — asking whether the equilibria are Pareto efficient, or Lindahl allocations, etc. — and even expanding the focus to the mechanisms’ stability properties as well, is too narrow. It suggests that we also need to take into account some desiderata for mechanisms’ out-of-equilibrium properties.

In this paper we take a very limited, preliminary step in that direction. Our objective is to devise a mechanism for public goods that *always*, whether in equilibrium or not, produces feasible and “acceptable” outcomes and still produces Lindahl allocations as equilibria. Along the way, we introduce a notion of acceptability that, as far as we know, has not appeared before. We’re partially successful in attaining this objective: the mechanism we introduce always produces feasible and acceptable outcomes, and does produce Lindahl allocations as Nash equilibria. Communication among the participants is via natural, market-like proposals involving quantities and Lindahl-like

¹For example, Bagnoli and Lipman (1989), de Trenqualye (1989, 1994), Kim (1993), Varian (1996), Peleg (1996), Corchon and Wilkie (1996), Tian (2000), Chen (2002), Healy and Mathevet (2013), and Van Essen (2013, 2015).

²For example, see Chen and Plott (1996), Chen and Tang (1998), Chen and Gazzale (2004), Healy (2006), Van Essen (2012), and especially Van Essen, Lazzati, and Walker (2012).

prices, which are turned into outcomes according to a transparent and simple outcome function. There are many other, non-Lindahl Nash equilibria as well, however the Lindahl equilibrium is the only strong equilibrium, and the mechanism always yields (weak) Pareto improvements, whether in or out of equilibrium. In the two-person case we also identify the coalition-proof Nash equilibria.

We begin by making two observations. The first is that for the problem of allocating or exchanging purely private goods, simple market institutions such as the double auction have enjoyed remarkable success in the laboratory, beginning with the landmark paper by Smith (1962). The second observation is that when there are only two traders, the public-goods allocation problem and the private-goods allocation problem are essentially identical.

Building on these two observations, we first define a simple market-like mechanism for implementing Walrasian allocations when there are only two consumers and two goods. In the spirit of the market games introduced by Shapley and Shubik (1977), Dubey (1982), and others,³ the actions the mechanism makes available to the players are natural, economically meaningful price-and-quantity proposals, and the proposals lead to outcomes in a natural and intuitive way. The mechanism's outcomes are always feasible, both for individuals and in the aggregate, whether the mechanism is in equilibrium or not, and the outcomes are always individually rational.

We then show that a straightforward reinterpretation of quantities and prices converts the mechanism into one for allocating a public good. And it's then straightforward to extend the mechanism to an arbitrary number of participants, preserving all the properties of the two-person private-goods and public-good versions of the mechanism.

We make liberal use of the Edgeworth Box to depict the arguments and the intuition for the private-goods exchange mechanism and the Kölm Triangle to provide intuition for the public-goods version of the mechanism.

The Pure Exchange Allocation Problem

There are two goods and two traders. Trader S wishes to sell good X in exchange for good Y, and Trader B wishes to purchase good X in exchange for good Y. It's convenient to think of Y as money.

The number of units of X the traders exchange will be denoted by q ; the price at which the units are exchanged is denoted by p ; and we write $m = pq$ for the amount of money exchanged. Thus, B pays $m = pq$ dollars to S in exchange for q units of X. Each trader $i \in \{B, S\}$ has a strictly quasiconcave utility function $u_i(\cdot)$ over trades $(q, m) \in \mathbb{R}_+^2$. We assume that u_S is strictly

³For example, Wilson (1978), Schmeidler (1980), and Binmore (1987).

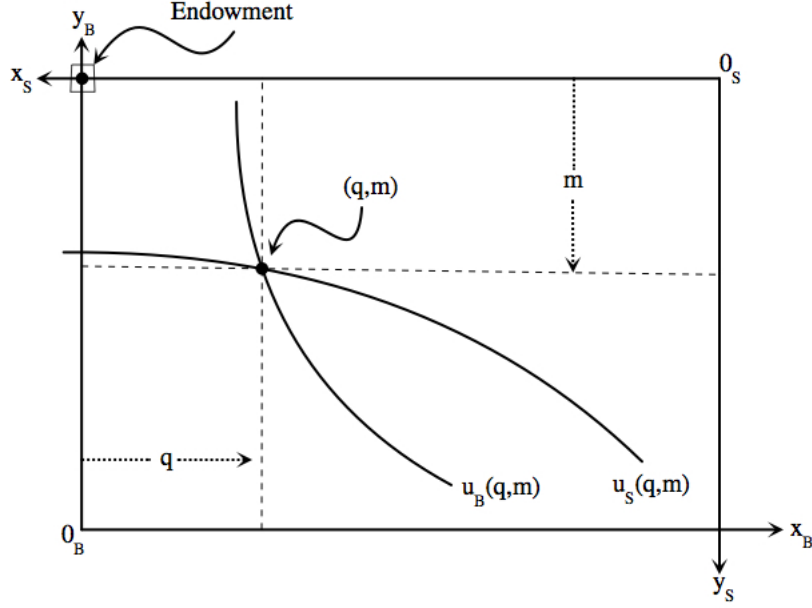


Figure 1: Preferences, endowments, and a trade (q, m)

decreasing in q and strictly increasing in m , and that u_B is strictly increasing in q and strictly decreasing in m . See Figure 1.

While we define the mechanism and carry out the analysis in terms of trades (q, m) and the associated prices p , these can of course be related to allocations and preferences in the usual way:

Each trader $i \in \{B, S\}$ owns an **endowment bundle** $(\hat{x}_i, \hat{y}_i) \in \mathbb{R}_+^2$, with $\hat{x}_S > 0$ and $\hat{y}_B > 0$, and the mechanism's outcome (q, m) yields the **allocation** $((x_B, y_B), (x_S, y_S))$ defined by

$$x_B = \hat{x}_B + q, \quad y_B = \hat{y}_B - m, \quad x_S = \hat{x}_S - q, \quad y_S = \hat{y}_S + m. \quad (1)$$

We assume that each trader has a strictly quasiconcave utility function U_i over bundles $(x_i, y_i) \in \mathbb{R}_+^2$, from which the functions u_i above are defined in the obvious way (see Figure 1):

$$u_B(q, m) = U_B(\hat{x}_B + q, \hat{y}_B - m) \quad \text{and} \quad u_S(q, m) = U_S(\hat{x}_S - q, \hat{y}_S + m).$$

For any price $p > 0$, let $\hat{q}_i(p)$ denote Trader i 's utility-maximizing quantity q_i — *i.e.*, the trade $(\hat{q}_i, p\hat{q}_i)$ maximizes $u_i(q_i, pq_i)$ for the given price p . Thus, $\hat{q}_B(\cdot)$ is Trader B's demand function and $\hat{q}_S(\cdot)$ is Trader S's supply function. Note that we use \hat{q}_i to denote both the function $\hat{q}_i(\cdot)$ and also the quantity $\hat{q}_i(p)$, when it's clear what the relevant price p is.

We restrict our attention to allocation problems in which there is a unique Walrasian allocation, which we assume is interior: the **Walrasian outcome**, denoted (q^W, p^W) , is the unique pair (q, p) that satisfies $q = \hat{q}_B(p) = \hat{q}_S(p)$, and we assume that $0 < q^W < \hat{x}_S$ and $0 < p^W q^W < \hat{y}_B$.

The Pure Exchange Mechanism

Each trader makes a **proposal** $\xi_i = (q_i, p_i) \in \mathbb{R}_+^2$. We place the following restrictions on the traders' proposals, to prevent a trader from offering more than he owns:

$$q_S \leq \hat{x}_S \quad \text{and} \quad p_B q_B \leq \hat{y}_B. \quad (2)$$

The proposal (q_i, p_i) can be interpreted as "I will buy/sell any amount up to q_i units of X (but no more) at the price p_i for each unit." When the **profile** of proposals is $\xi = ((q_B, p_B), (q_S, p_S))$, the **outcome** $(q, m) \in \mathbb{R}_+^2$ is given by

$$q = \begin{cases} \min\{q_B, q_S\}, & \text{if } p_S \leq p_B \\ 0, & \text{if } p_S > p_B \end{cases}$$

$$p = \frac{1}{2}(p_B + p_S)$$

$$m = pq$$

See Figures 2 and 3. We'll sometimes abuse this terminology a bit by referring to (q, p) as an outcome.

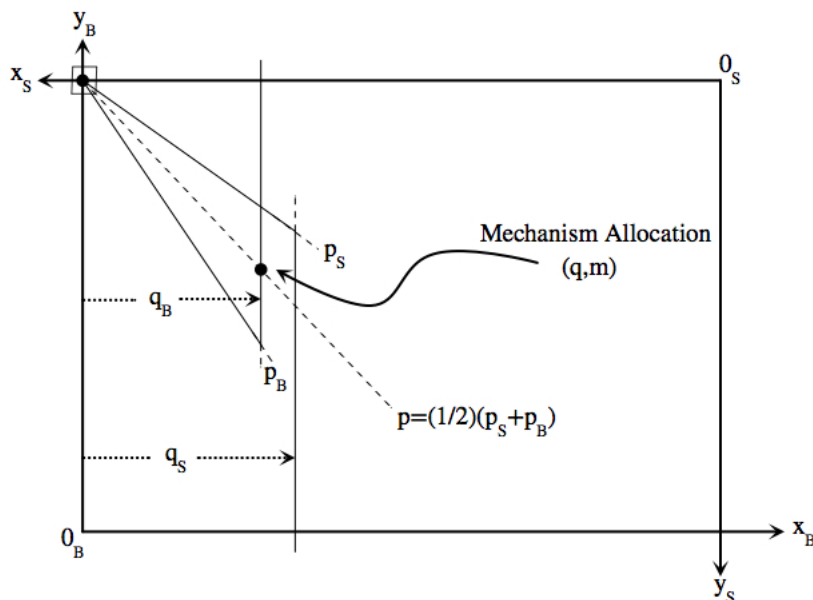


Figure 2: The outcome (q, m) if $p_S \leq p_B$

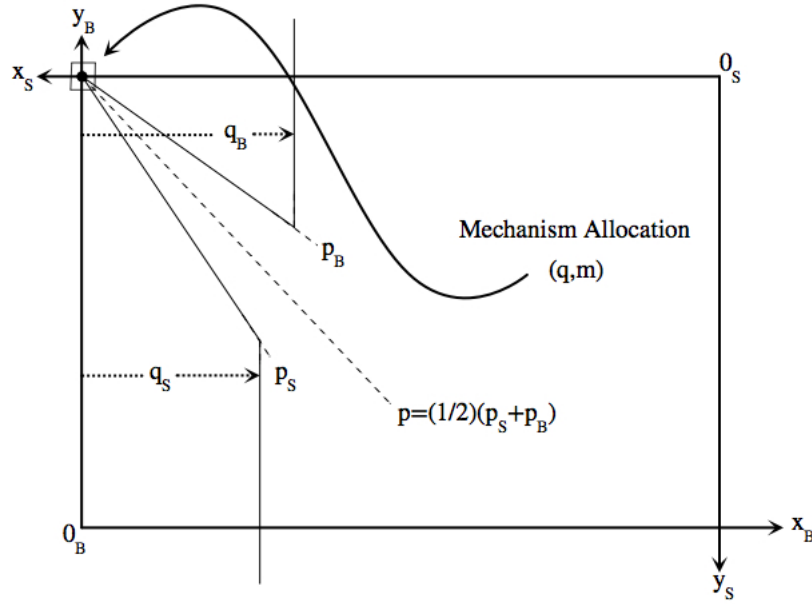


Figure 3: The outcome (q, m) if $p_S > p_B$.

Feasibility and Acceptability

It follows from (2) that each trader's strategy space, or message space, which we denote by Ψ_i , depends on his own endowment of one of the goods:

$$\Psi_B(\dot{y}_B) = \{ (q_B, p_B) \in \mathbb{R}_+^2 \mid p_B q_B \leq \dot{y}_B \} \quad \text{and} \quad \Psi_S(\dot{x}_S) = \{ (q_S, p_S) \in \mathbb{R}_+^2 \mid q_S \leq \dot{x}_S \}.$$

By restricting the traders' proposals in this way, we've built into the mechanism that for any admissible profile of proposals $\xi = (\xi_B, \xi_S)$ — whether the profile is an equilibrium or not — the outcome is **individually feasible**,

$$x_i, y_i \geq 0, \quad i \in \{B, S\},$$

and satisfies the **aggregate feasibility** condition,

$$x_B + x_S \leq \dot{x}_B + \dot{x}_S \quad \text{and} \quad y_B + y_S \leq \dot{y}_B + \dot{y}_S,$$

and is in fact always **non-wasteful**, and therefore **balanced** for every profile of proposals (not merely the equilibrium proposals):

$$x_B + x_S = \dot{x}_B + \dot{x}_S \quad \text{and} \quad y_B + y_S = \dot{y}_B + \dot{y}_S.$$

We denote the outcome function defined in (3) as $\varphi^* : \Psi_B(\dot{y}_B) \times \Psi_S(\dot{x}_S) \rightarrow \mathbb{R}_+^2$.

The outcome function φ^* has the following *acceptability property*, which as far as we know has not previously appeared in the literature. For this definition let (\mathbf{M}, φ) be an arbitrary mechanism, or

game form, for n players, with message space, or action space, $\mathbf{M} = \times_1^n M_i$ and outcome function $\varphi : \mathbf{M} \rightarrow Z$, where Z is the set of possible outcomes. For each i and each $\xi_i \in M_i$, let $\varphi_i(\xi_i)$ denote the set of all outcomes that can occur if i chooses the message ξ_i :

$$\varphi_i(\xi_i) := \{z \in Z \mid z = \varphi(\tilde{\xi}) \text{ for some } \tilde{\xi} \text{ s.t. } \tilde{\xi}_i = \xi_i\}.$$

Definition: For each player i , designate some outcome $\hat{z}_i \in Z$ as a **status-quo** outcome for i . An outcome $z \in Z$ is **acceptable** for the utility function u_i if $u_i(z) \geq u_i(\hat{z}_i)$. A message $\xi_i \in M_i$ is **uniformly acceptable** for u_i if every outcome in $\varphi_i(\xi_i)$ is acceptable for u_i . An outcome function φ has the **acceptability property for** (u_1, \dots, u_n) if there is a uniformly acceptable message ξ_i for each $i = 1, \dots, n$.

The outcome function φ^* has the acceptability property, but also the much stronger property described in the following remark:⁴

Remark: If preferences are quasiconcave, then under the outcome function φ^* every proposal $\xi_i = (q_i, p_i)$ that satisfies $u_i(q_i, p_i q_i) \geq u_i(0, 0)$ is uniformly acceptable to player i . In other words, any proposal that's acceptable to i is uniformly acceptable to i . Therefore, if i makes only proposals that are acceptable to him, then the outcome under φ^* (whether in equilibrium or not) will always be acceptable to him.

The outcomes we've defined as acceptable for i are the ones that are often called *individually rational* for i . But the notion of individual rationality is typically applied to a mechanism's *equilibrium* outcomes. An important feature of the outcome function φ^* , on the other hand, is the property described in the above Remark: an acceptable proposal by i can yield only acceptable outcomes for i , whether in or out of equilibrium, whatever proposal the other trader makes. We therefore have the following result:

Remark: If preferences are quasiconcave and each trader i 's proposal $\xi_i = (q_i, p_i)$ is acceptable for u_i , then the outcome under φ^* is weakly a Pareto improvement on the no-trade outcome, *i.e.*, $u_i(q_i, p_i q_i) \geq u_i(0, 0)$ for $i \in \{B, S\}$, whether (ξ_B, ξ_S) is an equilibrium or not.

In the public good setting the acceptability property and both the individual and aggregate feasibility properties of the outcome function φ^* are important features that distinguish it from other mechanisms, which, when out of equilibrium, can produce outcomes that are far from feasible

⁴The property described in this remark is well-defined only for mechanisms in which individuals' messages ξ_i are the arguments of their utility functions u_i (or map to those arguments, as here). This could perhaps serve as one possible definition of the notion that a mechanism uses "natural" messages. In that case we can say that a *message* (or *proposal*) ξ_i is acceptable if $u_i(\xi_i) \geq u_i(\hat{z}_i)$.

or far from acceptable. We'll have more to say about feasibility, acceptability, and the players' strategy spaces when we turn to the public-goods version of the mechanism.

Equilibrium

The following proposition is obvious:

Proposition 1.1: If (q, p) is a Walrasian outcome, and if $q_B = q_S = q$ and $p_B = p_S = p$, then $((q_B, p_B), (q_S, p_S))$ is a Nash equilibrium.

It's also obvious that there are many Nash equilibria — indeed, any profile $((q_B, p), (q_S, p))$ that satisfies $q_B = q_S \leq \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$ is a Nash equilibrium. We proceed to develop some conditions that a Nash equilibrium must satisfy.

Since each player i can unilaterally guarantee zero trade by proposing $q_i = 0$, the following proposition is immediate:

Proposition 1.2: Every Nash equilibrium is individually rational.

There are many Nash equilibria in which no trade takes place: for example, if $q_B = q_S = 0$, then $((q_B, p_B), (q_S, p_S))$ is an equilibrium for any p_B and p_S . There are also equilibria in which $p_S > p_B$, so that $q = 0$. Any equilibrium in which $q = 0$ will be called a **no-trade equilibrium**, and an equilibrium in which $q > 0$ will be called an **equilibrium with trade**.

Proposition 1.3: If $((q_B, p_B), (q_S, p_S))$ is a NE with trade, then $q_B > 0$, $q_S > 0$, and $p_B = p_S$.

Proof: If $q_B = 0$ or $q_S = 0$ or $p_S > p_B$, then $q = 0$; therefore a NE with trade must have $q_B > 0$, $q_S > 0$, and $p_S \leq p_B$. Suppose $p_S < p_B$; then neither player is maximizing his utility, as follows: if B , for example, proposes (q'_B, p'_B) , where $p'_B = p_S$ and $q'_B = q_B$, then the outcome will be $q' = q$ (*i.e.*, q is unchanged) and $p' = p_S < p < p_B$. Therefore $p'q' < pq$, and $u_B(q', p'q') > u_B(q, pq)$. (Alternatively, we could have applied this argument to S instead of B .) ■

Proposition 1.4: If $((q_B, p_B), (q_S, p_S))$ is a NE with trade, then the outcome (q, p) satisfies $q \leq \widehat{q}_B(p)$ and $q \leq \widehat{q}_S(p)$.

Proof: Suppose by way of contradiction that $q > \widehat{q}_B(p)$. Let $p'_B = p_B$ and $q'_B = \widehat{q}_B(p)$. Then the outcome (q', p') satisfies $p' = p$; and since $q > \widehat{q}_B(p)$, we have $q' = \min\{q'_B, q_S\} = \widehat{q}_B(p)$. Therefore

$$u_B(q', p'q') = u_B(\widehat{q}_B(p), p\widehat{q}_B(p)) > u_B(q, pq).$$

The same argument establishes that if $q > \widehat{q}_S(p)$, then S 's utility is increased by proposing $(q'_S, p'_S) = (\widehat{q}_S(p), p_S)$ instead of (q_S, p_S) . ■

Proposition 1.5: If $((q_B, p_B), (q_S, p_S))$ is a NE with trade and $q_B \neq q_S$, then $q = \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$.

Proof: Let $q_B < q_S$; then we have $q = q_B$, and Proposition 1.4 therefore guarantees that $q_B \leq \min\{\widehat{q}_S(p), \widehat{q}_B(p)\}$. We will show that $q_B = \widehat{q}_B(p)$. Suppose by way of contradiction that $q_B < \widehat{q}_B(p)$; we show that the proposal $(q'_B, p'_B) = (\widehat{q}_B(p), p_B)$ yields an outcome with greater utility for B than the outcome from the proposal (q_B, p_B) . There are two cases to consider: $q_S \geq \widehat{q}_B(p)$ and $q_S < \widehat{q}_B(p)$. In both cases the outcome (q', p') will satisfy $p' = p$ (*i.e.*, p will not change), because $p'_B = p_B$. In the first case, where $q_S \geq \widehat{q}_B(p)$, we have $q' = q'_B = \widehat{q}_B(p)$. B clearly prefers this outcome (q', p') to (q, p) . In the second case, where $q_S < \widehat{q}_B(p)$, we have $q' = q_S$, and therefore $(q', p') = (q_S, p)$. This gives us $q < q' < \widehat{q}_B(p)$, and since u_B is strictly quasiconcave and $u_B(q, pq) < u_B(\widehat{q}_B(p), p\widehat{q}_B(p))$, we have $u_B(q, pq) < u_B(q_S, pq_S)$ — *i.e.*, the proposal (q'_B, p'_B) does indeed yield B a higher utility than (q_B, p_B) , contradicting that $((q_S, p_S), (q_B, p_B))$ is a NE. Therefore we must have $q_B = \widehat{q}_B(p)$. The same argument establishes that if $q_S < q_B$, then we have $q = \widehat{q}_S(p)$. ■

Summarizing Propositions 1.2 - 1.5, every NE with trade $((q_B, p_B), (q_S, p_S))$ satisfies the following *necessary* conditions:

(E1) $((q_B, p_B), (q_S, p_S))$ is individually rational.

(E2) $p_B = p_S, q_B > 0, q_S > 0$.

(E3) Exactly one of the following is true:

(E3.1) $q = q_B = q_S \leq \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$, or

(E3.2) $q = q_S = \widehat{q}_S(p) < \min\{q_B, \widehat{q}_B(p)\}$, or

(E3.3) $q = q_B = \widehat{q}_B(p) < \min\{q_S, \widehat{q}_S(p)\}$.

In fact, the conditions (E1) - (E3) are also *sufficient* to ensure that a profile $((q_B, p_B), (q_S, p_S))$ is a NE with Trade, as the following proposition establishes.

Proposition 1.6: If $((q_B, p_B), (q_S, p_S))$ satisfies (E1) - (E3) then it is a NE with trade.

Proof: Suppose that $((q_B, p_B), (q_S, p_S))$ satisfies (E1) - (E3) and (q, p) is the outcome. Consider first whether the Buyer B can obtain a better outcome by deviating to a proposal (q'_B, p'_B) . (E2) guarantees that the outcome satisfies $q > 0$, and (E1) therefore guarantees that u_B will not be increased by either $q'_B = 0$ or $p'_B < p_B = p_S$, either one of which produces the outcome $q' = 0$. Consequently the outcome (q', p') must satisfy $q' > 0$ and $p' \geq p_B$, and therefore u_B is greatest if $p' = p_B$.

It remains to establish that u_B will not be increased by any $q'_B \neq q_B$. First suppose that $q'_B > q_B$. If either (E3.1) or (E3.2) holds, then this will not change the outcome: $q' = q$. If (E3.3) holds,

so that $q = q_B = \widehat{q}_B(p)$, then clearly u_B will not be increased by any $q'_B \neq q_S$. Now suppose that $q'_B < q_B$. If either (E3.1) or (E3.3) holds, this will reduce q and therefore reduce u_B ; if (E3.2) holds it will reduce u_B if $q'_B < q_S$ (because it will reduce q), and it will leave q and therefore u_B unchanged if $q'_B \geq q_S$. Thus, there is no $q'_B \neq q_B$ that will increase u_B , establishing that there is no profitable deviation from (q_B, p_B) for the Buyer. The parallel argument establishes that there is no profitable deviation from (q_S, p_S) for the Seller. ■

Together, Propositions 1.2 to 1.6 provide the following characterization of the Nash equilibria with trade:

Theorem 1.1: The proposal profile $((q_B, p_B), (q_S, p_S))$ is a NE with trade if and only if it satisfies (E1) - (E3).

Figure 4 depicts all the outcomes (q, m) at the NE profiles $\xi = ((q_B, p_B), (q_S, p_S))$. The curves \mathcal{Z}_S and \mathcal{Z}_B in Figure 4 are the Seller's and the Buyer's offer curves (in the quadrant in which S and B are indeed the seller and the buyer, rather than the reverse). The set \mathcal{Z}_{SB} is the region bounded by \mathcal{Z}_S and \mathcal{Z}_B — *i.e.*,

$$\mathcal{Z}_{SB} = \{(q, m) \mid \exists q', q'' : (q', m) \in \mathcal{Z}_B, (q'', m) \in \mathcal{Z}_S, q' \leq q \leq q''\}.$$

From the characterization conditions (E1) - (E3) one can easily see that the NE with trade yield exactly the outcomes (q, m) that lie in the region \mathcal{Z}_{SB} , with the exception of the no-trade outcome $(q, m) = (0, 0)$, which is clearly the outcome of any no-trade NE. Therefore \mathcal{Z}_{SB} is the set of NE outcomes.

Pareto Efficiency and Refinements

The following result, which is clear from Figure 4, is an immediate consequence of the assumption that the utility functions are strictly increasing and strictly quasiconcave.

Theorem 1.2: The unique Pareto efficient Nash equilibrium outcome is the Walrasian outcome.

Consequently the Walrasian outcome is the only strong Nash equilibrium (SNE) outcome: A Nash equilibrium is **strong**⁵ if no coalition can unilaterally improve upon it; therefore when there are only two players a Nash equilibrium is strong if and only if it is Pareto efficient.

Theorem 1.3: The unique strong Nash equilibrium outcome is the Walrasian outcome.

⁵Aumann (1959) introduced the concept of a strong Nash equilibrium.

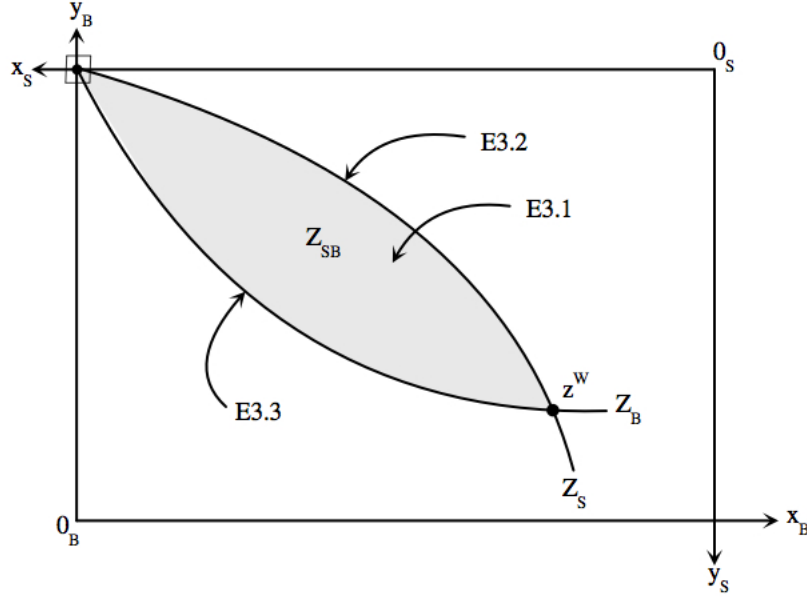


Figure 4: \mathcal{Z}_{SB} , the set of Nash equilibrium outcomes

This seems like an attractive result. However, it may be overly optimistic, for the same reason that strong Nash equilibrium is in many cases too *pessimistic*: the notion of strong Nash equilibrium assumes that a coalition can successfully deviate from an equilibrium if the deviation is preferred by all its members. In many games this leaves no SNE (thus the pessimism), while here it leaves us with the unique Walrasian equilibrium (optimism). But perhaps we should not regard such coalitional deviations as viable unless they're immune to further deviations by subcoalitions. This is the substance of **coalition-proof Nash equilibrium** (CPNE), introduced by Bernheim, Peleg, and Whinston (1987). When there are only two players, as we have here, the coalition-proof Nash equilibria are the Nash equilibria that are not Pareto-dominated by any other Nash equilibrium.

So the question we have to answer to determine the CPNE of our mechanism is which NE are Pareto-dominated by some other NE? It's easy to see that any allocation in the interior of the region \mathcal{Z}_{SB} is Pareto-dominated by other allocations in \mathcal{Z}_{SB} — *i.e.*, by other NE. The following result, which is depicted in Figure 5, establishes that indeed these allocations, as well as certain segments of the offer curves \mathcal{Z}_S and \mathcal{Z}_B , are not CPNE outcomes, but that other segments of the offer curves *are* CPNE outcomes.

Theorem 1.4: The set of coalition-proof Nash equilibrium outcomes is

$$\{(q, m) \in \mathcal{Z}_S \mid q_B^* \leq q \leq q^W\} \cup \{(q, m) \in \mathcal{Z}_B \mid q_S^* \leq q \leq q^W\},$$

where (q_B^*, m_B^*) maximizes u_B on S 's offer curve \mathcal{Z}_S and (q_S^*, m_S^*) maximizes u_S on B 's offer curve \mathcal{Z}_B .

Proof: Let $\xi = ((q_B, p_B), (q_S, p_S))$ be a NE, with outcome (q, p) . If $q = 0$ then any NE that

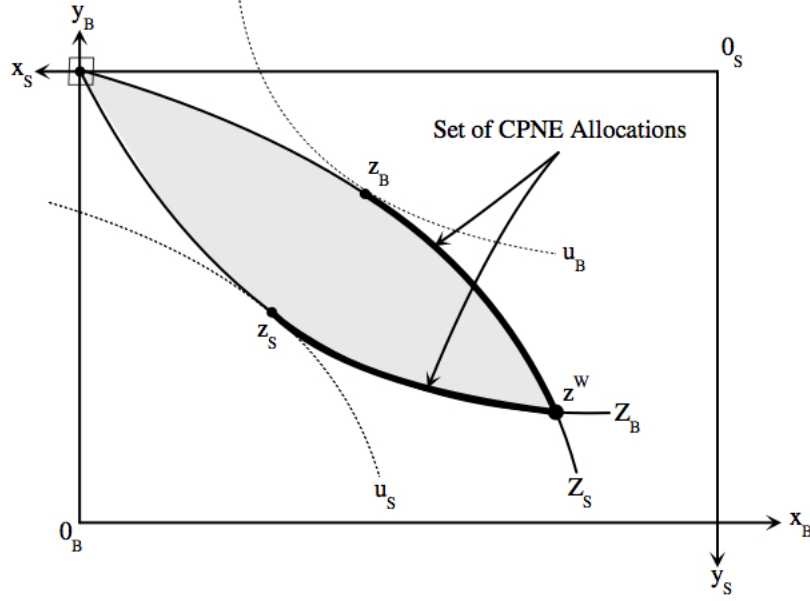


Figure 5: The coalition-proof Nash equilibria

yields the Walrasian outcome Pareto dominates ξ , so ξ is not a CPNE. If $0 < q < \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$, let $q' = \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$; the profile $\xi' = ((q', p), (q', p))$ yields the outcome (q', p) and is a NE; and the outcome (q', p) Pareto dominates (q, p) — the player i for whom $q = \widehat{q}_i(p)$ is clearly better off, and the other player is better off as well, due to the strict quasiconcavity of each u_j — so again ξ is not a CPNE. This leaves us with just those NE for which the outcome is a nonzero allocation on one of the offer curves, \mathcal{Z}_S or \mathcal{Z}_B — *i.e.*, $q = \min\{\widehat{q}_B(p), \widehat{q}_S(p)\}$. We complete the proof for \mathcal{Z}_S — *i.e.*, $q = \widehat{q}_S(p)$; the proof for \mathcal{Z}_B is identical.

There are two quite different cases to consider: (1) $q < q_S^*$, for which the NE is not a CPNE; and (2) $q \geq q_S^*$, for which the NE *is* a CPNE.

(1) Suppose that $(q, pq) \in \mathcal{Z}_S$ and $q < q_S^*$. Let p^* be the price at which $(q_S^*, p^* q_S^*) \in \mathcal{Z}_S$. This is the buyer's most-preferred NE outcome, so he prefers it to (q, pq) ; and it is on the seller's offer curve \mathcal{Z}_S , with $p^* > p$, so the seller prefers it to (q, pq) as well. Therefore ξ is not a CPNE.

(2) Suppose that $(q, pq) \in \mathcal{Z}_S$ and $q_S^* \leq q < q^W$, and let (q', m') be an allocation that Pareto dominates $(q, m) = (q, pq)$. We will show that (q', m') lies outside the region \mathcal{Z}_{SB} and is therefore not a NE outcome. Because $(q, m) = (q, pq) \in \mathcal{Z}_S$, we have $MRS_S = p$ at (q, pq) . From (E.3) we have $q \leq \widehat{q}_B(p)$, and since (q, m) is not Walrasian, we have $q < \widehat{q}_B(p)$, from which it follows that $MRS_B > p$ at (q, m) . Thus, $MRS_S < MRS_B$ at (q, m) , and therefore the Pareto improvement (q', m') must satisfy $q' > q$. Let σ be the slope of the offer curve \mathcal{Z}_S at (q, m) , and note that $MRS_B(q, m) < \sigma$: if instead $MRS_B(q, m) \geq \sigma$, then $u_B(q, m) > u_B(q^*, m^*)$, contradicting that

(q^*, m^*) maximizes u_B on \mathcal{Z}_S . Now we have $q' > q$, $u_B(q', m') > u_B(q, m)$, and $MRS_B(q, m) < \sigma$; therefore $(q', m') \notin \mathcal{Z}_{SB}$. Therefore there is no NE strategy profile ξ' that yields the outcome (q', m') , and since (q', m') is an arbitrary Pareto improvement on (q, m) , there is no NE that dominates $\xi = ((q_B, p_B), (q_S, p_S))$ — *i.e.*, ξ is a CPNE. ■

The Public Good Allocation Problem

Now we assume that the good X is a public good. We continue to think of the good Y as money, and we assume that it costs cq dollars to provide q units of the public good. The two-person allocation problem is to decide on the level q at which the public good will be provided, and how the cost of providing the q units, say cq , will be divided between two persons A and B . Allocations, or outcomes, are therefore triples (q, t_A, t_B) that satisfy the equation $t_A + t_B = cq$; the amounts t_A and t_B are in effect **taxes** paid by A and B . Figure 6 depicts the allocations/outcomes in a Kölm Triangle diagram.⁶

Each person has a positive amount \hat{y}_i of the Y-good and has a strictly quasiconcave utility function u_i over pairs (q, t_i) ; we assume that each u_i is strictly increasing in q and strictly decreasing in t_i . As in the private good case, we denote the players' demand functions by $\hat{q}_i(\cdot)$ — *i.e.*, for each $p > 0$, $\hat{q}_i(p)$ is the public-good level q that maximizes $u_i(q, pq)$.

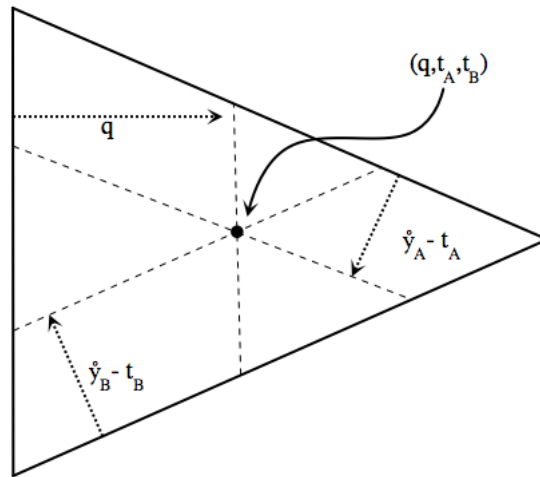


Figure 6: An allocation (q, t_A, t_B) depicted in the Kolm Triangle

⁶Expositions of the Kölm Triangle can be found in Thomson (1999) and Schlesinger (1989).

Definition: A **Lindahl outcome** is a triple (q, t_A, t_B) that satisfies $q = \hat{q}_A(p_A) = \hat{q}_B(p_B)$ and $p_A + p_B = c$, where $p_A = t_A/q$ and $p_B = t_B/q$ are **Lindahl prices**.

A typical Lindahl allocation/outcome is depicted in Figure 7. The slope of the “price line” is given by the Lindahl prices p_A and p_B . We assume there is a unique Lindahl outcome, (q^L, t_A^L, t_B^L) , which is interior: $q^L > 0$ and $0 < t_i^L < \dot{y}_i$ for $i = A, B$.

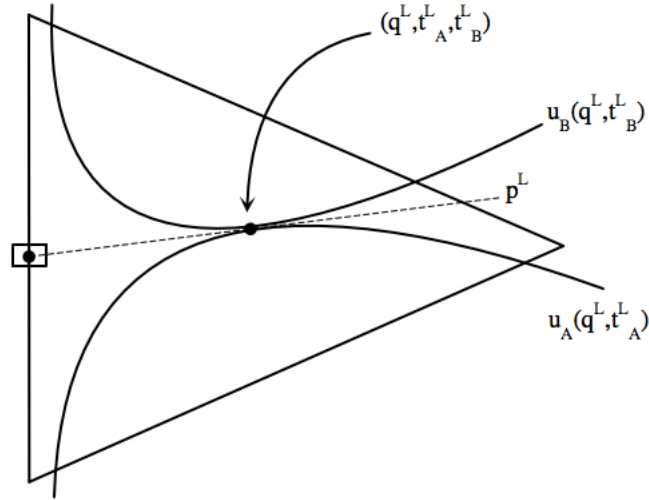


Figure 7: A Lindahl allocation and price-line

The Public Good Mechanism

Each person makes a proposal $\xi_i = (q_i, \pi_i) \in \mathbb{R}_+^2$, which, as we’ll see below, can be interpreted as “I will agree to any public good level up to q_i units (but no more) if the cost to me is no more than π_i for each unit.” As in the private-goods version of the mechanism, we require that each proposal satisfy the constraint $\pi_i q_i \leq \dot{y}_i$.

When the profile of proposals is $\xi = ((q_A, \pi_A), (q_B, \pi_B))$, the mechanism’s outcome (q, t_A, t_B) is given by

$$q = \begin{cases} \min\{q_A, q_B\}, & \text{if } \pi_A + \pi_B \geq c \\ 0, & \text{if } \pi_A + \pi_B < c \end{cases}$$

$$t_A = p_A q \text{ and } t_B = p_B q, \text{ where}$$

$$p_A = \frac{1}{2}c + \frac{1}{2}(\pi_A - \pi_B) \text{ and } p_B = \frac{1}{2}c + \frac{1}{2}(\pi_B - \pi_A).$$

Note that $p_A + p_B$ is identically equal to c , so for any pair of proposals $\xi = (\xi_A, \xi_B)$ the budget will

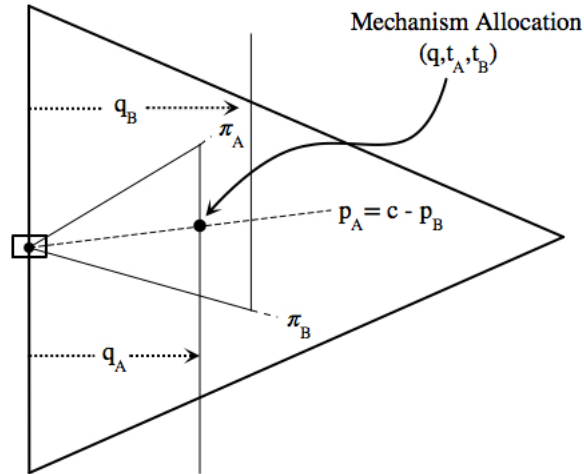


Figure 8: The outcome if $p_A + p_B \geq c$

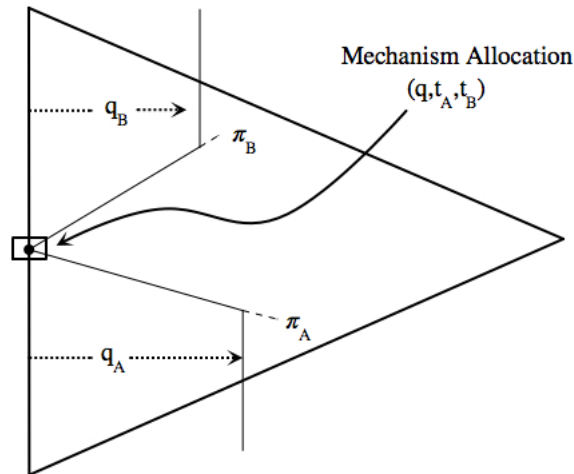


Figure 9: The outcome if $p_A + p_B < c$

be balanced. Moreover, if $q > 0$ then $\pi_A + \pi_B \geq c$, so each price p_i cannot be larger than Player i 's proposed price π_i — he pays no more than π_i per unit for the public good.

Note too the similarity of the mechanism's equations, above, to the ones for the exchange mechanism. Comparing Figures 8 and 9 with Figures 2 and 3 suggests that indeed the mechanism for the public-good problem is the same as the mechanism we've already introduced for the exchange problem. We verify that as follows.

We first show that given any proposal $\xi_A = (q_A, \pi_A)$ by Player A, the decision problem for Player B here is the same as the decision problem for the Buyer in the exchange mechanism:

Let $p'_B = \pi_B$ and $p'_A = c - \pi_A$. Then we have $p'_B \geq p'_A$ if and only if $\pi_A + \pi_B \geq c$. Therefore

$$q = \begin{cases} \min\{q_A, q_B\}, & \text{if } p'_B \geq p'_A \\ 0, & \text{if } p'_B < p'_A. \end{cases}$$

Furthermore, let $p' = \frac{1}{2}(p'_A + p'_B)$; then we have $p' = \frac{1}{2}(c + \pi_B - \pi_A) = p_B$. And finally, let $m' = t_B$, so that we have $m' = p'q$. Now if we substitute p'_A, p'_B, p' , and m' for p_S, p_B, p , and m in the earlier analysis of the exchange mechanism, we see that Player B's decision problem here is precisely the Buyer's decision problem in the exchange mechanism. Symmetrically, Player A's decision problem here is also the same as the Buyer's (not the Seller's) decision problem in the exchange mechanism.

Because each player's decision problem is the same as the Buyer's problem in the exchange mechanism, and neither one's is the same as the Seller's problem, we can't simply apply the propositions we've already established. But each proof of the exchange-mechanism propositions consisted of establishing an identical property of both the Buyer's and the Seller's decisions. Therefore, while we can't appeal directly to the propositions themselves, for each proposition we can directly reproduce the proof of the Buyer's behavior to establish the corresponding result for both players in the public good mechanism.

Now it's straightforward to simply enumerate the results for the public good mechanism that correspond to those for the exchange mechanism. We omit the proofs, which, as we've just described, are identical to the proofs for the exchange mechanism except for changes in notation. We also omit the intermediate results, the analogues of Propositions 1.3 to 1.6.

Proposition 2.1: If (q, t_A, t_B) is a Lindahl outcome, and if $q_A = q_B = q$ and $\pi_i q = t_i$ for each $i \in \{A, B\}$, then $((q_A, \pi_A), (q_B, \pi_B))$ is a Nash equilibrium for which the outcome is (q, t_A, t_B) .

Proposition 2.2: Every Nash equilibrium is individually rational.

Theorem 2.1: The profile $((q_A, \pi_A), (q_B, \pi_B))$ of proposals is a NE with $q > 0$ if and only if it satisfies the following conditions:

- (E'1) $((q_A, \pi_A), (q_B, \pi_B))$ is individually rational.
- (E'2) $\pi_A + \pi_B = c, q_S > 0, q_B > 0$.
- (E'3) Exactly one of the following is true:
 - (E'3.1) $q = q_A = q_B \leq \min\{\widehat{q}_A(p_A), \widehat{q}_B(p_B)\}$, or
 - (E'3.2) $q = q_A = \widehat{q}_A(p_A) < \min\{q_B, \widehat{q}_B(p_B)\}$, or
 - (E'3.3) $q = q_B = \widehat{q}_B(p_B) < \min\{q_A, \widehat{q}_A(p_A)\}$.

The equilibrium outcomes are depicted in Figure 10.

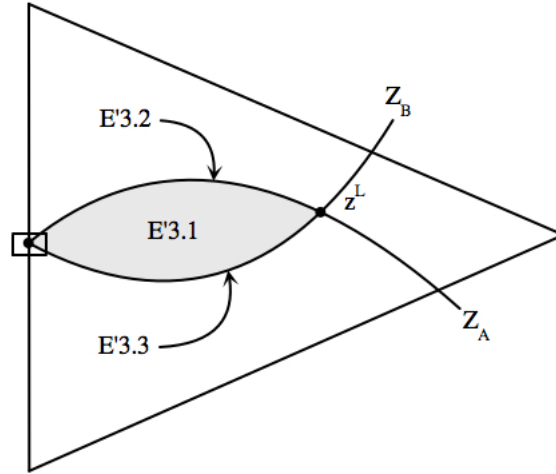


Figure 10: The Nash equilibrium outcomes

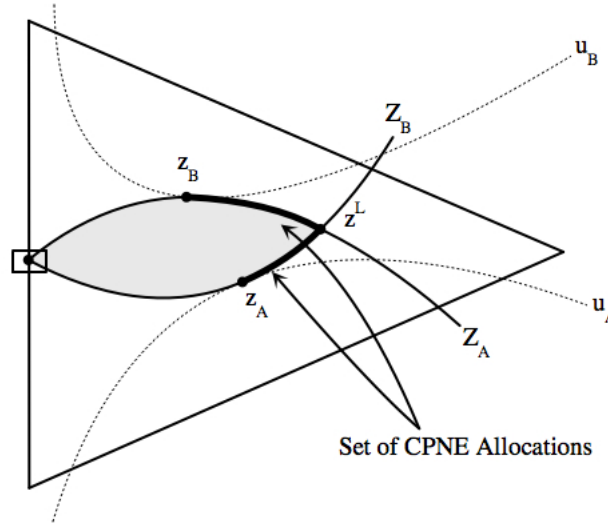


Figure 11: The coalition-proof Nash equilibria

Theorem 2.2: The unique Pareto efficient Nash equilibrium outcome is the Lindahl outcome.

Theorem 2.3: The unique strong Nash equilibrium outcome is the Lindahl outcome.

Theorem 2.4: The set of coalition-proof Nash equilibrium outcomes is

$$\{(q, t) \in \mathcal{Z}_A \mid q_B^* \leq q \leq q^L\} \cup \{(q, t) \in \mathcal{Z}_B \mid q_A^* \leq q \leq q^L\},$$

where (q_A^*, t_A^*) maximizes u_A on B 's offer curve \mathcal{Z}_B and (q_B^*, t_B^*) maximizes u_B on A 's offer curve \mathcal{Z}_A .

The CPNE outcomes are depicted in Figure 11.

The Mechanism with More Than Two Participants

So far, we've exploited the fact that the private-goods and public-goods allocation problems are identical in the two-person case to establish that a natural trading mechanism for private goods can be exactly duplicated when one of the goods is a public good, and that the mechanism has the same properties in both the private- and public-good problems. Here we extend the public-good version of the mechanism to an arbitrary number of participants.⁷

As in the two-person mechanism, each person makes a proposal $\xi_i = (q_i, \pi_i) \in \mathbb{R}_+^2$ which is interpreted in the same way as before: "I will agree to any public good level up to q_i units (but no more) if the cost to me is no more than π_i for each unit." We continue to require that each proposal satisfy the constraint $\pi_i q_i \leq \hat{y}_i$.

When the profile of proposals is $\xi = ((q_1, \pi_1), \dots, (q_n, \pi_n))$, the mechanism's outcome (q, t_1, \dots, t_n) is given by the outcome function $\varphi^* : \times_{i=1}^n \Psi_i(\hat{y}_i) \rightarrow \mathbb{R}_+^{n+1}$ defined as follows:

$$q = \begin{cases} \min\{q_1, \dots, q_n\}, & \text{if } \sum_{i=1}^n \pi_i \geq c \\ 0, & \text{otherwise ;} \end{cases}$$

$$t_i = p_i q, \text{ where } p_i = \frac{1}{n}c + \pi_i - \bar{\pi} \text{ and } \bar{\pi} = \frac{1}{n} \sum_{j=1}^n \pi_j \quad (i = 1, \dots, n).$$

Note that when $n = 2$ this is the mechanism we introduced above for two persons.

For every profile $\xi = ((q_1, \pi_1), \dots, (q_n, \pi_n))$ — whether ξ is an equilibrium or not — the mechanism's outcome $\varphi^*(\xi)$ has the following properties:

- (i) $\sum_i p_i \equiv c$, so the budget is always balanced.
- (ii) The outcome is always both collectively and individually feasible.
- (iii) Player i never pays more than his proposed price π_i per unit of the public good.

Therefore the mechanism's outcome can always be implemented, whether the mechanism is in equilibrium or not. Furthermore, if any of the public good is produced, no player's tax t_i will be more than he has proposed or more than the amount of the Y good he owns ($t_i \leq \hat{y}_i$), and the quantity of the public good will not exceed what any player has proposed. As in the private-goods version of the mechanism, the outcome function φ^* has the property that every acceptable proposal is uniformly acceptable:

⁷Extensions of the private-goods version of the mechanism to more than two traders is possible as well, but not in the same way, reflecting the fact that when $n > 2$ the private- and public-good allocation problems are no longer identical. Dubey (1984), for example, presents a "strategic market game" for any number of traders that is similar to the private-goods version of the mechanism presented here; see Giraud (2003) for a survey of such strategic market games.

Remark: If preferences are quasiconcave, then under the outcome function φ^* every proposal $\xi_i = (\pi_i, q_i)$ that satisfies $u_i(q_i, \pi_i q_i) \geq u_i(0, 0)$ is uniformly acceptable to player i ; therefore, if i makes only proposals that are acceptable to him, then the outcome under φ^* will always be acceptable to him. If each proposal ξ_i is acceptable for u_i , then the outcome $\varphi^*(\xi_1, \dots, \xi_n)$ is a weak Pareto improvement on the zero-provision allocation $(q, t_1, \dots, t_n) = (0, 0, \dots, 0)$, whether (ξ_1, \dots, ξ_n) is an equilibrium or not.

As with the two-person public-good mechanism, we can show that each player's decision problem is the same as the Buyer's problem in the private-goods exchange mechanism, where now the *aggregate* of all the $n - 1$ other players plays the role of the Seller: For each player i let $\check{q}_{-i} = \min_{j \neq i} q_j$, let $p'_{-i} = c - \sum_{j \neq i} \pi_j$, and let $p'_i = \pi_i$. Then we have $p'_i \geq p'_{-i}$ if and only if $\sum_i \pi_i \geq c$. Therefore, as in the exchange mechanism,

$$q = \begin{cases} \min\{q_i, \check{q}_{-i}\} & \text{if } p'_i \geq p'_{-i} \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, let $p' = \frac{(n-1)}{n} p'_i + \frac{1}{n} p'_{-i}$; then $p' = p_i$. This is again precisely the Buyer's problem in the two-player private-good mechanism (with the average price $\frac{1}{2} p_B + \frac{1}{2} p_S$ generalized to $\frac{(n-1)}{n} p'_i + \frac{1}{n} p'_{-i}$). Moreover, we can continue to depict the best response problem for each consumer in the Kölm triangle, as in Figure 7.

As before, it is straightforward to recover the equilibrium results for the n -player public-good mechanism that correspond to those in the exchange mechanism:

Proposition 3.1: If (q, t_1, \dots, t_n) is a Lindahl outcome, and if for all i , $q_i = q$ and $\pi_i q = t_i$, then the profile $\xi = ((q_i, \pi_i))_{i=1}^n$ is a Nash Equilibrium at which the outcome is (q, t_1, \dots, t_n) .

Proposition 3.2: Every Nash equilibrium is individually rational.

Theorem 3.1: The profile $\xi = ((q_i, \pi_i))_{i=1}^n$ is a NE with $q > 0$ if and only if it satisfies the following conditions:

(E''1) ξ is individually rational.

(E''2) $\sum_i \pi_i = c$ and $q_i > 0$ for all i .

(E''3) We have $q \leq \min\{\widehat{q}_1(p_1), \dots, \widehat{q}_n(p_n)\}$, and for each i , if $q_i < \check{q}_{-i}$, then $q_i = \widehat{q}_i(p_i)$.

Theorem 3.2: The unique Pareto efficient Nash equilibrium outcome is the Lindahl outcome.

Theorem 3.3: The unique strong Nash equilibrium outcome is the Lindahl outcome.

Discussion

We set out to devise a mechanism for allocating public goods in which the mechanism’s participants would communicate via natural, market-like proposals involving quantities and Lindahl-like prices, and in which a transparent and natural outcome function would always turn the price-quantity proposals into balanced and acceptable outcomes that would be Lindahl allocations when in equilibrium. The necessary insights were provided by focusing first on the problem when there are only two parties involved, where we could exploit the fact that in the two-person case the allocation problem for one public good and one private good is identical to the familiar Edgeworth Box two-person exchange problem with two private goods. We first devised a natural prices-and-quantities mechanism for the two-person private-goods problem, then performed a straightforward conversion of the mechanism to the two-person public good setting, and then generalized this public-good version of the mechanism so that it could be applied to an arbitrary number of participants. Here we point out some of the desirable features of the mechanism, and some of the trade-offs these features entail.

Clearly an attractive feature of the mechanism is that its outcomes are well-defined, feasible, and non-wasteful, no matter what proposals are made, whether the proposals are in equilibrium or not. Moreover, whether in equilibrium or not, the mechanism’s outcome will be both feasible and acceptable for each participant: while both the public good level q and a player’s tax t_i depend on all participants’ proposals, the feasibility and acceptability of the outcome to Player i depend only upon his own proposal (q_i, π_i) . Consequently, the mechanism’s outcomes can always be implemented — the public good can be produced, and taxes collected to finance it — without waiting until we arrive at an equilibrium. In this respect the mechanism operates in much the same way as exchange of private goods.⁸

Note that in the public-goods mechanisms devised by Groves & Ledyard (1977), Hurwicz (1979), and Walker (1981), as well as many others, the outcome functions yield out-of-equilibrium allocations that can be far from feasible, both individually and in the aggregate, and far from acceptable individually to the participants. Indeed, in Van Essen, et al. (2012), when the mechanisms of Kim (1993), Chen (2002), and Walker (1981), all of which attain Lindahl outcomes in equilibrium, were subjected to experimental tests, the mechanisms were rarely in equilibrium, and they all attained out-of-equilibrium outcomes that badly failed these feasibility and acceptability criteria.

Of course, this “always feasible, always acceptable” character of the mechanism comes at a cost. First is the restriction we impose on each player that he’s not allowed to propose more than he can deliver: $p_i q_i \leq \hat{y}_i$ in the private-goods version of the mechanism, and $\pi_i q_i \leq \hat{y}_i$ in the public good

⁸See Buchanan (1968, p. 88 ff.) for an informal but interesting foreshadowing of this idea.

mechanism. Each player’s action set, or message space, which we denoted by Ψ_i , therefore depends on one of the dimensions of the environment, \hat{y}_i , so the mechanism is not “totally decentralized” in the terminology of Schmeidler [1980]. However, we don’t regard this as a shortcoming of the mechanism: it’s a commonplace in actual trading that one is required to be able to “carry out his end of the bargain” in any trade he proposes, a requirement that has particular cogency when proposals can always be implemented. Note that this dependence of Ψ_i on \hat{y}_i does not require any player to have any information about any other player’s part of the environment; this is what Hurwicz (1972) calls “privacy-preserving” and therefore “informationally decentralized.”

The reader is referred to Schmeidler [1980, Section 3] for a discussion and references concerning aggregate and individual feasibility in allocation mechanisms, as well as alternative notions of decentralization.

While the Lindahl allocation is always an equilibrium outcome and is the only Pareto efficient equilibrium, there are also many equilibria that are not Pareto efficient. In particular, the mechanism provides each player with a veto over provision of any of the public good: the outcome function φ^* produces the public-good provision level $q = \min\{q_1, \dots, q_n\}$, so by choosing $q_i = 0$ a player ensures that the public good is not produced. Moreover, non-provision of the public good is an *equilibrium* outcome when as few as two players propose $q_i = 0$. The mechanism thus has a somewhat Wicksellian character.⁹

Both the veto-power property and the multiplicity of non-Pareto equilibria must be weighed against the advantages of a mechanism that always, in or out of equilibrium, yields feasible, non-wasteful, and acceptable outcomes, outcomes that are therefore (weakly) Pareto improving, and that can produce Lindahl allocations in equilibrium (and in the unique strong equilibrium). The Van Essen et al. paper (2012) strongly suggests that these properties are important, and perhaps even essential, for a satisfactory public-goods allocation mechanism.

⁹Wicksell (1896); see also Buchanan (1968) and Ledyard (2014).

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