

The dynamics of leverage in a Minskyan model with heterogeneous firms*

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July 31, 2015

Abstract

This paper introduces heterogeneous microeconomic behavior into a demand-driven macroeconomic model in order to study the joint dynamics of leverage and capital accumulation. By identifying the links between firm level variables and aggregate quantities, the paper contributes toward a reformulation of the Minskyan formal analysis that explicitly considers the role of microeconomic factors in generating macroeconomic instability. The aggregation of heterogeneous agents is not only performed numerically, as in traditional agent-based models, but also by means of an innovative analytical methodology, originally developed in statistical mechanics and recently imported into macroeconomics. The distinctive feature is in that the joint analysis of the numerical and analytical solutions of the model sheds light on the effects of financial fragility at the firm level on the dynamics of the macroeconomy. In particular, the analysis of steady-state and stability properties of the system provide additional insights on the role of behavioral and size heterogeneity of firms for the stocks of aggregate debt and capital.

Keywords: financial fragility, aggregate demand, agent-based model, master equation.

JEL classification: E16, E32, G01

1 Introduction

The 2008 financial turmoil itself, and the process of de-leveraging by the private sector observed in the following years, have drawn the attention to the crucial role of credit as a factor leading both to the instability of the system and to a strengthening of real-financial linkages in the economy. This view, which was central to the work of Hyman Minsky, is also supported by the vast historical evidence presented in Schularick and Taylor

*We thank André Diniz for the superb research assistance and Matthieu Charpe, Duncan Foley, Andrea Roventini, Peter Skott, Daniele Tavani, the participants to the Eastern Economic Association conference 2013 and the Analytical Political Economy workshop 2014 for valuable comments on previous versions of this paper. Financial support by the Institute for New Economic Thinking is also gratefully acknowledged.

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(2012), which highlights that credit booms tend to be followed by deeper recessions when compared to other financial crises episodes.

A formal investigation of these phenomena from a Minskyan standpoint requires the integration of the financial and economic systems in a demand-driven macroeconomic model. A seminal formalization of a Minsky crisis generated by self-fulfilling expectations was made by Taylor and O'Connell (1985) using a Kaleckian model, but without dealing explicitly with the role of debt. Delli Gatti et al. (1993) and Fazzari et al. (2008), among others, use aggregative dynamical models to represent the interaction between business debt and aggregate fluctuations while Chiarella and Di Guilmi (2011) and Lima and Meirelles (2007) introduce a micro-level of analysis that explicitly considers leverage heterogeneity of agents¹. An alternative, which has received renewed attention after the crisis (Khalil and Kinsella, 2011; Bezemer, 2010), builds on the traditions of Tobin (1969) and Godley and Lavoie (2007) to take into account all flows of income between sectors in the economy, as well as their accumulation into financial and tangible assets in a Keynesian setup. Besides allowing for formal Minskyan analyses of corporate debt and financial fragility (Dos Santos, 2005), the so-called Stock-Flow Consistent (SFC) models have recently been used to study the macroeconomic effects of shareholder value orientation and financialization (Van Treeck, 2009), household debt accumulation (Kim and Isaac, 2010), and other related issues.

The traditional limitation of this type of models, however, is that of dealing with economic behavior only in aggregative terms, thus excluding the heterogeneity of agents as a source of financial instability. The relevance of a microeconomic analysis in modeling financial fragility was stressed by Minsky himself: *“an ultimate reality in a capitalist economy is the set of interrelated balance sheets among the various units”* (Minsky, 2008, 116). Taylor and O'Connell (1985) remark that *“shifts of firms among classes as the economy evolves in historical time underlie much of its cyclical behavior. This detail is rich and illuminating but beyond the reach of mere algebra”*. It is thus for understanding the economy as an “out-of-equilibrium” system while allowing for heterogeneous microeconomic behavior (Delli Gatti et al., 2005, 2010), that the literature based on the so-called agent-based models (ABMs) has proven to be so useful for the analysis of financial instability. Nevertheless, for using a fully bottom-up approach, this type of models often lack a clear macroeconomic closure and are hard to connect to other macro analyses.

The starting point of this paper is to see agent-based, SFC and, more generally, Keynesian-Kaleckian macroeconomic approaches as complementary in their understanding of the crucial role of real-financial linkages for the instability of the economic system, as well as its macroeconomic dynamics². Moreover, we aim to contribute in filling what we consider to be a weakness in both the SFC and ABM approaches, namely its full reliance on numerical and computational solutions. The main problem with solving these models only numerically is that model results are usually very sensitive to parameter configurations, making it very difficult to establish general relationships between macro and micro-variables, as well as causality links within the system. In order to provide a few analytical insights that will add to the numerical simulations of the model, we perform an aggregation of heterogeneous agents by means of an innovative analytical methodology originally developed in statistical mechanics and recently imported into macroeconomics (see Aoki and Yoshikawa, 2006; Di Guilmi, 2008; Foley, 1994; Weidlich, 2000, among others). This modeling approach builds from the idea that, as the economy is populated by a very large number of dissimilar agents, an analytical model cannot keep track of the conditions of every single agent at each point in time. As Aoki and Yoshikawa (2006) remark: *“the point is that precise behavior of each agent is irrelevant. Rather we need to recognize that microeconomic behavior is fundamentally stochastic.”* Therefore, a microfounded analytical model should look at how many agents are in a certain condition, rather than at which agents, and represent their evolution in probabilistic

¹ See Chiarella and Di Guilmi (2012) and Lavoie (2009) for more exhaustive surveys of the Minskyan formal analysis.

² Such complementarities are clear in Godin and Kinsella (2012), which combines the two approaches in a study of the interaction of banks and firms in the leverage cycle. A dynamic micro-macro analysis is also developed in the work by Dosi et al. (2013), which builds an agent-based Keynesian model for the study of interactions between income distribution and monetary and fiscal policies.

terms. This approach is particularly suitable to microfound macroeconomic models, since it is able to endogenously derive the macro-equations and the dynamics of flows from the microeconomic behavioral rules, without imposing ad-hoc constraints.

Hence, the contributions of this paper are basically three. The first is mainly methodological, and consists in the integration of the numerical and analytical solutions of an agent-based model, further developing the seminal insights provided by Chiarella and Di Guilmi (2011) who show that the master equation solution can replicate the results of the simulations of a multi-agent model. Besides helping us open the “black box” of the simulations, the analytical approach allows for a few generalizations and a better assessment of the role of heterogeneity in levels and dimensions in relation to a more homogeneous microeconomic setup.

The second contribution, also methodological, concerns the innovative approach to the microfoundation of stock-flow consistent modeling. The macro-equations of the aggregate model are here generated endogenously in a bottom-up approach, starting from the behavioral equation for agents. Consequently, the steady-state analysis is able to assess the impact of the balance sheet structures of firms on the macroeconomy.

The third contribution is to explore micro and macroeconomic aspects leading to the emergence of short-run fluctuations and long-run instability of the economic system. In particular, the paper shows how the diverse financial structures of firms can determine the evolution of the economy, as stressed in Minsky’s narrative. As the objective is not to fully represent the behavior of any particular economy over time, but rather to provide a few methodological and theoretical insights for the analysis of leverage and financial instability, the model we present is particularly simple and stylized, but it is still able to point toward the usefulness of the method for more realistic settings.

The paper will be structured as follows. Section 2 presents the structure of the agent-based model. Results of the numerical simulations are then illustrated in section 3. Section 4 specifies the dynamics of firms’ transition between states, as well as the mean-field approximation procedure we use for the analytical solution of the model. In Section 5, we perform steady-state and stability analyses of the system of differential equations for the joint evolution of firms’ aggregate leverage and the proportion of firms in different financial states. Section 6 concludes the paper and points toward future developments.

2 The model

The economy described in this paper is composed by firms, households and a financial sector. Following Kalecki (1971), firms are assumed to set the price as a mark-up on the unit cost of labor, while holding excess capacity³ The mark-up, and hence the functional distribution of income between wages and profits, will be assumed to (exogenously) depend on the degree of industrial concentration and the relative bargaining power of workers and capitalists in the labor market. The model is demand-driven in nature: output is always below potential, which allows for investment behavior to be determined independently from savings. The degree of capacity utilization of firms, which depends on the quantity they sell in each period, will take the adjusting role for the macroeconomic equilibrium in the goods market.

Firms are divided into two classes and can switch between them. While *hedge* firms pay all their interest out of gross profits, *speculative* firms’ profits are lower than their financial commitments in the current period⁴. The method proposed in Section 5 will analytically identify the relationship between the share of firms in each class and the aggregate leverage dynamics. There is no microfoundation for the household sector, which is treated as an aggregate for simplicity. Their income is composed of wages and a component of profits distributed by firms. Households accumulate their entire stock of wealth in the form of money deposits.

³See Rowthorn (1982), Dutt (1984) and Taylor (1985) for early neo-Kaleckian models in this tradition.

⁴In the Mynskian terminology, *Ponzi* agents are distinguished from *speculative* agents for their income is never sufficient to make interest payments, requiring new borrowing at every period. This distinction will not be made here for simplification purposes.

Finally, the financial sector is also considered as an aggregate: its basic role is to provide loans, hence holding debt (or firm bonds) as an asset and to create money deposits endogenously as liabilities, while setting an exogenous interest rate.

2.1 Firms

A single firm is identified by the superscript j , while its state (speculative or hedge) by the subscript $z = 1, 2$. Firm level variables are indicated by small letters while aggregate while variables are represented by capital letters. Thus when a variable is written as x_1^j , it refers to the firm j belonging to state 1. A lower case variable with only the subscript, x_1 , indicates the *mean-field value*, which, as detailed in section 4 below, is a single value that replaces the vector of observations in each group. Symbols without superscript or subscript refer to aggregate variables. The numbers of firms in each group are indicated by N_1 and N_2 . In a simplification of the Minskyan terminology, we define two classes of firms, depending on the sign of their retained profits a^j , here defined as the difference between gross profits and interest payments:

- **$z = 1$:** *speculative* firms, whose gross profits in the current period are less than sufficient to pay interest on accumulated debt, requiring a new increase in debt.

$$a_t^j < 0; \quad (1)$$

- **$z = 2$:** *hedge* firms, whose profits in the current period are more than sufficient to pay interest on accumulated debt.

$$a_t^j \geq 0 \quad (2)$$

The investment function for the firm j is given by⁵

$$i_{z,t}^j = i_0 K_{z,t-1}^j + \alpha_z \pi p q_{t-1}^j + \beta p q_{t-1}^j - \lambda b_{t-1}^j \quad (3)$$

$$= i_0 K_{z,t-1}^j + (\alpha_z \pi + \beta) p q_{t-1}^j - \lambda b_{t-1}^j \quad (4)$$

where q^j is the quantity sold by the firm, π is the share of profits in output, p is the final good's price, b is the firm's outstanding debt and $i_0, \alpha_z, \beta, \lambda \geq 0$. This specification is the equivalent in levels to the standard neo-Kaleckian specification in which the gross percentage change in capital before depreciation is a positive function of the gross profit rate and the degree of capacity utilization⁶.

The positive effect of the profit rate on investment decisions can be justified based on two different channels: (i) it determines the level of internal finance available for investment, and (ii) it gives the return on productive capital, thus providing an incentive for firms to invest. When it comes to the magnitude of the parameter α , the formulation recalls the one in Delli Gatti et al. (1993), where the sensitivity to internal finance analytically devices the Minskyan borrower's and lender's risk. Following Fazzari et al. (1988) and Delli Gatti et al. (1993),

⁵For computational needs, in the multi-agent simulations we consider a sequential economy that evolves in discrete time, and the model will be presented accordingly.

⁶When technology exhibits fixed coefficients and the labor supply is infinitely elastic, it is possible to define the potential output only as a function of capital so that

$$\bar{q}_t^j = 1/\gamma k_t^j \quad (5)$$

where the inverse of capital productivity γ is a constant parameter, which can be normalized to one for simplicity. The degree of capacity utilization u^j of each firm, defined as the ratio of actual output q^j sold by the firm to potential output \bar{q}^j can thus be approximated to the actual output-to-capital ratio $u = \frac{q}{k}$.

we assume that $\alpha_1 > \alpha_2$, that is speculative firms are more sensitive to internal finance as they face the risk of bankruptcy and change their behavior in order to minimize it.

Investment decisions to expand the available capital stock also respond positively to firms' sales, as higher demand and a higher use of the existing capacity cause firms to expand their desired productive capacity and potential output. Finally, the third term in 3 allows the level of outstanding debt to inhibit investment decisions, as firms face higher balance sheet constraints. Equation (3) thus involves three factors: a micro-effect (b^j), a meso-effect (α_z) and a variable combining micro and macro effects (q^j).

The quantity actually sold by a firm is subject to a *preferential attachment* shock. Total demand $pQ(t)$ is determined as the macroeconomic level from aggregate investment and consumption, but its distribution among firms needs to be identified. We assume that this distribution is partially stochastic. In particular, demand is allocated based on the relative of size of firms (proxied by their stock of capital), to which a stochastic idiosyncratic shock s is added. The expected market share of firm j is hence

$$\mathbb{E}[q_t^j] = Q_t \frac{k_t^j}{K_t} \quad (6)$$

where Q_t is the total demand, given by the consumption of managers and wage earners, and K represents the aggregate capital for the whole economy⁷. Defining \tilde{s} as a uniformly distributed stochastic variable with $\mathbb{E}[\tilde{s}] = 0$, we set

$$s_t^j = \tilde{s}_t^j \left[1 - \frac{k_t^j}{K_t} \right] \quad (7)$$

in order for $\sum^N q^j = Q$. Accordingly the quantity actually sold by the single firm is

$$q_t^j = \mathbb{E}[q_t^j] \left[1 + s_t^j \right] \quad (8)$$

Firms will set prices in oligopolistic markets by applying a mark-up over unit labor costs:

$$p = (1 + \mu) \frac{w}{\xi} \quad (9)$$

where μ is the mark-up rate, w is the nominal wage and $\xi = Q/L$ is labor productivity.

With μ taken as a parameter, the mark-up rate is constant and the labor share of output Ψ is also given exogenously from:

$$\Psi = \frac{w}{p} \frac{1}{\eta} = \frac{1}{1 + \mu} \quad (10)$$

The gross profit share of aggregate output Π will then be given by

$$\Pi = 1 - \Psi = \frac{\mu}{1 + \mu} \quad (11)$$

As previously described, each firms' retained profits are computed as the difference between its gross profits and the net interest it pays on debt, at an exogenous interest rate \bar{i} . Since the flow of gross profits is given by a constant share Π of each firms' output from the mark-up rule, retained profits can be expressed as:

$$a_t^j = \Pi p q_t^j - \bar{i} b_{t-1}^j \quad (12)$$

⁷Hence, larger firms are expected to have a greater market share and, consequently, to grow faster. This implicitly introduces a market competition mechanism with evolutionary features in which the selection process is financial and not technological as in Nelson and Winter (1982).

A firm will be assumed to default if $b^j/k^j \geq c$, where c is set as a positive constant. The probability for a bankrupted firm of being replaced is modeled as directly proportional to the performance of the economy in the previous period.

The increase in the stock of debt in the case of *hedge* firms will only occur when desired investment $i^j > a^j$, which requires firms to seek external finance for capital accumulation. Whenever $a^j > i^j$, firms will (partially or totally) repay their stock of debt and distribute any remaining profits to managers.

Hence, the law of motion of firms' bonds when $a_t^j < i_t^j + b_{t-1}^j$ can be written as:

$$\Delta b^j = i_t^j - a_t^j \quad (13)$$

Whenever $a_t^j \geq i_t^j + b_{t-1}^j$, the previous stock of debt is entirely repaid, so that

$$\Delta b^j = -b_{t-1}^j \quad (14)$$

and the remainder $\Theta_t^j = a_t^j - i_t^j - b_{t-1}^j \geq 0$ is distributed to firms' managers (households), as will be described in subsection 2.2.

2.2 The household sector

As described in the previous subsection, workers earn wages w which add up to a constant share Ψ of total output Q , and managers receive an amount $\Theta = \sum_j \Theta^j$ from firms' excess profits.

Households' total disposable income Y is composed by wage and profit earnings:

$$Y_t = \Psi p Q_t + \Theta_t \quad (15)$$

Households' wealth is accumulated as an effect of savings S in the form of money deposits M ,

$$\Delta W_t = \Delta M_t = S_t \quad (16)$$

where savings S are defined as the difference between households' disposable income and consumption levels $S_t = Y_t - C_t$.

Finally, consumption spending will be assumed to be a fixed proportion of disposable income and wealth, as in standard in SFC models (Godley and Lavoie, 2007). We will further assume that the propensity to consume out of wages is different (higher) than the propensity to consume out of distributed profits $\Theta = \sum_j \Theta^j$. The hypothesis of differential savings among income groups or social classes have been a key feature in neo-Kaleckian models since Kaldor (1955 - 1956) and Kalecki (1971).

$$C_t = (1 - s_\Psi) \Psi p Q_t + (1 - s_\Pi) \Theta_t + (1 - \sigma) W_{t-1} \quad (17)$$

where $0 \leq s_\Psi, s_\Pi, \sigma < 1$ and σ are the propensities to save out of wages, distributed profits and wealth, respectively.

2.3 The financial sector

The financial sector is considered as an aggregate. It gives loans to firms, hence holding bonds as an asset, and creates money deposits endogenously. It charges an exogenous interest rate \bar{i} on bank loans.

Since money deposits are assumed to earn no interest, net interest payments by firms allow the financial intermediary to accumulate net worth Ω_b according to:

$$\Delta \Omega_t^b = \bar{i} B_{t-1} \quad (18)$$

2.4 Goods market equilibrium

As the model is demand-driven, total output Q adjusts to the sum of aggregate consumption C , from equation (17), and investment I :

$$pQ_t = (1 - s_\Psi)\Psi pQ_t + (1 - s_\Pi)\Theta_t + (1 - \sigma)W_{t-1} + I_t$$

As it is not possible to establish analytically how many *hedge* firms fall in the sub-category in which retained profits are more than sufficient to repay the total stock of debt, and thus the amount of profits that are distributed to managers, we will assume hereafter that the propensity to consume out of profits is zero ($s_\Pi = 1$) for simplicity⁸.

Solving for pQ_t and substituting Ψ from expression (10) yields

$$pQ_t = \frac{1 + \mu}{\mu + s_\Psi} [I_t + (1 - \sigma)W_{t-1}] \quad (19)$$

Section 5 will specify the dynamics of the number of firms in each state N_z , as well as the mean-field value for each macro-variable within the sub-population of *speculative* or *hedge* firms. Using equation (3) and following this notation, the aggregate level of investment I_t in (19) can be written as:

$$I_t = N_1[(\alpha_1 \Pi + \beta) pq_{1,t-1} - \lambda b_{1,t-1}] + N_2[(\alpha_2 \Pi + \beta) pq_{2,t-1} - \lambda b_{2,t-1}] \quad (20)$$

3 Numerical simulations

We run numerical simulations of the above described agent-based model in Matlab⁹. The parameter set of the benchmark scenario is presented in Table 2.

Figure 1 displays the simulation results for a single run: a cyclical pattern arises from this simple setup in line with the Minskyan characterization of the business cycle. In other words, the system goes through periods of debt-fueled booms, followed by busts triggered by firms' deleveraging and bankruptcies. During expansions, the share of speculative units increases together with aggregate leverage. At the peak of the cycle, the amount of debt becomes unsustainable and heavily leveraged firms start reducing investment, while financially distressed speculative firms default. The decline in aggregate demand only stops when firms' debt is small enough for investment to recover. The economy does not display any growth trend, given that technology and labor productivity are constant.

Despite the fact that fluctuations are perhaps too regular to be compared with a real economy, the model mimics the empirically observed pro-cyclicality of business debt and profits, as well as a few other regularities. First, Monte Carlo replications of the model yield a realistic value of 4 for the ratio between the variance of investment and the variance of consumption. The autocorrelation coefficients of output and investment are both equal to 0.89 over 50 Monte Carlo replications, being comparable to estimates in the literature for de-trended series. Second, the distribution of variations of output from period to period can be well fitted by a Weibull, in line with the evidence presented in Di Guilmi et al. (2005). Third, the distribution of firms is right-skewed according to different proxies of their size (debt and profits), as also empirically detected.

The sensitivity of the agent-based model to parameter configurations was tested by means of Monte Carlo simulations. The parameters $\alpha_1, \beta, \delta, \mu, \lambda$ and \bar{i} are varied within a given interval and for each value we run 1,000 replications, so as to make sure that results are not affected by the particular random seed of a single run.

⁸In a more complete model, the distribution of profits to managers could be replaced by share buybacks by firms, for instance, and would also not enter the determination of aggregate demand.

⁹The codes are available upon request.

The intervals of variations we use are: $\alpha_1 \in [0.2, 0.45]$, $\beta \in [0.05, 0.35]$, $\delta \in [0.025, 0.15]$, $\mu \in [0.3, 0.7]$, $\lambda \in [0.01, 0.13]$ and $\bar{i} \in [0.05, 0.3]$ ¹⁰.

Results of Monte Carlo simulations for α_1 (figure 2) allow for a numerical assessment of the role of behavioral heterogeneity, considering equation (3) and the fact that α_2 is kept constant at 0.2. Not surprisingly, an increase in α_1 , the sensitivity of speculative firms' investment to internal funds, reduces the degree of financial fragility of the system, as it lowers the share of speculative firms and the debt-to-GDP ratio. However the debt-to-capital ratio is slightly increasing in α_1 for both speculative and hedge firms, and hence in aggregate terms, implying that the rate of accumulation of debt is faster than the accumulation of capital in these scenarios. Similar results are obtained from an increase in β , the sensitivity of investment to demand, but in this case the increase in the aggregate debt-to-capital ratio is more dramatic, due to the rise in speculative firms' leverage (figure 3).

The results for a raise in the interest rate \bar{i} (figure 4) show that a higher rate raises the bankruptcy ratio and, consequently, increases aggregate demand by eliminating the financially constrained firms, while slowing down the accumulation of debt. This result can be explained both through the lower negative component in the investment function (3), due to lower debt in the system, and through the higher share of speculative firms, which have a higher sensitivity to internal resources and therefore are expected to invest more, *ceteris paribus*. At a first sight, the results seem to support a *lean-against-the-wind* type of policy, but only if one neglects the increase in volatility and the higher frequency and magnitude of fluctuations that are generated in the system.

In order to test the role of the firms' demography in determining this result, we run a different set of Monte Carlo simulations in which the maximum debt-to-capital ratio is increased to $c = 25$ and all bankrupted firms are immediately replaced by new ones¹¹. Results are reported in figure 5. In this setting, aggregate debt and the debt-to-capital ratio are increasing in \bar{i} and almost all firms are speculative. Hence, results for aggregate demand and the debt-to-output ratio can thus be seen as consequence of the assumption of a constant number of firms N throughout the simulations.

An increase in λ implies a greater sensitivity of firms' investment to their outstanding debt, or a higher degree of risk-aversion. Debt and debt-to-capital ratios are shown to be lower for higher λ (figure 6), while debt-to-output ratios increase, due the lower level of aggregate demand.

An increase in the profit share of output, as generated by a larger mark-up μ , reduces aggregate demand, aggregate leverage and the proportion of speculative firms in the economy. These firms display a higher average debt-to-output ratio (figure 7) but a lower debt-to-capital ratio, as they invest more due to higher profits, but sell less due to lower aggregate demand. At the aggregate level, however, the increase in profits more than compensates for the decline in aggregate demand, leading to lower aggregate leverage. The lower level of aggregate demand also inhibits investment decisions, partially offsetting the positive effect of higher profitability on capital accumulation. Overall, the debt-to-capital ratio decreases both for speculative firms and at the aggregate level.

Finally, as displayed in figure 8, a higher rate of depreciation of capital has a proportionally larger impact on debt, via bankruptcy, than on capital and demand, causing a reduction in the overall leverage. In contrast, the debt-to-capital ratio of speculative firms is increasing in δ .

¹⁰The other parameters are set as in the benchmark scenario, with the exception of $c = 3$, which smooths the cycle without qualitatively affecting the results.

¹¹The simulations without changes in firms' demography will also permit a consistent comparison with the analytical solution, which does not explicitly consider bankruptcy, in section 5.

4 Firms' dynamics

As illustrated in sections 1 and 2, the analytical solution method adopted by this paper operates through a reduction in the heterogeneity by grouping the agents into clusters. In order to study the stochastic evolution of the system analytically we use a mean-field approximation, which essentially involves reducing the vector of observations of a variable to a single value for a given population (Brock and Durlauf, 2001). In the present treatment, the mean-field approximation is performed by simply taking the average of the relevant micro-variables within each group.

The following step is presented in subsection 4.1 and concerns the definition of the probabilistic rules that determine the switch of firms between groups. The dynamics of the number of agents in each cluster is assumed to follow a Markovian stochastic process. This class of processes can be described by a master equation, namely a stochastic differential equation that depends on the probabilities of transition between states. The main steps of this procedure and the solution of the master equation are presented in subsection 4.2. In particular, this solution can be expressed by an ordinary differential equation plus a stochastic component given by a Wiener process. Both the ordinary differential equation and the noise term are formulated as functions of the micro-variables that determine the transition of agents between groups. This result is then used in section 5 for the derivation of the laws of motion of relevant macroeconomic variables.

As for the notation, since the population of firms is reduced to just two (one average hedge firm and one average borrowing firm), the superscript j is no longer used for firm-level variables.

4.1 Transition probabilities

The probability of a single firm changing state can be derived using (1) or (2). In particular, a hedge firm turns to speculative if its net profit becomes negative according to condition (1), while a speculative firm becomes hedge if it earns a positive net profit, based on condition (2). The magnitude of each firms' net profit depends on the idiosyncratic shock s , which determines the quantity they sell. As the distribution of the shock s is known by assumption, it is convenient to quantify the probabilities of transition as function of s . Let us preliminarily define the variable Γ_z as

$$\Gamma_{z,t} = \left(\frac{\bar{i}_t b_{z,t}}{\Pi p q_{z,t}} - 1 \right) \frac{K_t}{K_t - k_{z,t}} \quad (21)$$

Using equations (1), (2), (8) and (12), it follows that a speculative firm j becomes hedge if

$$s_t^j \geq \Gamma_{1,t} \quad (22)$$

and a hedge firm becomes speculative when

$$s_t^j < \Gamma_{2,t} \quad (23)$$

The first probability will be denoted by η and the second by ζ . Accordingly, we can write

$$\eta_t^j = Pr[s_t \geq \Gamma_{1,t}] \quad (24)$$

$$\zeta_t^j = Pr[s_t < \Gamma_{2,t}] \quad (25)$$

Assuming s to be uniformly distributed in the interval $[-0.5, 0.5]$, we are able to quantify the two probabilities using the cumulative distribution function of s , so that:

$$\eta_t = 0.5 - \Gamma_{1,t} \quad (26)$$

$$\zeta_t = 0.5 + \Gamma_{2,t} \quad (27)$$

4.2 Solution to the master equation: stochastic dynamics of trend and fluctuations

If we assume that the number of firms in each of the two states follows a jump Markov process, we can use the master equation to describe the stochastic evolution of the system. Taking N_1 as our state variable, the master equation allows us to represent the dynamics of the probability of a certain number of firms N_1 being in state 1 as the balance equation between the aggregate transition to and from state 1. Considering an interval of time Δt small enough to be approximated by a continuous time representation, we can express the master equation for the dynamics of firms in state 1 as

$$\frac{dP(n_1, t)}{dt} = \eta(t)P(n_1 - 1/N)(t) + \zeta(t)P(n_1 + 1/N)(t) + [\eta(t) + \zeta(t)]P(n_1)(t) \quad (28)$$

where $n_1 = N_1/N$.

Equation (28) describes the dynamics of the probability of having a fraction n_1 of speculative firm as the sum of the probability of transitioning from a number $N_1 - 1$ to N_1 and the probability of transitioning from a number $N_1 + 1$ to N_1 , less the probability of already having a number N_1 of speculative firm times the probability of a transition into or from the speculative state. The master equation can be solved using approximation techniques. The solution algorithm introduced by Aoki (2002) and further developed by Landini and Uberti (2008) and Di Guilmi (2008) has the advantage of returning a solution that is composed by two equations: an ordinary differential equation, which describes the time evolution of the trend of the stochastic process and a partial differential equation (known as Fokker-Planck equation), whose general solution identifies the probability distribution of the fluctuations around the drift component¹². For this algorithm to be applied, the state variable must be split into trend and cyclical components. Following Aoki (2002) and indicating, respectively, with ϕ the deterministic trend and v the deviation, the state variable can be reformulated as

$$n_1 = \phi + N^{-1/2}v \quad (29)$$

so as to normalize the standard deviation v . The development of the solution algorithm yields the following ODE for the trend

$$\dot{\phi} = \zeta\phi - (\eta + \zeta)\phi^2 \quad (30)$$

The stationary solution of the second equation presumes a Gaussian distribution for the stochastic noise around the trend, so that:

$$P(v) = C \exp\left(-\frac{v^2}{2\sigma^2}\right) \quad (31)$$

such that $P(v) \sim \mathcal{N}\left(0, \frac{\eta\zeta}{(\eta+\zeta)^2}\right)$ The dynamics of n_1 can thus be described by

$$\dot{n}_1(t) = \zeta\phi - (\eta + \zeta)\phi^2 + \sigma dV(t) \quad (32)$$

where dV is a stationary Wiener increment and σdV is the stochastic fluctuation component in the proportion of speculative firms, coming from the distribution (31).

The solution of the master equation provides a good approximation for the underlying agent-based model despite the reduction in heterogeneity. Figure 9 shows the results for the share of speculative firms, obtained for the master equation solution by using the results of the simulations of the agent based model in equation (32), and the level of aggregate debt obtained from the product of the mean-field values of firms' debt b_1 and b_n and the corresponding number of firms. The proportion n_1 is weighted by the number of surviving firms, so as

¹²The full derivation of the solution is omitted here. We refer the interested reader to Di Guilmi (2008) and Chiarella and Di Guilmi (2011).

to take into account the effect of bankruptcy¹³. Not considering this effect, the two series move together during expansions while, during recessions, the master equation solution n_1 does not drop until the cycle is very close to its trough. Having tested the reliability of the analytical solution, we can now use it to gain further insights into the behavior of the model.

5 Analytical solution

This section presents an analytical assessment of the links between debt dynamics at the micro and macro levels. In particular, it highlights the role of firms' heterogeneity for the dynamics of aggregate debt and leverage.

We will thus start by studying separately the evolution of the stock of debt held by *speculative* and *hedge* firms, before proceeding to a macro analysis. The results of the analysis are also compared with the outcomes of the Monte Carlo simulations, yielding further insights on the causal relationships that determine the dynamics of the model.

5.1 Debt dynamics

In this subsection we focus on the joint dynamics of b_1, b_2 and N_1 , looking first at the steady state values and then at the conditions for stability.

5.1.1 Steady-state analysis

Proposition 1. *At the steady-state where $\dot{b}_1 = \dot{b}_2 = 0$, the mean-field debt-to-output ratio of speculative firms is higher than that of hedge firms. Formally: $\frac{\tilde{b}_1}{pq_1} > \frac{\tilde{b}_2}{pq_2}$.*

Proof.

Using the law of motion (13) and substituting for i and a from (3) and (12), the evolution of the mean-field level of debt for speculative firms is given by

$$\dot{b}_1(t) = [(\alpha_1 - 1)\Pi + \beta]pq_1(t) - (\lambda - \bar{i})b_1(t) \quad (33)$$

The steady-state value for b_1 is:

$$\tilde{b}_1 = \frac{[(\alpha_1 - 1)\Pi + \beta]pq_1}{\lambda - \bar{i}} \quad (34)$$

At the steady-state, the debt-to-output ratio for speculative firms is thus:

$$\left(\frac{\tilde{b}_1}{pq_1} \right) = \frac{(\alpha_1 - 1)\Pi + \beta}{\lambda - \bar{i}} \quad (35)$$

The debt-to-output ratio is then the higher, the higher the responses of investment to profits α_1 and output β , the profit share Π itself and the interest rate \bar{i} , and the lower is the sensitivity of investment to outstanding debt λ .

¹³Di Guilmi et al. (2012) proposes and uses a solution method for the master equation for a system with a variable number of agents. Given the considerable analytical complexity of that treatment and the scope of the present paper we adopted the simplification of considering N fixed and consider this difference in the comparison of the numerical and analytical treatment.

Analogously, for hedge firms, the steady-state value for the debt-to-output ratio is given by¹⁴:

$$\left(\frac{\tilde{b}_2}{pq_2}\right) = \frac{(\alpha_2 - 1)\Pi + \beta}{\lambda - \bar{i}} \quad (36)$$

Since we have considered that speculative firms are more sensitive than hedge firms to their profit rate when deciding to invest, namely that $\alpha_1 > \alpha_2$, the equilibrium debt-to-output ratio is also higher for the former category. ■

Proposition 2. *At the equilibrium where $\dot{b}_1 = \dot{b}_2 = \dot{N}_1 = 0$, which constitutes a steady-state for the aggregate level of debt B , the number of speculative firms \tilde{N}_1 is the higher, the lower is α_1 , the sensitivity of speculative firms' investment to profits. The direction of the response of \tilde{N}_1 to parameters \bar{i} , α_2 , β , λ and Π depends on the specific parametric configuration.*

Proof.

Using the law of motion for the number of speculative firms¹⁵ (32) and substituting for η and ζ using expressions (26), (27) and (21), we get:

$$\dot{N}_1(t) = N \left\{ \left(-0.5 + \frac{\bar{i}}{\Pi} \frac{b_2(t)}{pq_2(t)} \right) N_1(t) - \left[1 + \frac{\bar{i}}{\Pi} \left(\frac{b_2(t)}{pq_2(t)} + \frac{b_1(t)}{pq_1(t)} \right) \right] N_1(t)^2 \right\} \quad (37)$$

The number of speculative firms is constant when $\dot{N}_1 = 0$, which yields the steady-state

$$\tilde{N}_1 = N \frac{-0.5 + \frac{\bar{i}}{\Pi} \frac{b_2}{pq_2}}{1 + \frac{\bar{i}}{\Pi} \left(\frac{b_2}{pq_2} + \frac{b_1}{pq_1} \right)} \quad (38)$$

The number for speculative firms hence depends ambiguously on the interest rate on loans and the debt-to-output ratios of hedge firms, while it decreases with the level of the debt-to-output ratio of speculative firms.

As the total number of firms N is fixed, the steady-state value for the number of hedge firms is $\tilde{N}_2 = N - \tilde{N}_1$. Hence, in the equilibrium where N_1 , b_1 and b_2 are constant, the total stock of firms' debt B is also constant¹⁶.

The equilibrium value of the number of speculative firms \tilde{N}_1 at this steady-state can be obtained by substituting (35) and (36) in (38), so that:

$$\tilde{N}_1 = N \frac{-\frac{\lambda}{2} + \bar{i}(\alpha_2 + \frac{\beta}{\Pi} - \frac{1}{2})}{\lambda + \bar{i}(\alpha_2 + \alpha_1 + 2\frac{\beta}{\Pi} - 3)} \quad (39)$$

Proposition 3. *At the equilibrium where $\dot{b}_1 = \dot{b}_2 = \dot{N}_1 = 0$, the steady-state value of aggregate debt \tilde{B} is the higher, the higher is, ceteris paribus, the degree of heterogeneity between the two subgroups of firms as for the sensitivity of investment to profits ($\alpha_1 - \alpha_2$), while, for $\alpha_1 = \alpha_2$, it is unaffected by the difference in the magnitude of output, or firms' size, ($q_1 - q_2$).*

¹⁴The correspondence between the two cases only holds because the steady-state value for hedge firms for which debt evolved according to the alternative law of motion (14) is zero.

¹⁵The stochastic noise is not considered in the present analysis since its expected value is null. The term $\frac{K_t}{K_t - k_{z,t}}$ that appears in equation (21) can be considered equal to 1 for a large number of firms N and can therefore be omitted.

¹⁶This is only a sufficient, not a necessary condition, as $\dot{B} = N_1\dot{b}_1 + \dot{N}_1 b_1 + N_2\dot{b}_2 + \dot{N}_2 b_2$.

Proof.

$$\begin{aligned}\tilde{B} &= \tilde{N}_1 \tilde{b}_1 + (N - \tilde{N}_1) \tilde{b}_2 \\ &= \tilde{N}_1 (\tilde{b}_1 - \tilde{b}_2) + N \tilde{b}_2\end{aligned}$$

By substituting \tilde{b}_1 and \tilde{b}_2 from (34) and (36), respectively:

$$\tilde{B} = \frac{\Pi p}{\lambda - \bar{i}} [\tilde{N}_1 [(\alpha_1 q_1 - \alpha_2 q_2) + (\beta/\Pi - 1)(q_1 - q_2)] + N(\alpha_2 - 1 + \beta/\Pi)q_2] \quad (40)$$

where \tilde{N}_1 is only as a function of parameters, as expressed in (39).

From (40), the steady-state value of aggregate debt in the special case of full homogeneity between subgroups of firms in both (i) the sensitivity of investment to profits ($\alpha = \alpha_1 = \alpha_2$), and (ii) firms' size ($k = k_1 = k_2$), which from (7) and (8) is equivalent to assuming the same output $q = q_1 = q_2$, would be given by:

$$\tilde{B}(\alpha, q) = \frac{[(\alpha - 1)\Pi + \beta]pQ}{\lambda - \bar{i}}$$

Now, assuming that firms have the same sensitivity of investment to profits ($\alpha = \alpha_1 = \alpha_2$), but are heterogeneous in size ($q_1 \neq q_2$), we get:

$$\begin{aligned}\tilde{B}(\alpha, q_1, q_2) &= \frac{N[(\alpha - 1)\Pi + \beta]pq_2}{\lambda - \bar{i}} + \frac{\tilde{N}_1[(\alpha - 1)\Pi + \beta]p(q_1 - q_2)}{\lambda - \bar{i}} \\ &= \frac{(\alpha - 1)\Pi + \beta}{\lambda - \bar{i}} [Npq_2 + \tilde{N}_1p(q_1 - q_2)] \\ &= \frac{[(\alpha - 1)\Pi + \beta]pQ}{\lambda - \bar{i}} = \tilde{B}(\alpha, q)\end{aligned}$$

Finally, assuming that firms have the same size ($q = q_1 = q_2$) but are heterogeneous in the sensitivity of investment to profits ($\alpha_1 \neq \alpha_2$),

$$\begin{aligned}\tilde{B}(\alpha_1, \alpha_2, q) &= \frac{N[(\alpha_2 - 1)\Pi + \beta]pq}{\lambda - \bar{i}} + \frac{\tilde{N}_1\Pi pq(\alpha_1 - \alpha_2)}{\lambda - \bar{i}} \\ &= \frac{[(\alpha_2 - 1)\Pi + \beta]pQ}{\lambda - \bar{i}} + \frac{\tilde{N}_1\Pi pq(\alpha_1 - \alpha_2)}{\lambda - \bar{i}} \\ &= \tilde{B}(\alpha, q) + \frac{\tilde{N}_1\Pi pq(\alpha_1 - \alpha_2)}{\lambda - \bar{i}}\end{aligned}$$

It is thus clear that the higher is heterogeneity in investment behavior, namely the higher is α_1 relative to α_2 , the higher is aggregate debt. ■

The results of the steady state analysis generally match the outcomes of numerical simulations previously described¹⁷. In fact, in all the simulations $\frac{b_1}{pq_1} > \frac{b_2}{pq_2}$ as for proposition 1. The analytical approximation of the agent-based model reveals that this regularity is emerging due to behavioral heterogeneity ($\alpha_1 > \alpha_2$). Proposition 2 proves that the inverse correlation between α_1 and N_1 showed in figure 2 is independent of the parameter

¹⁷In the present analysis, we consider the outcome of the Monte Carlo simulations as an approximation of the steady state.

configuration, while the patterns that emerge in the Monte Carlo simulations for \bar{i} , β and λ have relied on a particular parameter setting. Even with a number of different Monte Carlo simulations using different parameter settings, the full reliance on numerical solutions would not be sufficient to provide this kind of certainty. Finally, given that α_2 is constant in the Monte Carlo simulations, proposition 3 illustrates the causal relationship through which the model generates the direct correlation between α_1 and aggregate debt displayed by figure 2. The second statement of proposition 3 stresses the role of behavioral heterogeneity relative to size heterogeneity for the outcome of the simulations. Again, this is a result that could not be achieved with the sole numerical analysis, given the impossibility of forcing size homogeneity across firms.

Summing up, the level of heterogeneity in the elasticity of the investment function to internal finance appears to be crucial for the long run dynamics of the economy. For a given level of α_2 , a larger α_1 implies not only a larger difference in the leverage ratio of the two types of firms but also a greater stock of aggregate debt. Speculative firms invest more and, consequently, resort to credit. At the same time, a larger α_1 reduces the steady-state level of the number of speculative firms. Given that $B = N_1 b_1 + N_2 b_2$, the effect is larger on b_1 than on N_1 . The decrease in N_1 can be explained from the fact that speculative firms rely more on internal finance for capital accumulation, which also raises their chances of becoming hedge.

5.1.2 Stability analysis

In this subsection we will focus on the joint stability of b_1 and N_1 around the steady-state, given the more crucial role of speculative firms' leverage for the financial stability of the system as a whole, from a Minskyan standpoint. A stability analysis for hedge firms, or b_2 and N_2 , can be derived in similar fashion. This study provides additional qualitative insights on the results of the Monte Carlo simulations.

The analysis will be based on the system of differential equations composed by expressions (37) and (33).

Proposition 4. *The interest rate \bar{i} has an ambiguous effect on the stability of the dynamical system.*

The uncertainty about the effect of the interest rate is a consequence of the different role it has on the dynamics of the number of speculative firms and on their individual level of debt. On the one hand, a higher interest rate increases the number of speculative firms through its effect profits in equation (12). Due to the logistic functional form of the dynamics of speculative firms (32), as the level of N_1 increases, its rate of growth decelerates, such that $\frac{\partial N_1}{\partial N_1} < 0$. On the other hand, the interest rate positively affects b_1 , through its negative effect on retained profits. As a consequence $\frac{\partial b_1}{\partial b_1}$ responds positively to \bar{i} . The two effects are embodied in the expression for the trace of the system's Jacobian matrix, as analyzed below.

Proof.

The Jacobian of the system composed by expressions (37) and (33) can be derived from the following partial derivatives:

$$\frac{\partial \dot{N}_1}{\partial N_1} = \left(-0.5 + \frac{\bar{i} b_2}{\Pi p q_2} \right) - 2N_1 \left[1 + \frac{\bar{i}}{\Pi} \left(\frac{b_2}{p q_2} + \frac{b_1}{p q_1} \right) \right] \quad (41)$$

$$\frac{\partial \dot{N}_1}{\partial b_1} = \frac{\bar{i}}{\Pi p q_1} N_1^2 \quad (42)$$

$$\frac{\partial \dot{b}_1}{\partial b_1} = -(\lambda - \bar{i}) \quad (43)$$

$$\frac{\partial \dot{b}_1}{\partial N_1} = 0 \quad (44)$$

From (41), we can evaluate the Jacobian of the system (\dot{N}_1, \dot{b}_1) at the steady-state, by substituting from (35) and (39), which yields

$$J_{\bar{N}_1, \bar{b}_1} = \begin{bmatrix} 0.5 - \frac{\bar{i}}{\Pi} \frac{b_2}{pq_2} & \frac{\bar{i}}{\Pi pq_1} \left[\frac{-0.5 + \frac{\bar{i}}{\Pi} \frac{b_2}{pq_2}}{1 + \frac{\bar{i}}{\Pi} \left(\frac{b_2}{pq_2} + \frac{(\alpha_1 - 1)\Pi + \beta}{\lambda - \bar{i}} \right)} \right]^2 \\ -(\lambda - \bar{i}) & 0 \end{bmatrix}$$

The trace of J is thus given by:

$$Tr(J) = 0.5 - \frac{\bar{i}}{\Pi} \frac{b_2}{pq_2} - (\lambda - \bar{i})$$

By further substituting for the steady-state of b_2 from (36), we have that in the vicinity of the steady-state for total debt B analyzed in the previous subsection, the trace is:

$$Tr(J) = 0.5 - \frac{\bar{i}}{\Pi} \frac{(\alpha_2 - 1)\Pi + \beta}{\lambda - \bar{i}} - (\lambda - \bar{i})$$

The system is hence unstable whenever the following expression holds:

$$\frac{\bar{i}}{\Pi} \frac{(\alpha_2 - 1)\Pi + \beta}{\lambda - \bar{i}} + (\lambda - \bar{i}) > 0.5 \quad (45)$$

Debt instability for speculative firms is thus more likely when the profit share Π is high, and the response of hedge firms' investment to profits and demand, α_2 and β , respectively, are low. The effect of the interest rate on debt stability depends on parameter configurations. ■

Proposition 5. *Given the constraints on the parameters, the steady states for \dot{b}_1 and \dot{N}_1 are most likely on a saddle path.*

Proof. After a few algebraic simplifications, the determinant of J can be written as:

$$Det(J) = \bar{i} \left(\alpha_2 - 1 + \frac{\beta}{\Pi} \right) - \frac{(\lambda - \bar{i})}{2\Pi}$$

The case of saddle-path (in)stability is thus obtained whenever the interest rate is sufficiently low¹⁸ and the term in the right and side of expression (46), which increases with the sensitivity of investment to debt λ , and decreases with the profit share Π , the sensitivity of *hedge* firms' investment to profits α_2 , and the response of investment to output, β .

$$\bar{i} < \frac{\lambda}{2\Pi(\alpha_2 - 1) + 2\beta + 1} \quad (46)$$

We numerically assess the effect of the interest rate on the stability of the system for the following values and ranges of parameters: $\alpha_2 = 0.2$, $\beta = 0.5$, $\lambda \in [0.15, 0.35]$, $\Pi \in [0.3, 0.5]$, $\bar{i} \in [0.05, 0.1]$. The stabilizing effect coming from a lower \dot{N}_1 appears to dominate the destabilizing effect of an increase in the average debt of speculative firms, such that a larger interest rate makes a stable trajectory more likely. In particular, as shown by figure 10, for $\lambda = 0.15$, $\bar{i} > 0.8$, $\Pi = 0.5$ the determinant becomes positive. A lower λ increases N_1 , thus reducing \dot{N}_1 , due to the logistic functional form of (32). Figure 11 illustrates that the determinant is also more likely to be positive for low values of Π . In particular we verified it is always positive for $\lambda = 0.35$, $\bar{i} > 0.5$, $\Pi = 0.3$. Thus, a higher profit share negatively affects the stability of the system \dot{N}_1, \dot{b}_1 . ■

¹⁸In the benchmark scenario of the simulations the interest rate is in the stability range, not satisfying conditions (45) and (46).

Propositions 4 and 5 complete the study of the sensitivity of the system to parameters, identifying ranges of stability. Again, a numerical approach could only lead to this type of interpretation through multiple attempts, by reporting for example the values of the parameters for which the simulation crashes or returns inconsistent results, without any possibility of generalization.

5.2 Dynamics of capital and the leverage ratio

With $0 < \delta < 1$ as the rate of capital depreciation and investment from (20), the law of motion for speculative firm's capital stock \dot{k}_1 , in value, can be expressed as:

$$p\dot{k}_1(t) = (\alpha_1\Pi + \beta)pq_1(t) - \lambda b_1(t) - p\delta k_1(t) \quad (47)$$

Consequently, k_1 reaches the steady state at:

$$p\tilde{k}_1 = \frac{(\alpha_1\Pi + \beta)pq_1 - \lambda b_1}{\delta} \quad (48)$$

Proposition 6. *At the steady-state for both b_1 and k_1 , the leverage ratio for the speculative firm, $\tilde{b}_1/(p\tilde{k}_1)$, is the higher, (i) the higher is the interest rate \bar{i} , (ii) the higher is the depreciation rate δ , (iii) the lower is the sensitivity of investment to leverage λ , (iv) the higher is the sensitivity of investment to profits α_1 , (v) the higher is the sensitivity of investment to output β , and (vi) the lower is the profit share Π .*

Proof

By expressing pq_1 as a function of \tilde{b}_1 in (34) and substituting in (48), we obtain the steady-state leverage ratio for the *speculative* firm:

$$\frac{\tilde{b}_1}{p\tilde{k}_1} = \frac{\delta(\alpha_1 + \beta/\Pi - 1)}{\lambda - \bar{i}(\alpha_1 + \beta/\Pi)} \quad (49)$$

Proposition 6 also expresses some of the results of the Monte Carlo simulation, and demonstrates that these outcomes do not rely on specific parameter configurations. Given that the analytical solution does not consider bankruptcy of firms, the relevant plot for this comparison is figure 5. ■

Proposition 7. *At the steady-state for b_1 , b_2 , k_1 and k_2 , the leverage ratio of speculative firms is higher than that of hedge firms. Formally:*

$$\frac{\tilde{b}_1}{p\tilde{k}_1} > \frac{\tilde{b}_2}{p\tilde{k}_2}$$

Proof

Analogously, the steady-state leverage ratio for *hedge* firms is given by:

$$\frac{\tilde{b}_2}{p\tilde{k}_2} = \frac{\delta(\alpha_2 + \beta/\Pi - 1)}{\lambda - \bar{i}(\alpha_2 + \beta/\Pi)} \quad (50)$$

Proposition 7 hence follows from comparing equation (49) to (50), on the assumption that $\alpha_1 > \alpha_2$. ■

Proposition 8. *At the steady-state where $\dot{k}_1 = \dot{k}_2 = \dot{b}_1 = \dot{b}_2 = \dot{N}_1 = 0$, the steady-state level of aggregate capital stock \tilde{K} decreases with firms' heterogeneity in the sensitivity of investment to profits ($\alpha_1 - \alpha_2$), but is unaffected by the heterogeneity in firms' size ($q_1 - q_2$).*

Proof.

The aggregate level of capital K reaches a steady-state level \tilde{K} given by:

$$\tilde{K} = \tilde{N}_1(\tilde{k}_1 - \tilde{k}_2) + N\tilde{k}_2 \quad (51)$$

By substituting \tilde{k}_1 from (48) and \tilde{b}_1 from (34), and \tilde{k}_2 and \tilde{b}_2 from the analogous expressions for *hedge* firms, we obtain:

$$\tilde{K} = \frac{\Pi}{\delta(\lambda - \bar{i})} \left\{ \tilde{N}_1 \left[\left(\lambda - \bar{i} \frac{\beta}{\Pi} \right) (q_1 - q_2) - \bar{i}(\alpha_1 q_1 - \alpha_2 q_2) \right] + N \left[\lambda - \bar{i} \left(\alpha_2 + \frac{\beta}{\Pi} \right) \right] q_2 \right\} \quad (52)$$

- *Full homogeneity:* if $q = q_1 = q_2$ and $\alpha = \alpha_1 = \alpha_2$.

The first term in (52) vanishes, and

$$\tilde{K}(\alpha, q) = \frac{\Pi}{\delta(\lambda - \bar{i})} \left[\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) \right] Nq = \frac{\Pi}{\delta(\lambda - \bar{i})} \left[\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) \right] Q$$

Hence, in the scenario of full homogeneity, the aggregate output-to-capital ratio $U = Q/\tilde{K}$, which as previously mentioned can be interpreted as a proxy for the firms' degree of capacity utilization, would respond positively to the sensitivity of investment to both profits (α) and output (β).

- *Homogeneity in investment behavior:* if $q_1 \neq q_2$ and $\alpha = \alpha_1 = \alpha_2$.

$$\begin{aligned} \tilde{K}(\alpha, q_1, q_2) &= \frac{\Pi}{\delta(\lambda - \bar{i})} \left[\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) \right] [\tilde{N}_1(q_1 - q_2) + Nq_2] \\ &= \frac{\Pi}{\delta(\lambda - \bar{i})} \left[\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) \right] Q = \tilde{K}(\alpha, q) \end{aligned}$$

- *Homogeneity in investment size:* if $q = q_1 = q_2$ and $\alpha_1 \neq \alpha_2$.

$$\tilde{K}(\alpha_1, \alpha_2, q) = \frac{\Pi}{\delta(\lambda - \bar{i})} \left\{ \tilde{N}_1 [-\bar{i}q(\alpha_1 - \alpha_2)] + \left[\lambda - \bar{i} \left(\alpha_2 + \frac{\beta}{\Pi} \right) \right] Q \right\}$$

Proposition 8 integrates the results of the Monte Carlo simulations by separately studying the role of heterogeneity in, respectively, behavior and size, and confirm the stronger relevance of the former in shaping the long-run dynamics of the system. Interestingly, results for $\alpha_1 - \alpha_2$ are not consistent with the numerical simulations, which implies that the result of proposition 8 cannot be generalized to the case in which heterogeneity in size is present. From this perspective the Monte Carlo simulations and the analytical solution can be seen as complementary for the understanding of the role of heterogeneity. ■

Proposition 9. *At the steady-state where $\dot{k}_1 = \dot{k}_2 = \dot{b}_1 = \dot{b}_2 = \dot{N}_1 = 0$, the aggregate leverage ratio $\tilde{B}/(p\tilde{K})$ is the higher, (i) the higher is the interest rate \bar{i} , (ii) the lower is the response of investment to debt λ , and (iii) the higher is the degree of heterogeneity in the response of investment to profits $\alpha_1 - \alpha_2$, while it is unaffected by the heterogeneity in firms' size $q_1 - q_2$.*

Proof

Proposition 9 is a corollary of Propositions 3 and 8.

From (40) and (52), the equilibrium aggregate leverage ratio is given by:

$$\frac{\tilde{B}}{p\tilde{K}} = \frac{\delta [\tilde{N}_1[(\alpha_1 q_1 - \alpha_2 q_2) + (\beta/\Pi - 1)(q_1 - q_2)] + N(\alpha_2 - 1 + \beta/\Pi)q_2]}{\tilde{N}_1 \left[\left(\lambda - \bar{i} \frac{\beta}{\Pi} \right) (q_1 - q_2) - \bar{i}(\alpha_1 q_1 - \alpha_2 q_2) \right] + N \left[\lambda - \bar{i} \left(\alpha_2 + \frac{\beta}{\Pi} \right) \right] q_2}$$

- *Full homogeneity:* if $q = q_1 = q_2$ and $\alpha = \alpha_1 = \alpha_2$.

$$\frac{\tilde{B}}{p\tilde{K}}(\alpha, q) = \frac{\delta \left[\left(\alpha + \frac{\beta}{\Pi} - 1 \right) \right]}{\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right)}$$

- *Homogeneity in investment behavior:* if $q_1 \neq q_2$ and $\alpha = \alpha_1 = \alpha_2$.

$$\begin{aligned} \frac{\tilde{B}}{p\tilde{K}}(\alpha, q_1, q_2) &= \frac{\delta \left[\tilde{N}_1 \left(\alpha + \frac{\beta}{\Pi} - 1 \right) (q_1 - q_2) + N \left(\alpha - 1 + \frac{\beta}{\Pi} \right) q_2 \right]}{\tilde{N}_1 \left[\left(\lambda - \bar{i} \frac{\beta}{\Pi} - \bar{i} \alpha \right) (q_1 - q_2) \right] + N \left[\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) \right] q_2} \\ &= \frac{\delta \left[\left(\alpha + \frac{\beta}{\Pi} - 1 \right) [\tilde{N}_1 (q_1 - q_2) + N q_2] \right]}{\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right) [\tilde{N}_1 (q_1 - q_2) + N q_2]} \\ &= \frac{\delta \left[\left(\alpha + \frac{\beta}{\Pi} - 1 \right) \right]}{\lambda - \bar{i} \left(\alpha + \frac{\beta}{\Pi} \right)} = \frac{\tilde{B}}{p\tilde{K}}(\alpha, q) \end{aligned}$$

- *Homogeneity in size:* if $q = q_1 = q_2$ and $\alpha_1 \neq \alpha_2$.

$$\frac{\tilde{B}}{p\tilde{K}}(\alpha_1, \alpha_2, q) = \frac{\delta \left[\tilde{N}_1 q (\alpha_1 - \alpha_2) + \left(\alpha_2 + \frac{\beta}{\Pi} - 1 \right) Q \right]}{\tilde{N}_1 [-\bar{i} q (\alpha_1 - \alpha_2)] + \left[\lambda - \bar{i} \left(\alpha_2 + \frac{\beta}{\Pi} - 1 \right) \right] Q}$$

Hence, the higher is $(\alpha_1 - \alpha_2)$, the higher is the numerator and the lower is the denominator of the expression above. ■

Given α_2 , a higher α_1 reduces the accumulation of capital and, consistently with the results of proposition 3, increases the aggregate leverage ratio. This is a consequence of the larger reliance of capital accumulation on the less financially sound firms. Proposition 9 confirms the results of the Monte Carlo simulations for α_1 and \bar{i} , which do not depend on the parameter configuration. The analysis points to the role of behavioral

heterogeneity, and in particular to the sensitivity of investment to internal finance and debt, as the key drivers of the system along a more sustainable long-run path. The statements of proposition 9 are consistent with figures 2, 4 and 6, thus suggesting that point (iii) of the proposition holds also with size heterogeneity.

The results of this subsection, when taken together with the stability analysis of the system, help illuminate the implications of the Monte Carlo simulations for the role of the interest rate. It becomes clear that the handling of the interest rate \bar{i} can have perverse effects on the system dynamics by accelerating the accumulation of debt and, over a certain threshold, can lead the system to instability. Consequently, a lean-against-the-wind policy can add to the fragility of the system and does not represent an ideal tool to halt a speculative boom.

6 Concluding remarks

This paper introduces microeconomic foundations into a demand-driven macroeconomic model in order to study the role of real-financial linkages in generating economic growth and business cycles endogenously. The paper shows that the analysis of systemic financial fragility is greatly enriched by the introduction of heterogeneous microeconomic behavior, in line with Minsky's approach. To this aim, the model was initially built in a bottom-up fashion and solved numerically with full heterogeneity of agents. An aggregation method imported from statistical mechanics then helped us reduce the degree of heterogeneity between firms to only two classes – inspired by Minsky's terminology of hedge and speculative financial behavior – thus allowing for the study of the relationship between micro behavioral rules and macroeconomic dynamics of firms' leverage and capital.

The analytical study has added to the results of the numerical simulations in at least four different ways. First, it demonstrates which results from the Monte Carlo simulations can be generalized, and which ones rely on particular parametric configurations. Second, it illuminates the causal relationships that lead to the observed effects of changing parameters on the macro variables in the simulations. Third, it allowed for a systematic study of stability conditions that cannot be performed in non-trivial agent-based models. Fourth, it permits experiments that are not possible in simple numerical simulations, such as examining the role of heterogeneity of agents along a specific dimension, while keeping other dimensions homogeneous. The analytical representation of the stability conditions and causal chain within the model can also assist in the calibration the numerical simulations, by identifying parameter ranges in which the model is unstable or yields inconsistent results.

With reference to the specific results of this model, the analysis shows that a higher difference in the response of firms' investment to internal finance between speculative and hedge firms pushes the system along a long-run path characterized by higher leverage and lower capital accumulation. In fact, a larger difference $\alpha_1 - \alpha_2$ implies that growth relies more on speculative firms and therefore on borrowing. In terms of policy indications, the analysis shows that a higher interest rate on loans is likely to have a stabilizing role, but at the price of a higher share of speculative firms with a high leverage ratio in the economy. It is worth noting that a higher sensitivity of speculative firms to internal finance leads, *ceteris paribus*, to a larger accumulation of debt at micro and macro-level. The increase in external financing appears to offset the benefit of a more conservative attitude of speculative firms. In terms of income distribution, a larger share of profit has a potentially destabilizing effect on the dynamics of the system.

The integration of Monte Carlo simulations with the analysis of the dynamical system derived from this analytical approximation of the agent-based model represents an absolute primer in the literature, and we hope that it will pave the way for a different modeling approach. While at this stage of developments, numerical simulations are still essential for a complete analysis of model outcomes, especially due to nonlinearities in the system, the proposed analytical solution provides a benchmark for interpreting and generalizing these results. Besides the introduction of a multi-state master equation, which can provide a more accurate approximation of the underlying agent-based model, a possible further development concerns a more refined study of the

conditions under which bubbles and busts are generated in the present setting. When it comes to adding more realistic features to the model setup, possible extensions include, for instance, incorporating a stock market, endogenizing the mark-up rate and the functional distribution of income, as well as allowing for heterogeneity in the household and/or financial sector.

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	Households	Firms		Banks	Total
		Current	Capital		
Consumption	$-C$	$+C$			0
Investment		$+I$	$-N_1[(\alpha_1 \pi + \beta)pq_1 - \lambda b_1] - N_2[(\alpha_2 \pi + \beta)pq_2 - \lambda b_2]$		0
Wages	$+\Psi pQ$	$-\Psi pQ$			0
Profits	$\Pi pQ - (1+r)B$	$-(N_1 a_1 + N_2 a_2)$			0
Distributed profits	$A - B$		$B - A$		0
Loan interests		$-rB$		rB	0
Change in loans			$N_1 [i_1 - a_1] + N_2 [i_2 - a_2]$	$-\dot{B}$	0
Total	0	0	0	0	0

Table 1: Matrix of flows.

Symbol	Value	Description
c	6	bankruptcy parameter
ξ	1	output-labour ratio
w	1	salary
μ	0.6	price mark-up
α_1	0.25	speculative firms' parameter for rate of profit in the investment function
α_2	0.2	hedge parameter for rate of profit in the investment function
β	0.16	elasticity of investment to past demand
λ	0.05	elasticity of investment to debt
s_Ψ	0.5	propensity to save out of salary
s_π	1	propensity to save out of profit
σ	1	propensity to save out of wealth
\bar{i}	0.05	interest rate
δ	0.1	capital depreciation
i_0	0.05	autonomous investment

Table 2: Parameters value and description.

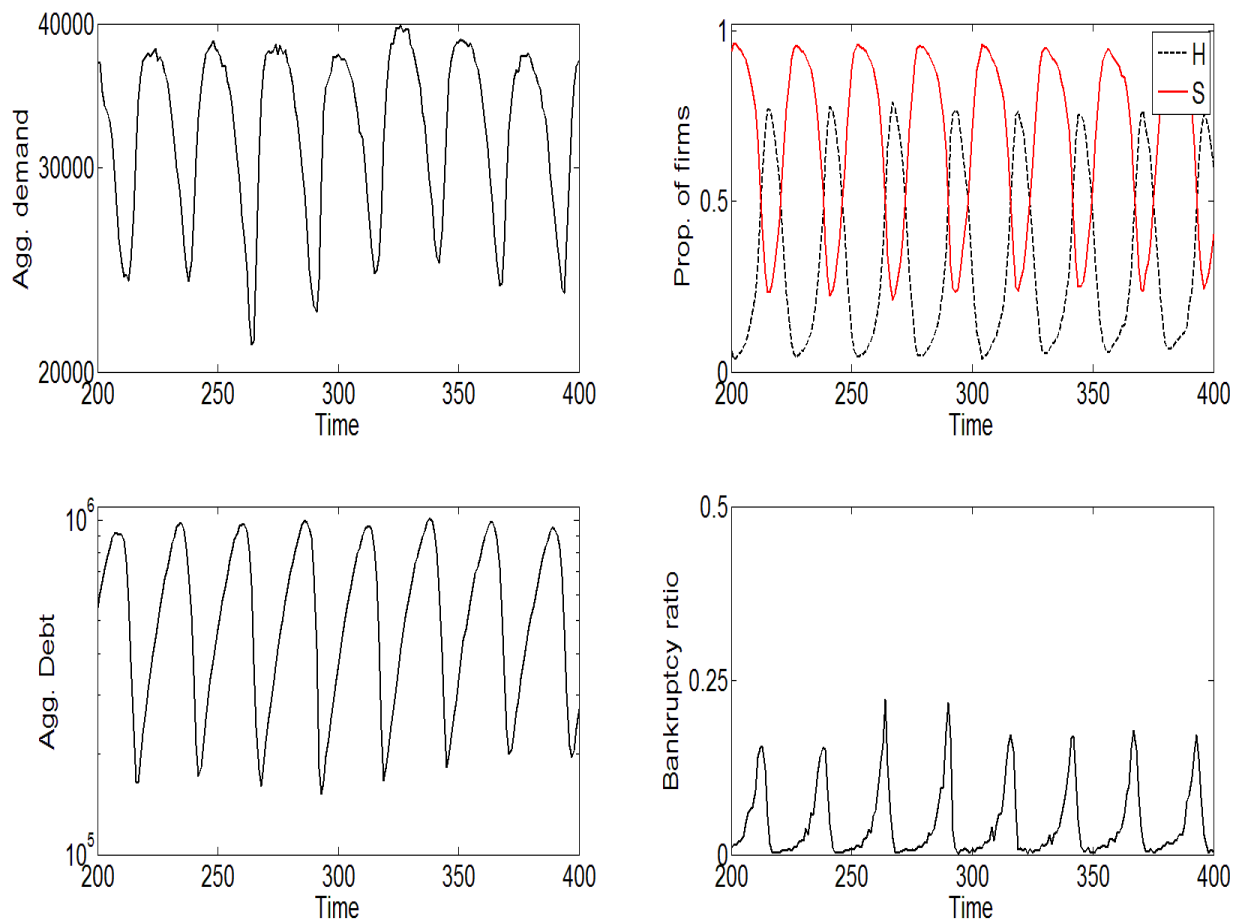


Figure 1: Single run results: aggregate demand, proportions of hedge and speculative firms, debt and bankruptcy ratio.

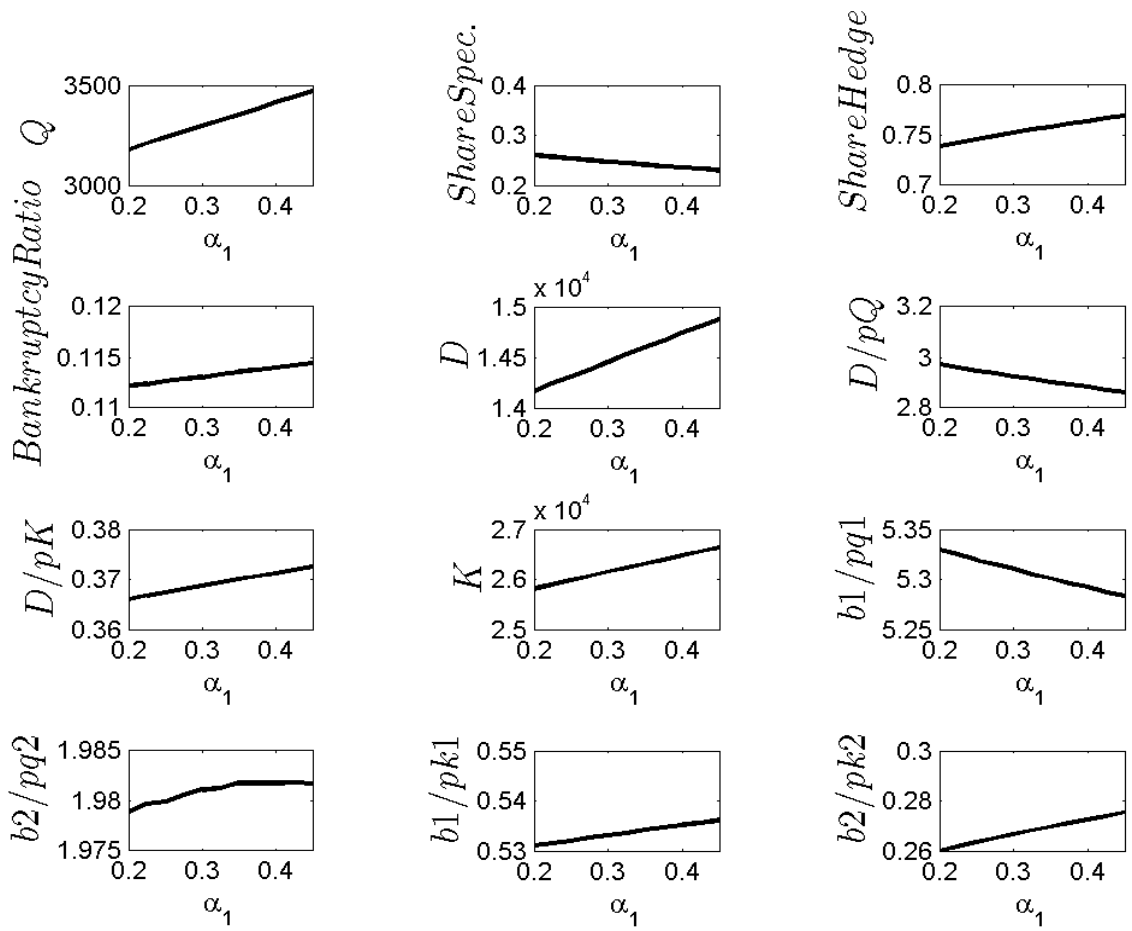


Figure 2: Results of Monte Carlo simulations for α_1 .

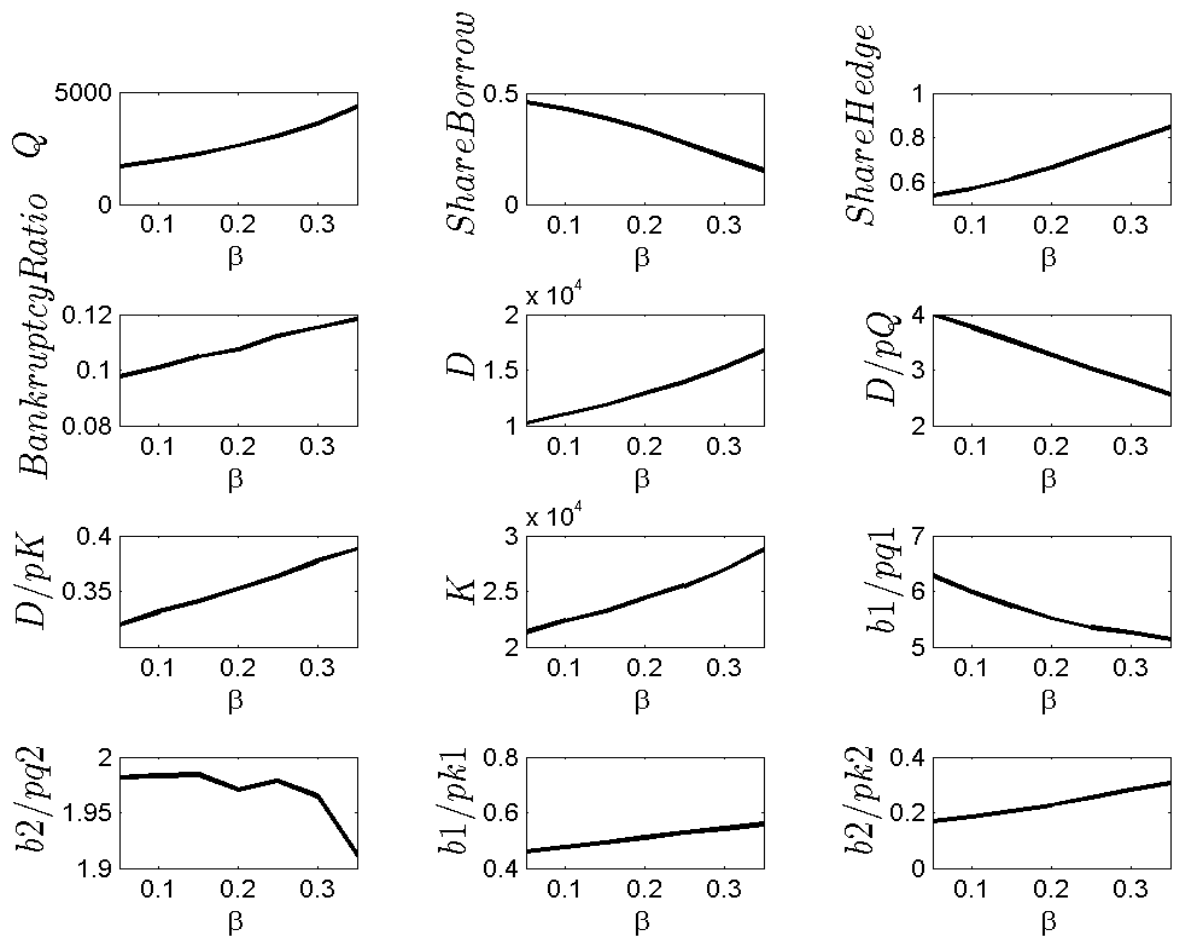


Figure 3: Results of Monte Carlo simulations for β .

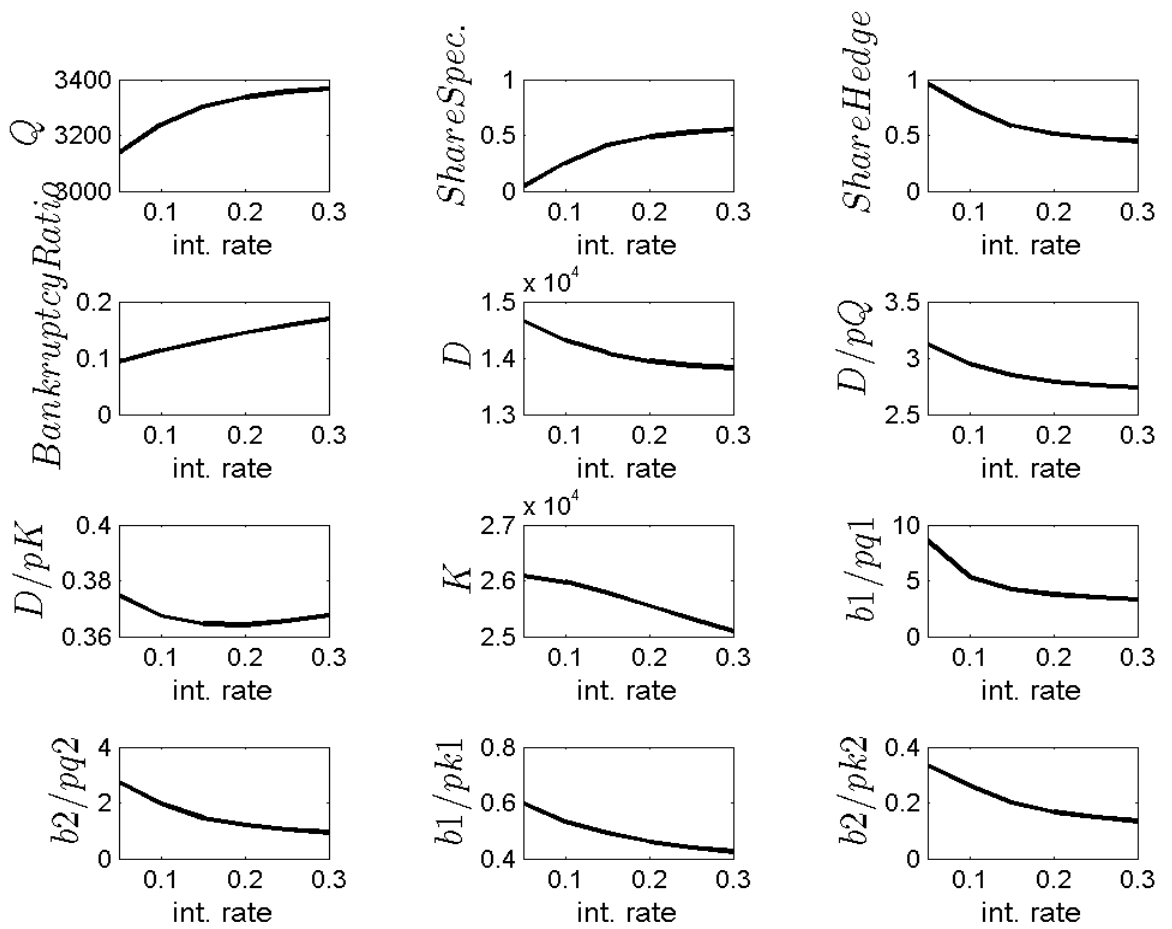


Figure 4: Results of Monte Carlo simulations for \bar{i} .

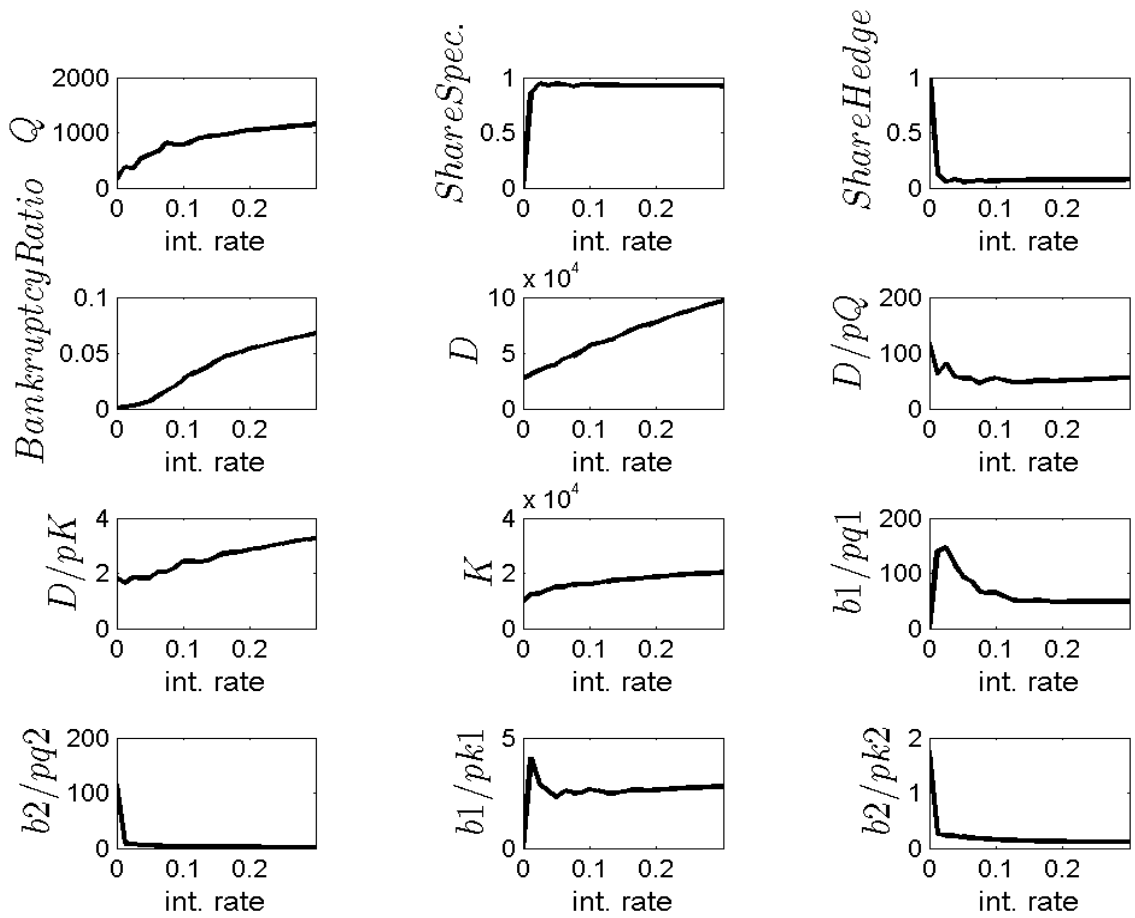


Figure 5: Results of Monte Carlo simulations for \bar{i} with $c = 25$ and immediate replacement of bankrupted firms.

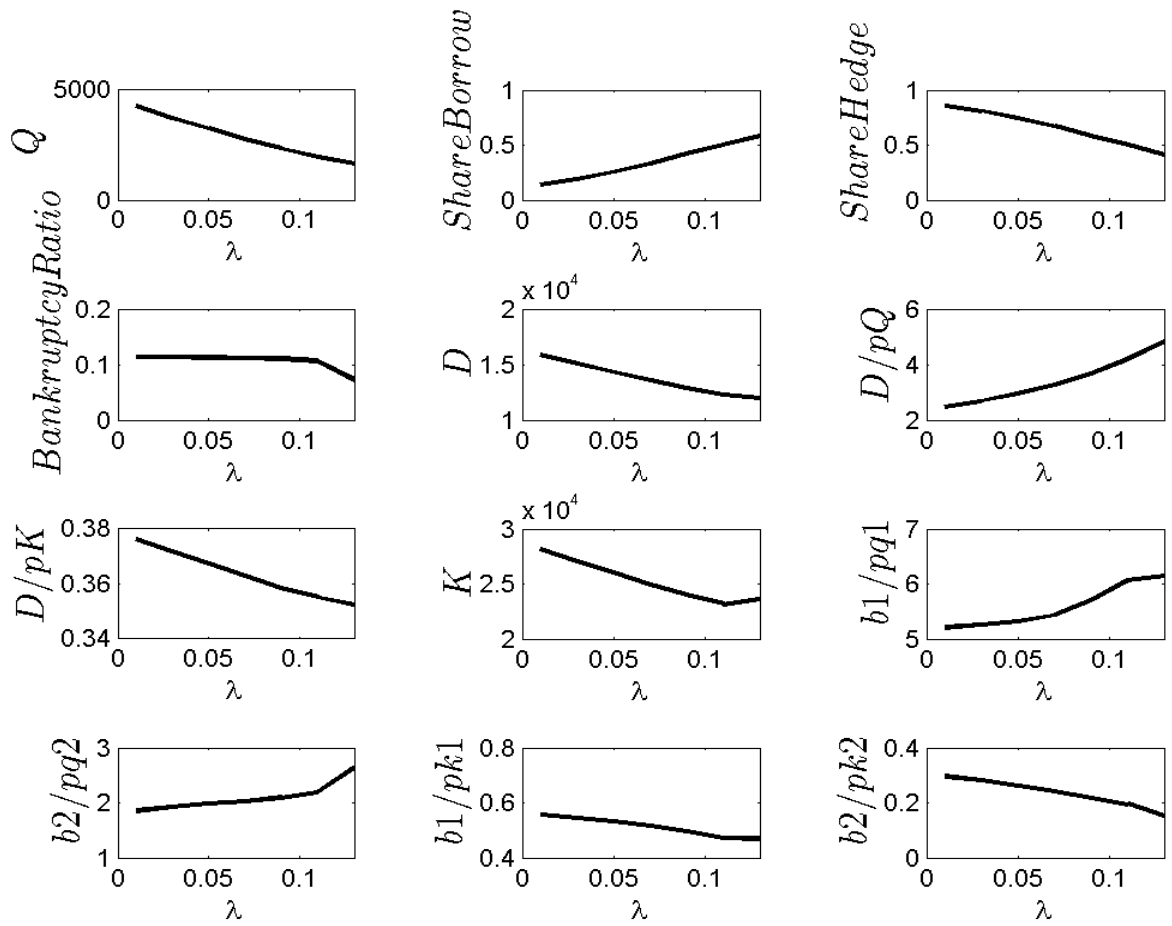


Figure 6: Results of Monte Carlo simulations for λ .

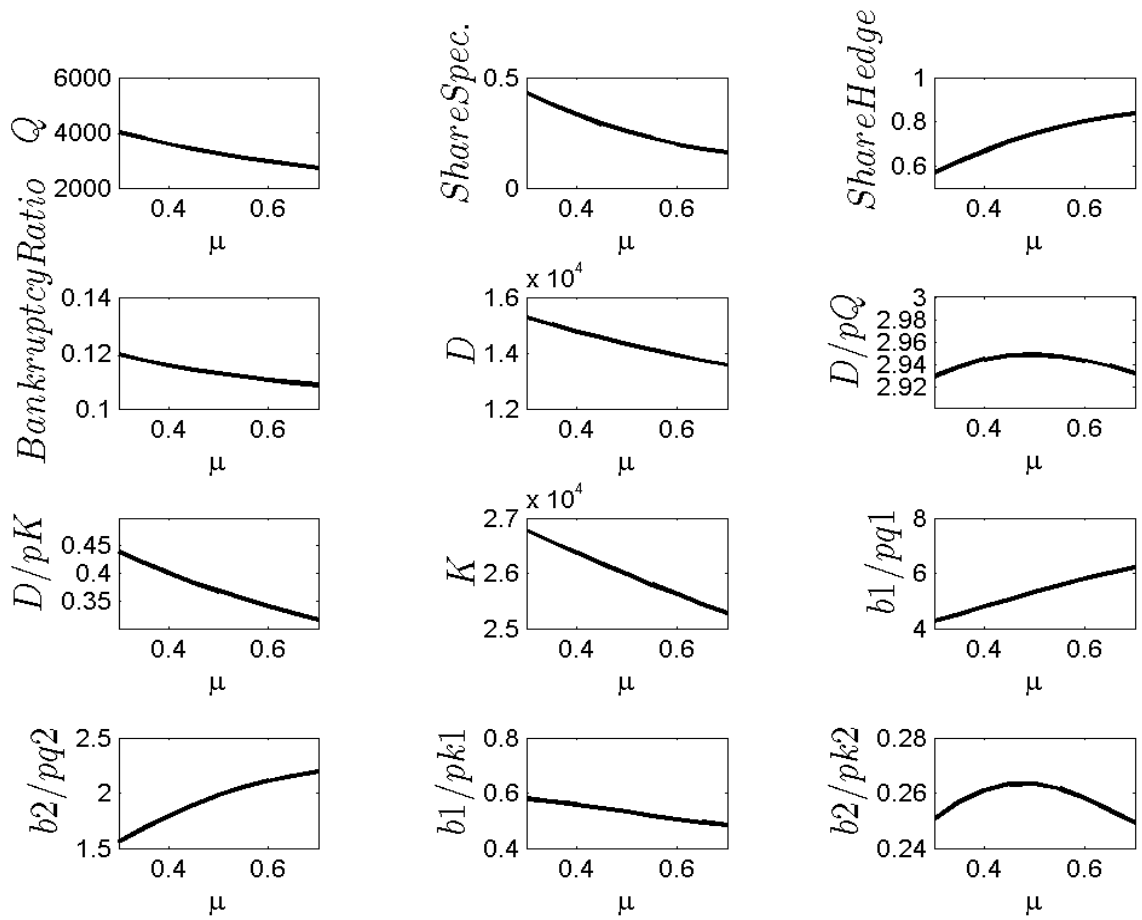


Figure 7: Results of Monte Carlo simulations for μ .

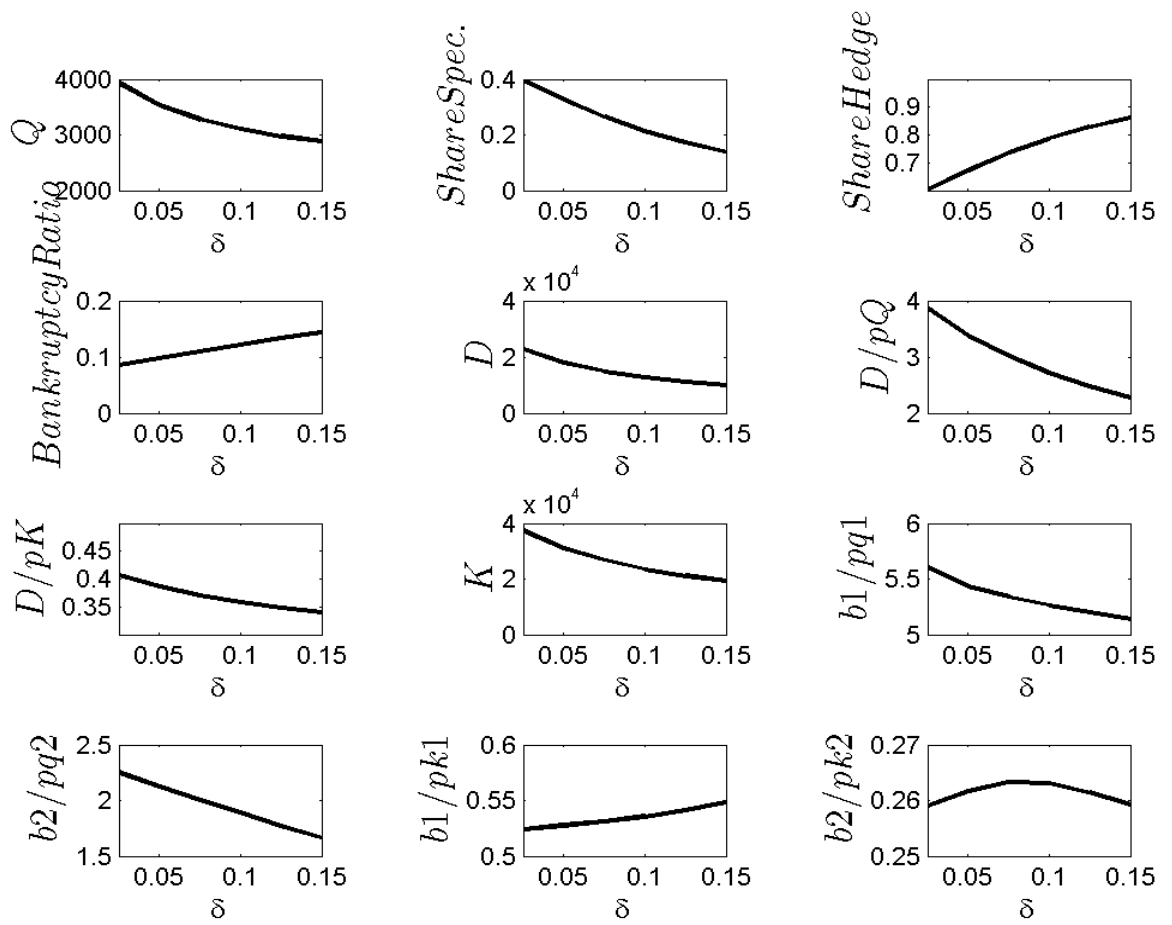


Figure 8: Results of Monte Carlo simulations for δ .

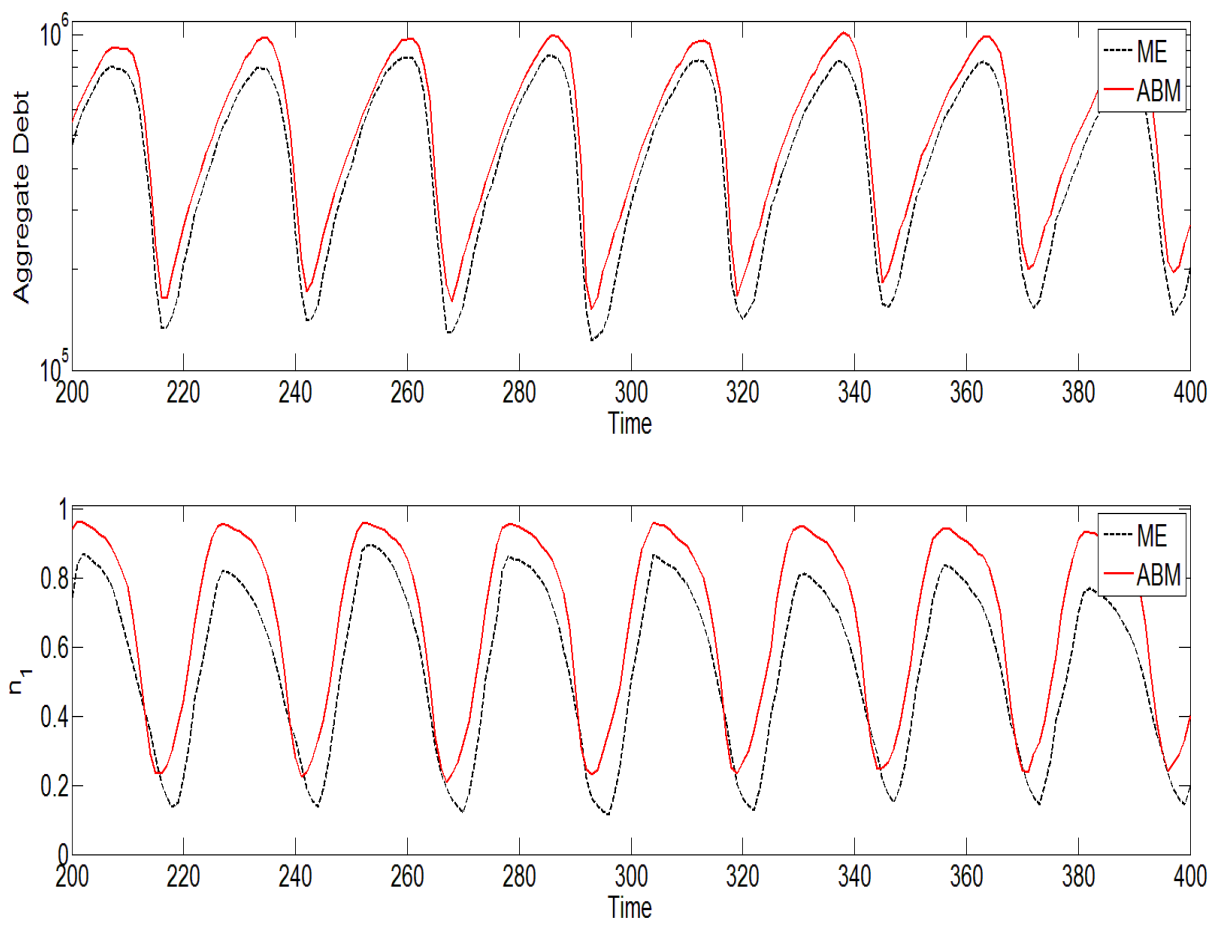


Figure 9: Comparison of the results of the agent-based model and the analytical solution for aggregate debt (upper panel) and share of speculative firms (lower panel).

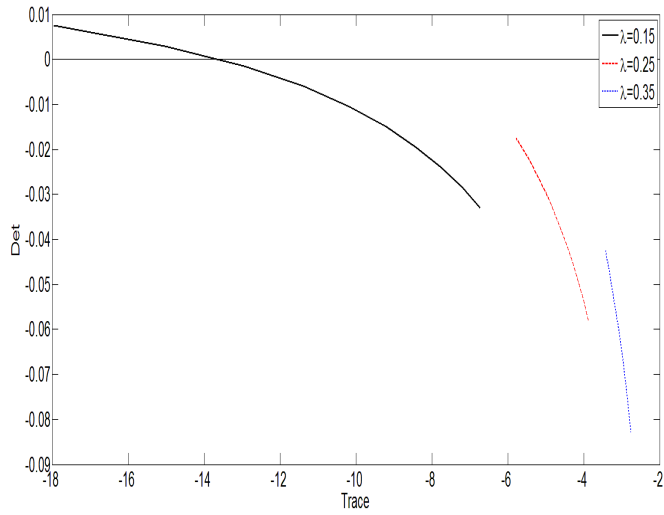


Figure 10: Trace and determinant for different values of λ and \bar{i} . *The interest rate varies from 0.01 to 0.1 as we move north along each contour.*

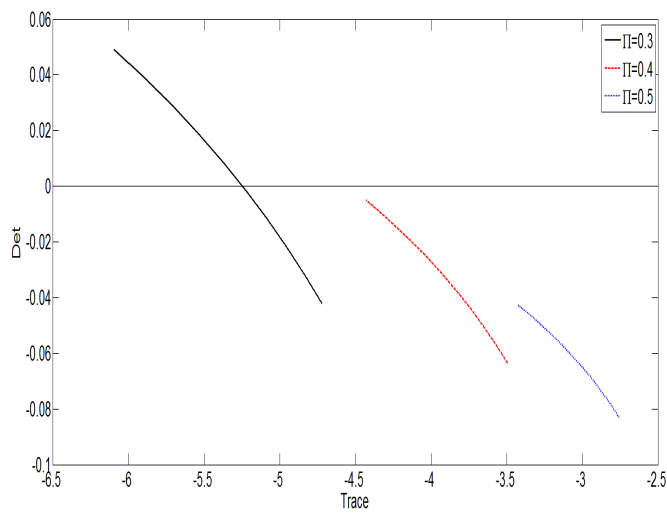


Figure 11: Trace and determinant for different values of Π and \bar{i} . *The interest rate varies from 0.01 to 0.1 as we move north along each contour.*