Price Reaction and Equilibrium Disagreement over Public Signal Interpretation *

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Abstract

We develop a theory of endogenous disagreement over interpretation of public news based on the optimal expectation model proposed by Brunnermeier and Parker (2005). In our model, each agent forms an optimal interpretation, and agrees to disagree with others in equilibrium. Endogenous disagreement and trade can arise following public news events. The model predicts that market price overreacts to uninformative news and underreacts to informative news, thus providing a unified account for the drift in price following significant news events, and the excessive price volatility in response to noisy information.

1 Introduction

Standard models of rational expectation have a hard time explaining the following well-documented phenomena: (i) trading volume spikes immediately following a public news announcement;[1] (ii) price exhibits medium-run momentum following a public news announcement such as earning reports;[2] and (iii) price volatility is too high to be justified by

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[1] See for example, Kandel and Pearson (1995), and Hong and Stein (2007).

[2] See for example Jagadeesh and Titman (1993), Bernard and Thomas (1989), and Fama and French (1988).

changes in underlying fundamental variables.[3] These empirical regularities are inconsistent with standard models that assume agents commonly hold the correct prior belief and interpretation of public news. As proposed by Hong and Stein (2007), a promising path to gain a better understanding of these phenomena is considering models of disagreement, which dispense with the assumptions of a common prior and/or news interpretation, and allow agents to "agree to disagree". In this paper, we develop a novel theory of equilibrium disagreement over news interpretation that offers predictions on the reaction of price and volume to public news announcement that are consistent with the empirical findings above.

We endogenize disagreement among agents by allowing each of them to "choose" his own interpretation of a public signal, in a spirit similar to the optimal expectation model proposed by Brunnermeier and Parker (2005). In their model, each agent (i) derives *anticipatory utility* from the optimism of enjoying high consumption in the future; (ii) is able to choose to hold a subjective belief that differs from the objective distributions, and "agree to disagree" with other agents. The basic trade-off facing each agent is the benefit of optimism (higher anticipatory utility) versus the cost of making bad decisions. As noted in their paper, the optimal expectation model can be viewed as a "theory for prior belief for Bayesian rational agents". They apply their model to an asset pricing setting without news arrival, and show that in equilibrium, an asset can be mispriced relative to its expected value if its payoff distribution is skewed. We adapt their model to build *a theory for news interpretation for Bayesian rational agents.* Agents in our model face a similar problem and tradeoff in deciding their subjective interpretation of a public news event. We show that if agents put high enough weights on anticipatory utilities, disagreement in news interpretation arises endogenously. Moreover, the equilibrium asset price may over-react or under-react to the news event depending on the news' objective informativeness.

Our contribution is twofold. First, on the theoretical front, we develop a novel model of endogenous disagreement over news interpretation with a solid psychological and economic foundation. Second, our model generates useful comparative statics results that can shed light on a number of well-documented empirical findings.

Below we briefly describe our model and results. There is a single risky asset and three periods. Market is open for trading at the end of period 0 and 1, and the asset's final payoff is realized in period 2. At the beginning of period 1, a piece of informative news concerning the asset's final payoff is publicly announced. Before any trading takes place (i.e., at the beginning of period 0), agents decide how they would like to interpret the forthcoming news, and their subjective news interpretations are held fixed for the rest of the game. After independently choosing their beliefs, they trade at the end of period 0. At the beginning of period 1, the news arrives and each agent updates her belief about the asset's payoff based on her chosen news interpretation. After the updating, there is another round of trading at

[3] See for example, Shiller (1981), and LeRoy and Parke (1992).

the end of period 1. Finally, the asset pays off. Each agent's choice of subjective news interpretation affects both her anticipatory utility and trading behaviors in period 0 and 1.

For analytical tractability, we assume the asset's payoff is normally distributed, and the public signal is the true payoff plus an independently and normally distributed white noise. Therefore, the public signal is characterized by its mean value and precision. To simplify exposition, we consider two cases regarding the choice of news interpretation separately. In the first case, agents agree over the public signal precision; but are free to choose their beliefs over the signal mean. We find that there always exists an equilibrium in which prices fully reflect the information content of the public signal. Even though the price level is fully rational, endogenous disagreement arises if agents put high enough weights on anticipatory utilities, and the public news is informative enough. In this case, one group of agents over-estimates the signal mean, while the other group under-estimates it. At the end of period 0, the former (latter) group holds a long (short) position. After the public signal realizes, the former group becomes relatively pessimistic about the asset and holds a short position, while the latter group becomes relatively optimistic and holds a long position.

Things get more interesting in the second case where agents agree over the signal's mean but are free to choose their beliefs over its precision. We find that if agents put high enough weights on the anticipatory utilities, equilibrium disagreement arises and the market price does not fully reflect the signal content. In this case, one group of agents overestimates the signal precision, whereas the other group underestimates it, and the two groups trade against each other upon the arrival of the public news. The main result is that if the objective signal precision is sufficiently high, then the equilibrium price underreacts to the signal; if the objective signal precision is sufficiently low, then the price overreacts to the signal. In other words, our model predicts that price exhibits momentum following informative news event, and exhibits reversal following uninformative news event. This result is interesting because it provides a unified account for the drift in price following significant news event (such as announcement on merger, buy-back, or new stock issuance), and the excessive price volatility in response to rumors and noisy information.

The outline of the paper is as follows. We first review the theoretical and empirical literature most related to the current study. The model is set up in Section 2. Section 3 outlines the procedure of solving the model and establishes equilibrium existence. The cases of disagreement over signal mean and precision are considered in Section 4 and 5 respectively. The last section concludes. Lengthy proofs are relegated to the appendix.

1.1 Related Literature

Our theory of endogenous disagreement is based on Brunnermeier and Parker (2005). The novel feature of their model is that agents can choose their belief to maximize a weighted sum of flow utilities and anticipatory utilities.[4] They show that in a static asset pricing setting without any news arrival, positively skewed assets admits a high level of anticipatory utility, so features a high equilibrium price. We modify their asset-pricing setting to allow for the arrival of an interim public news, and investigate its effect on the equilibrium price. To ensure tractability, a restriction is imposed on the set of feasible subjective news interpretations. We arrive at a counterpart result of that in Brunnermeier and Parker (2005): an objectively informative news causes the market price to underreact, while an uninformative news causes the market price to overreact. Brunnermeier, Gollier, and Parker (2007) contains a more comprehensive treatment of the pricing implications of the optimal expectation model.

Our model belongs to the class of disagreement models. The seminal works of Harris and Raviv (1993) and Kandel and Pearson (1995) use a model of heterogenous news interpretations to explain the jump in trading volume following public news announcement. These papers exogenously specify two groups of traders who hold different beliefs about the generation process of a public signal. Consequently, these two groups update their beliefs on the true state of the world differently, and are willing to trade against each other. A limitation of these work is that, without additional assumptions on the model of the asset market, the pricing outcome is almost completely determined by the exogenously specified disagreement pattern. We extend this line of work by endogenizing traders' disagreement over news interpretation. This allows us to pin down the pattern of equilibrium disagreement, and offer testable predictions on the consequent mispricing.

The implication of disagreement on market prices have been studied in a number of articles. Banerjee, Kaniel, and Kremer (2009) show that disagreement over the information contained in the equilibrium price is necessary to generate a price drift. Scheinkman and Xiong (2003) show that in the presence of a short-sale constraint and an exogenous disagreement over news interpretation, the asset has a speculative value and is thus overpriced relative to the fundamentals. We impose no constraint on short-selling, and focus on the pricing implication of the objective informativeness of the public news. Ottaviani and Sørensen (2015) shows that disagreement in prior beliefs, when combined with either a wealth constraint or a decreasing absolute risk aversion preference, causes market prices to exhibit initial momentum and eventual reversal. In contrast, we abstract away from any wealth effect; our results are driven solely by investors' equilibrium disagreement over news interpretation.

A number of papers explain stock price anomalies, such as short-run momentum and long-run reversal, using behavioral biases documented in the psychology and behavioral economics literature. Daniel, Hirshleifer and Subrahmanyam

[4] The effect of anticipatory utility on behaviors has also been investigated in Loewenstein and Elster (1992), Kahneman, Wakker and Sarin (1997), and Caplin and Leahy (2001).

(1998) assume investors are overconfident in estimating the precision of a privately observed signal, which leads to a price overreaction. Barberis, Shleifer, and Vishny (1998) assume agents have a misspecified model of the earning dynamics, which leads them to overestimate the probability of mean-reversion following a single positive shock, as well as the probability of price momentum following a series of positive shocks. Hong and Stein (1999) show that price exhibits underreaction if the information flow among investors is gradual, and they fail to extract information from market prices. The latter assumption can be interpreted as a form of overconfidence: investors overestimate the quality of their own information, and underestimate the information content of the market price.[5] We differ from these papers by considering a different behavioral bias, so our mechanism is very different. As multiple mechanisms are likely to be at work in reality, we view our approach as complementary to the existing literature. Moreover, whereas these models do not address the effect of public news on investor disagreement and trading volume, we derive implications on the effect of public news announcement on both the price dynamics and trading volumes.

2 Model Setup

There are three periods: period 0, 1, and 2. There is one continuum of ex-ante identical agents, indexed by $i \in [0,1]$. Each agent has an initial endowment of one unit of wealth. Each agent evaluates her final wealth W according to the CARA utility function: $u(W) = -\exp(-W)$. There are two tradable assets: a risk-free asset with gross return equal to one at period 2; and a risky asset which pays out ω at period 2, where ω is normally distributed with mean μ and precision α_ω.[6] The risky asset is in zero net supply. All asset trading occurs in a perfectly competitive market, in which there is no short-sale constraint.

The market is open for trading at the end of period 0 and period 1. The market price in period 0 is denoted by P^0. At the beginning of period 1, a public signal, denoted by s, realizes and is observed by all agents. The public signal is the sum of the payoff of risky asset ω and a noise term ε, i.e., $s = \omega + \varepsilon$. The noise is independently and normally distributed with mean 0 and precision α_ε. Another round of trading occurs after s is publicly observed. The market price corresponding to signal s is denoted by P_s. We write $P \equiv \left\{P^0, \{P_s\}\right\}$ to denote the price vector. In period 2, ω realizes and each agent consumes her final wealth.

Two nonstandard features of the model are: (i) in addition to consumption, agents also derive utility from anticipation; (ii) agents can hold beliefs about precision of the public signal that differs from the objective value, and "agree to disagree" with each other. Specific details are described below.

[5] Hirshleifer and Teoh (2003) and Peng and Xiong (2006) show that limited attention to public information can generate results similar to gradual information flow.

[6] The precision of a random variable x, α_x, is the reciprocal of its variance.

At the beginning of period 0, each agent chooses a subjective belief about the interpretation of the public signal. Once chosen, her belief is fixed for the rest of the game, and her subsequent trading behaviors are governed by this fixed belief. It is important to stress that although the agent can choose her interpretation of the signal different from the true signal generation process, she must update her belief using Bayes' rule. For analytical tractability, the set of feasible subjective beliefs is restricted to be normal distributions. Specifically, each agent i can choose to believe that the signal noise ε is independently distributed according to $N\left(\hat{\mu}_{\varepsilon}^{i}, \left(\hat{\alpha}_{\varepsilon}^{i}\right)^{-1}\right)$ for some $\hat{\mu}_{\varepsilon}^{i} \in \mathbb{R}$ and $\widehat{\alpha}_{\varepsilon}^{i} \in \mathbb{R}_{+}$. Denote her choice of signal interpretation by $\hat{\sigma}^{i} \equiv \left(\hat{\mu}_{\varepsilon}^{i}, \hat{\alpha}_{\varepsilon}^{i}\right)$. It differs from the objective values whenever she chooses $\hat{\sigma}^{i} \neq (0, \alpha_{\varepsilon})$. An expectation operator defined by agent i's subjective belief is denoted by $\hat{E}^{i}$. In period 1 after signal s has been publicly observed, each agent i sets her position ℓ_{s}^{i} in the risky asset in order to maximize her expected utility $\hat{E}_{\omega|s}^{i}\left[u\left(W\right)\right]$ subject to her budget constraint. Similarly, in period 0 after subjective beliefs are formed, each agent i chooses L^{i} in order to maximize $\hat{E}_{\omega,s}^{i}\left[u\left(W\right)\right]$, taking into account her trading behavior in period 1.

Now we discuss the agent's criterion of choosing her subjective belief $\hat{\sigma}^{i}$. At the beginning of period 0, she chooses her subjective belief with the goal of maximizing her well-being defined by

$$E_{\omega,s}\left\{\lambda_{0}\hat{E}_{\omega,s}^{i}\left[u\left(W\right)\right] + \lambda_{1}\hat{E}_{\omega|s}^{i}\left[u\left(W\right)\right] + u\left(W\right)\right\}, \tag{1}$$

where $\lambda_0, \lambda_1 > 0$. The first two terms in the expression above are the agent's anticipatory utility in period 0 and 1 respectively. Weights λ_0 and λ_1 measure how much each agent care about her respective anticipatory utilities. Note that the outside expectation operator is based on the objective probability distribution. At the beginning of period 0, each agent chooses her subjective belief to maximize her well-being, taking into account the equilibrium market prices and her subsequent (sequentially optimal) trading behaviors.

We adopt the solution concept of competitive equilibrium. Denote the measure of agents holding belief $\hat{\sigma}$ by $\beta\left(\hat{\sigma}\right)$. The competitive equilibrium is defined as a tuple of market price vector $P \in \mathbb{R} \times \mathbb{R}^{\mathbb{R}}$ and a measure of agents' subjective belief $\beta : \mathbb{R} \times \mathbb{R}_{+} \to [0, 1]$, such that (i) each agent's strategy is optimal given her belief; (ii) each agent's belief is chosen optimally to maximize her well-being; and (iii) market clears in every period and following every signal realization. A more formal definition can be found in Definition 1 in the next section.

There are two major differences between our model and the optimal expectation model of Brunnermeier and Parker (2005). First, while they allow agent's subjective belief to be any probability distribution over the signal and the state, our formulation restricts each agent's belief to be normally distributed. This restriction makes the analysis tractable. Second, while they restrict the weights on anticipatory utilities to be one for all periods, we allow λ_0 and λ_1 to be distinct and different from one. A heavy weight can be interpreted as either the agents care a lot about anticipatory utilities, or the "duration of anticipation" is long.

3 Preliminaries

The model can be solved in three steps: first, fixing the market price vector and an agent's belief, we compute her optimal trading plan; second, we formulate the agent's problem of choosing the optimal belief; finally, we define formally the competitive equilibrium, establish its existence and study its basic property.

3.1 Asset Trading Given Beliefs

Fix a price vector P and a period-1 signal realization s. Suppose an agent holds subjective belief $\hat{\sigma}^i = \left(\hat{\mu}^i_\varepsilon, \hat{\alpha}^i_\varepsilon\right)$ and enters period 1 with wealth W^i_s. She adjusts her asset holding of the risky asset ℓ^i_s by solving the following problem:

$$U\left(W^i_s, \hat{\sigma}^i; P, s\right) \equiv \max_{\ell_s} \hat{E}^i_{\omega|s}\left[-\exp\left(-\left[W^i_s + \ell_s\left(\omega - P_s\right)\right]\right)\right]. \tag{2}$$

The objective function in (2) can be simplified into:

$$U\left(W^i_s, \hat{\sigma}^i; P, s\right) = \max_{\ell_s} -\exp\left(-W^i_s - \ell_s\left(\frac{\hat{\alpha}^i_\varepsilon\left(s - \hat{\mu}^i_\varepsilon\right)}{\alpha_\omega + \hat{\alpha}^i_\varepsilon} - P_s\right) + \frac{1}{2}\left(\ell_s\right)^2 \frac{1}{\alpha_\omega + \hat{\alpha}^i_\varepsilon}\right),$$

giving the optimal position:

$$\ell_s\left(\hat{\sigma}^i; P\right) \equiv \frac{\hat{E}^i_{\omega|s}[\omega] - P_s}{\hat{V}^i_{\omega|s}[\omega]} = \hat{\alpha}^i_\varepsilon\left(s - \hat{\mu}^i_\varepsilon\right) - P_s\left(\alpha_\omega + \hat{\alpha}^i_\varepsilon\right). \tag{3}$$

Observe that the optimal holding is independent of wealth, a result that arises from the CARA utility.

Next, consider the agent's period-0 investment problem. If agent i holds L^i units of the risky asset in period 0, and the signal is s, then her period-1 wealth is

$$W^i_s\left(L^i; P\right) = 1 + L^i\left(P_s - P^0\right). \tag{4}$$

Substituting the optimal holding (3) and wealth (4) into program (**??**), we get

$$U\left(W^i_s\left(L^i; P\right), \hat{\sigma}^i; P, s\right) = -\exp\left(-1 - L^i\left(P_s - P^0\right) - \frac{1}{2}\frac{\left(\hat{E}^i_{\omega|s}[\omega] - P_s\right)^2}{\hat{V}^i_{\omega|s}[\omega]}\right). \tag{5}$$

Denote by $\hat{F}^i$ the agent's subjective distribution over the signal.[7] Her period-0 asset choice problem is thus

$$\Psi_0\left(\hat{\sigma}^i; P\right) \equiv \max_{L^i} \int U\left(W^i_s\left(L^i; P\right), \hat{\sigma}^i; P, s\right) d\hat{F}^i(s). \tag{6}$$

Straightforward computation gives the optimal asset holding in period 0:

$$L\left(\hat{\sigma}^i; P\right) \equiv \frac{1-\eta}{\eta}\hat{\alpha}_\varepsilon \hat{\mu}_\varepsilon.$$

To summarize, given a price vector P and a subjective belief $\hat{\sigma}^i$, the agent holds $L\left(\hat{\sigma}^i; P\right)$ of the risky asset in period 0, and adjusts to $\ell_s\left(\hat{\sigma}^i; P\right)$ after public signal s is realized in period 1.[8]

[7] The subjective distribution of s is normal with mean $\hat{\mu}^i_\varepsilon$ and precision $\left(\frac{1}{\alpha_\omega} + \frac{1}{\hat{\alpha}^i_\varepsilon}\right)^{-1}$.

[8] A more detailed derivation can be found in Lemma 11 in the Appendix.

3.2 Optimal Belief

Denote by $\Omega\left(\hat{\sigma}^i, P; s, \omega\right)$ the final wealth of agent i if the signal is s and the state is ω, provided that she invests optimally under belief $\hat{\sigma}^i$ as described in the subsection above:

$$\Omega\left(\hat{\sigma}^i, P; s, \omega\right) \equiv 1 + \left(P_s - P^0\right) L\left(\hat{\sigma}^i; P\right) + \ell_s\left(\hat{\sigma}^i; P\right)(\omega - P_s).$$

The expected consumption utility, calculated based on objective distributions of the state and the signal, is

$$\Gamma\left(\hat{\sigma}^i; P\right) \equiv \int\int u\left(\Omega\left(\hat{\sigma}^i, P; s, \omega\right)\right) dF(s|\omega)\, d\Phi(\omega), \tag{7}$$

where Φ is the distribution function of a normal distribution with mean 0 and precision α_ω; and $F(s|\omega)$ is the objective distribution of the public signal conditional on the state being ω. Belief is chosen to optimize the ex-ante well-being (1), a weighted sum of the anticipatory utilities and the consumption utility. The anticipatory utility for period 0 is given by $\Psi_0\left(\hat{\sigma}^i; P\right)$, defined in the previous subsection. The anticipatory utility for period 1 is given by:

$$\Psi_1\left(\hat{\sigma}^i; P\right) \equiv \int U\left(W_s^i\left(L\left(\hat{\sigma}^i; P\right)\right), \hat{\sigma}^i; P, s\right) dF(s), \tag{8}$$

where F is the objective signal distribution.[9] Agent i's well-being of holding belief $\hat{\sigma}^i$, provided that the price vector is P, is given by

$$V\left(\hat{\sigma}^i; P\right) \equiv \Gamma\left(\hat{\sigma}^i; P\right) + \lambda_0\Psi_0\left(\hat{\sigma}^i; P\right) + \lambda_1\Psi_1\left(\hat{\sigma}^i; P\right). \tag{9}$$

Recall λ_0 and λ_1 are weights associated with anticipatory utilities in period 0 and period 1 respectively. Given the equilibrium price vector, the agent's period-0 problem is to choose a belief $\hat{\sigma}^i$ to maximize her well-being $V\left(\hat{\sigma}^i; P\right)$.

3.3 Competitive Equilibrium

In a competitive equilibrium, the price vector $P = \left\{P^0, \{P_s\}\right\}$ is such that the market clears in every period and following every signal realization. Recall $\beta(\hat{\sigma}_\varepsilon)$ is the measure of agents with belief $\hat{\sigma}_\varepsilon$. As the asset is in zero net supply, the market-clearing condition for period 1 requires that for every signal $s \in \mathbb{R}$,

$$\int \ell_s\left(\hat{\sigma}^i; P\right) d\beta\left(\hat{\sigma}^i\right) = 0. \tag{10}$$

Likewise, in period 0, the market-clearing condition requires

$$\int L\left(\hat{\sigma}^i; P\right) d\beta\left(\hat{\sigma}^i\right) = 0. \tag{11}$$

Now we are ready to state the precise definition of the competitive equilibrium:

[9] Note that the outside expectation over the signal realization is taken with respect to the objective distribution (See (1)). On the other hand, the definition of U makes use of the subjective expectation of the state ω conditional on signal s (recall the definition of U in (2)).

Definition 1 *A pair of price vector and belief measure (P, β) constitutes a* ***competitive equilibrium*** *if and only if the following conditions hold:*

1. Every agent i acts optimally given her subjective belief $\hat{\sigma}^i$, that is, in period 0, her risky asset holding is $L\left(\hat{\sigma}^i; P\right)$; in period 1 after signal s is realized, her holding is $\ell_s\left(\hat{\sigma}^i; P\right)$.

2. The subjective belief of every agent is optimal given the price vector P, that is, for all $\hat{\sigma} \in supp\left(\beta\right)$,

$$\hat{\sigma} \in \arg\max_{\hat{\sigma}'} V\left(\hat{\sigma}'; P\right).$$

3. Market clearing conditions hold in each period and following each signal realization, i.e., (11) holds, and (10) holds for every $s \in \mathbb{R}$.

The following lemma is very useful in simplifying our problem of computing the competitive equilibrium:

Lemma 2 *The equilibrium price function P_s is linear. Precisely, there exists $\eta_0 \in \mathbb{R}$ and $\eta \in \mathbb{R}_+$ such that $P_s = \eta_0 + \eta s$.*

Proof. Note that equation (3) can be written as

$$\ell_s\left(\hat{\sigma}^i; P\right) = \hat{\alpha}_\varepsilon\left(s - \hat{\mu}_\varepsilon\right) - \left(\alpha_\omega + \hat{\alpha}_\varepsilon\right) P_s.$$

Substitute it into (10) and rearranging

$$\begin{aligned} & \int \left[\hat{\alpha}_\varepsilon\left(s - \hat{\mu}_\varepsilon\right) - \left(\alpha_\omega + \hat{\alpha}_\varepsilon\right) P_s\right] d\beta\left(\hat{\sigma}^i\right) = 0 \\ \Leftrightarrow \quad & P_s = \frac{-\int \hat{\alpha}_\varepsilon \hat{\mu}_\varepsilon d\beta\left(\hat{\sigma}\right) + s\int \hat{\alpha}_\varepsilon d\beta\left(\hat{\sigma}\right)}{\int \left(\alpha_\omega + \hat{\alpha}_\varepsilon\right) d\beta\left(\hat{\sigma}\right)}. \end{aligned}$$

Clearly, it is linear in s with a positive slope. ■

We refer to η as the price sensitivity. It captures how responsive the market price is to the public signal. If all agents in the market hold the rational belief, then the market-clearing condition implies that the price in period 1 necessarily equals the expected value of ω conditional on the signal realization. Consequently, the price sensitivity is at the rational level, defined by $\eta_R \equiv \frac{\alpha_\varepsilon}{\alpha_\omega + \alpha_\varepsilon}$. If $\eta < (>) \eta_R$, then we say the equilibrium price underreacts (overreacts) to the public information.

A symmetry property in the equilibrium is natural: in the absence of any informative news, i.e., in period 0 and following $s = 0$ in period 1, the market price equals the expected value of ω under the prior belief. We say the competitive equilibrium is **symmetric** if and only if $P^0 = \eta_0 = 0$. The price vector of a symmetric equilibrium is fully characterized by η. Note that even if $P^0 = 0$, trade may still take place if agents disagree over the mean signal in the next period. In the remainder of our analysis, we focus on symmetric competitive equilibrium.

Proposition 3 *A symmetric competitive equilibrium exists.*

In the subsequent sections, we separate the analysis into two distinct cases. In Section 4, the agents are allowed to choose belief on the signal mean $\hat{\mu}_\varepsilon^i$ only, whereas their belief on the signal precision is fixed at the objective value (i.e., all agents have $\hat{\alpha}_\varepsilon^i$ fixed at α_ε). In Section 5, we consider the opposite case: the agents are allowed to choose belief on the signal precision $\hat{\alpha}_\varepsilon^i$ only, whereas their belief on the signal mean is fixed at the objective value (i.e., all agents have $\hat{\mu}_\varepsilon^i$ fixed at 0). This separation of analysis help illustrate the source of the pricing and volume anomalies in our model. In Section 4, we find that if the weights on anticipatory utilities are large enough, endogenous disagreement over the signal mean can arise, resulting in a positive trading volume. However, the equilibrium price sensitivity necessarily remains at the rational level η_R. On the other hand, it is shown in Section 5 that if agents are allowed to choose their beliefs about the signal precision, the equilibrium price sensitivity can be different from the rational level η_R. Therefore, while optimal beliefs over the expected signal content can result in positive trading volume, it does not result in mispricing on its own. Price over or under reactions occur only if the signal informativeness can be subjectively interpreted.

4 Disagreement over Signal Mean

In this section, we investigate the case in which agents can form their own belief about the mean of the public signal, but not its precision. As every agent has $\hat{\alpha}_\varepsilon^i = \alpha_\varepsilon$, $\hat{\sigma}^i$ is characterized by his belief about the signal mean $\hat{\mu}_\varepsilon^i$ only. Thus, we write $V(\hat{\mu}_\varepsilon; \eta)$ to stand for an agent's well-being by holding belief $\hat{\mu}_\varepsilon$, provided that the price sensitivity is η.

Lemma 4 *Suppose each agent is free to choose $\hat{\mu}_\varepsilon^i$, but $\hat{\alpha}_\varepsilon^i$ is fixed at the objective precision α_ε. In a symmetric competitive equilibrium, $V(\hat{\mu}_\varepsilon; \eta)$ is continuous and symmetric in $\hat{\mu}_\varepsilon$ around* 0.

According to the lemma, either (i) $V(\hat{\mu}_\varepsilon; \eta)$ achieves its maximum at $\hat{\mu}_\varepsilon = 0$; or (ii) $V(\hat{\mu}_\varepsilon; \eta)$ achieves its maximum at both $\hat{\mu}_\varepsilon = x$ and $\hat{\mu}_\varepsilon = -x$, for some $x \in \mathbb{R}_+$. In case (i), there is neither equilibrium disagreement nor trade. In case (ii), equilibrium disagreement arises: one group of agents over-estimates the signal mean whereas the other group under-estimates it. The former group holds a long position in period 0 as they are relatively optimistic about the period-1 asset price; their expected price is $\eta \hat{E}^i[s] = \eta \hat{\mu}_\varepsilon^i > 0$. In period 1 after the public signal realizes, they revert to a short position as they are relatively pessimistic about the state of the world; their expected state is $\hat{E}^i[\omega|s] = \frac{\alpha_\omega}{\alpha_\omega + \alpha_\varepsilon}\left(s - \hat{\mu}_\varepsilon^i\right)$. The trading behavior of the group that under-estimates the signal mean is exactly opposite.

The following proposition identifies the precise condition for equilibrium disagreement, i.e., case (ii), to arise, and shows that the equilibrium price sensitivity is necessarily rational.

Proposition 5 *In a symmetric competitive equilibrium, the price sensitivity is unique and is rational. Equilibrium* *disagreement over the signal mean and thus trade arises if*

$$\lambda_0 + \lambda_1 \frac{\alpha_\varepsilon - \alpha_\omega}{\alpha_\varepsilon + \alpha_\omega} > 1.$$

The reason for the uniqueness of price sensitivity is as follows. First, it is clear that if case (i) arise, the market clears in period 1 if and only if the price sensitivity is rational.[10] Next, consider case (ii). Following a signal realization s, using (3), an agent with belief $\hat{\mu}_\varepsilon \neq 0$ holds a position:

$$\ell_s(\hat{\mu}_\varepsilon; P) = -\alpha_\varepsilon \hat{\mu}_\varepsilon + (\alpha_\omega + \alpha_\varepsilon)\left(\frac{\alpha_\varepsilon}{\alpha_\omega + \alpha_\varepsilon} - \eta\right) s.$$

Denote the measure of agents with belief $\hat{\mu}_\varepsilon$ by $\beta(\hat{\mu}_\varepsilon)$. The market clears following a realization of s if and only if

$$\begin{aligned} & \int \left[-\alpha_\varepsilon \hat{\mu}_\varepsilon + (\alpha_\omega + \alpha_\varepsilon)\left(\frac{\alpha_\varepsilon}{\alpha_\omega + \alpha_\varepsilon} - \eta\right) s\right] d\beta(\hat{\mu}_\varepsilon) = 0 \\ \Leftrightarrow \quad & (\eta_R - \eta) s = \eta_R \int \hat{\mu}_\varepsilon d\beta(\hat{\mu}_\varepsilon). \end{aligned}$$

As the right-hand side of the equality is independent of s, market-clearing holds for all s if and only if $\eta = \eta_R$.

The weights on anticipatory utilities, λ_0 and λ_1, determines whether equilibrium disagreement and trade arises or not. Clearly, if the weights λ_0 and λ_1 on anticipatory utilities are zero, then each agent only cares about the actual consumption utility. Consequently, there is no reason to choose subjective belief different from the objective distribution, as doing so would only result in suboptimal trading behaviors. We thus obtain the prediction of standard rational expectation model. The proposition above shows that the equilibrium remains fully rational if agents' weights on anticipatory utility are sufficiently small. Equilibrium disagreement and trade arises if the weights on anticipatory utility get large enough. Nonetheless, the equilibrium prices necessarily remain at the rational level. Thus, the disagreement over signal mean alone does not result in any price anomaly.

5 Disagreement over Signal Precision

In this section, we investigate the case in which agents can form their own belief about the precision of the public signal, but not its mean. As every agent has $\hat{\mu}^i_\varepsilon = 0$, his belief $\hat{\sigma}^i$ is characterized by his belief about the signal precision $\hat{\alpha}^i_\varepsilon$ only. Thus, we write $V(\hat{\alpha}_\varepsilon; \eta)$ to stand for an agent's well-being by holding belief $\hat{\alpha}_\varepsilon$, provided that the price sensitivity is η.

Lemma 6 *Suppose each agent is free to choose $\hat{\alpha}^i_\varepsilon$, but $\hat{\mu}^i_\varepsilon$ is fixed at the objective mean 0. In a symmetric competitive equilibrium, $V(\hat{\alpha}_\varepsilon; \eta)$ is continuous in $\hat{\alpha}_\varepsilon$ for all $\hat{\alpha}_\varepsilon \in [0, B(\eta))$, where*

$$B(\eta) \equiv \alpha_\varepsilon + \frac{\sqrt{(\alpha_\varepsilon + \alpha_\omega)\left(\eta^2 \alpha_\omega + (1-\eta)^2 \alpha_\varepsilon\right)}}{1-\eta}.$$

[10] Otherwise, either all agents hold a long position or all agents hold a short position.

Moreover, $V(\hat{\alpha}_\varepsilon;\eta) = -\infty$ *for all* $\hat{\alpha}_\varepsilon \geq B(\eta)$.

The upper bound $B(\eta)$ arises because if the agent's choice of $\hat{\alpha}_\varepsilon$ was too high, her asset position in period 1 would be very extreme (recall (3)) on average. As a result, the integrand in (7) would have a fat negative tail, and the integral equals negative infinity. Note that there is no positive lower bound on $\hat{\alpha}_\varepsilon$ because agents are risk-averse: a small value of $\hat{\alpha}_\varepsilon$ inherently limits their asset holding.

By the theorem of maximum, for each $\eta \in [0,1)$, an optimal belief exists and lies in the interval $[0, B(\eta))$. Define the optimal belief correspondence for each level of price sensitivity by $\alpha^*(\eta)$, i.e.,

$$\alpha^*(\eta) = \arg\max_{\hat{\alpha}_\varepsilon \in [0, B(\eta)]} V(\hat{\alpha}_\varepsilon;\eta).$$

An intuition similar to that of Proposition 5 applies here: equilibrium disagreement and trade arises if and only if the weights on anticipatory utilities are large enough. More precisely,

Lemma 7 *There is a boundary on the* λ_0-λ_1 *plane such that if* (λ_0, λ_1) *falls below the boundary, then there exists an equilibrium in which* $\eta = \eta_R$ *and all agents hold rational belief. On the other hand, if* (λ_0, λ_1) *is above the boundary, then disagreement necessarily arises in equilibrium. Generically,* η_R *is NOT an equilibrium price sensitivity.*

According to the lemma, if the weights on anticipatory utilities are relatively small, it is an equilibrium for all agents to hold the objective belief $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$. In this case, no trade occurs and the equilibrium outcome coincide with the standard rational expectation model. On the other hand, if the weights on anticipatory utilities are large enough, it is no longer optimal to hold the objective belief. In this case, one group of agents holds belief $\hat{\alpha}^i_\varepsilon > \alpha_\varepsilon$; while another group holds belief $\hat{\alpha}^i_\varepsilon < \alpha_\varepsilon$. Following a positive public signal $s > 0$, the former group becomes over-optimistic about the state (relative to the objective mean) and holds a long position; whereas the latter group becomes over-pessimistic about the state and holds a short position. The reverse happens if $s < 0$. The equilibrium proportion of each group is such that the market always clears.

We devote the remainder of this section to investigate the price reaction to the public signal. We are particularly interested in how the direction of the price reaction depends on the objective signal informativeness α_ε. The main result is that if the objective signal informativeness α_ε is large enough, then the price underreacts to the public signal, i.e. $\eta < \eta_R$; if α_ε is small enough, then the price overreacts to the public signal, i.e., $\eta > \eta_R$. This can be viewed as the counterpart result of Brunnermeier and Parker (2005) that the skewness of the payoff distribution affects the asset price. The following lemma is intuitive and useful for understanding our main result.

Lemma 8 *(i) If* $\alpha_\varepsilon \in \alpha^*(\eta_R)$*, then there exists an equilibrium with* $\eta = \eta_R$.

(ii) If $\hat{\alpha}_\varepsilon > \alpha_\varepsilon$ *for all* $\hat{\alpha}_\varepsilon \in \alpha^*(\eta_R)$*, then there exists an equilibrium with* $\eta > \eta_R$.

(iii) If $\hat{\alpha}_\varepsilon < \alpha_\varepsilon$ *for all* $\hat{\alpha}_\varepsilon \in \alpha^*(\eta_R)$*, then there exists an equilibrium with* $\eta < \eta_R$.

Consider case (ii). If the price sensitivity is at the rational level, then it is optimal for all agents to over-estimate the signal informativeness. Consequently, all agents hold a long (short) position following a positive (negative) news. Therefore, in order to clear the market, the market price reacts by more than the rational level. The converse occurs if agents find it optimal to under-estimate the signal informativeness at the rational price sensitivity, as in case (iii) above.

To gain better intuition, consider the following numerical example with $\alpha_\omega = \alpha_\varepsilon = 1$, $\lambda_0 = 0.5$ and $\lambda_1 = 2$. The dotted curve in Figure 1 depicts the agent's well-being versus the choice of belief $\hat{\alpha}_\varepsilon$, fixing the price sensitivity at the rational level $\eta_R = 0.5$. Apparently, the optimal belief is $\hat{\alpha}_\varepsilon = 0$, i.e., the public signal is completely uninformative. At this price sensitivity, the market fails to clear: all agents holds a long (short) position following a negative (positive) public signal realization. In order to clear the market, the price sensitivity can go down, making it less appealing to "go against" the public signal. The solid curve in the figure depicts the agent's well-being at a price sensitivity of $\eta = 0.306$. At this level of η, there are two optimal beliefs: $\hat{\alpha}_\varepsilon = 0$ and $\hat{\alpha}_\varepsilon = 1.73$. Following the realization of a positive public signal, agents with belief $\hat{\alpha}_\varepsilon = 0$ hold a short position; whereas agents with belief $\hat{\alpha}_\varepsilon = 1.73$ hold a long position.

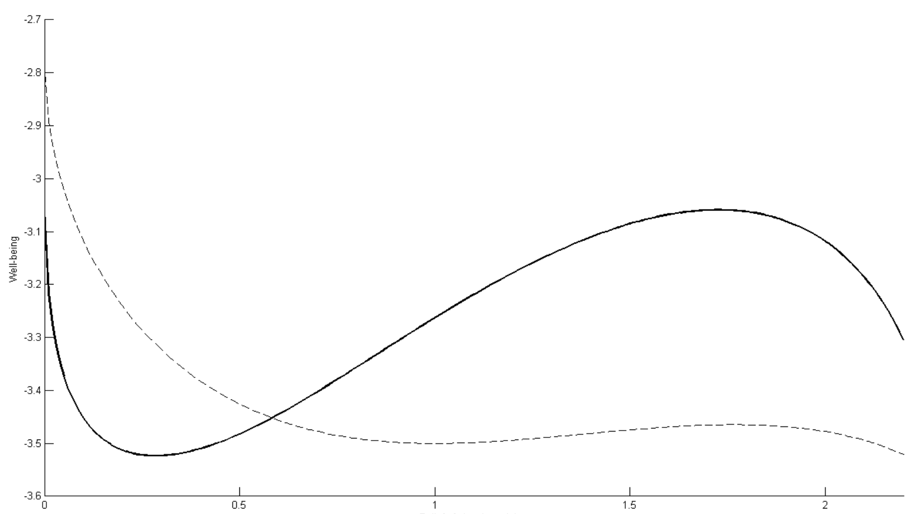

Figure 1: Well-being a function of subjective belief

We are now ready to state the main results:

Proposition 9 *(i) Suppose* $\lambda_0 > 0$ *or* $\lambda_1 > 0$*. There exists a* $\underline{\alpha}_\varepsilon \in (0, \infty)$ *such that if* $\alpha_\varepsilon > \underline{\alpha}_\varepsilon$*, the equilibrium price necessarily underreacts to the public signal.*

(ii) Suppose $\lambda_1 > 1$ *and* λ_0 *small relative to* λ_1*. There exists a* $\bar{\alpha}_\varepsilon \in (0, \infty)$ *such that if* $\alpha_\varepsilon < \bar{\alpha}_\varepsilon$*, the equilibrium price necessarily overreacts to the public signal.*

The intuition of the proposition is as follows. Suppose the realized signal is $s > 0$, and the period-1 price is rational. Therefore, after the signal arrival, the price and the objective expected value of the asset coincide and equal $\eta_R s$. Imagine for the time-being that an agent can choose the signal interpretation *after* the signal arrival. Recall the benefit of holding a belief different from the objective distribution is that the agent can now (mistakenly) perceive a positive expected gain from trade; whereas the cost of doing so is that she would make bad investment decision under that belief. The optimal interim belief is one that optimizes the associated tradeoff. If the agent chooses to over-estimate the signal precision, i.e., $\hat{\alpha}_\varepsilon > \alpha_\varepsilon$, she ends up having a subjective valuation for the asset in the interval $(\eta_R s, s)$. On the other hand, if the agent chooses to under-estimate the signal precision, i.e., $\hat{\alpha}_\varepsilon < \alpha_\varepsilon$, her subjective valuation is in the interval $[0, \eta_R s)$. Now if η_R is very large, the latter interval is much larger than the former interval, and the optimal belief is very likely to lie in the interval $[0, \eta_R s)$. Averaging over all $s \in \mathbb{R}$, the optimal ex-ante belief is to under-estimate the signal precision. Consequently, the market fails to clear, as all agents hold a short position after a positive signal, and a long position after a negative signal. To make it less appealing to under-estimate the signal precision, the price sensitivity must adjust downwards: an informative signal gives rise to an underreaction of market price.

Conversely, if $s > 0$, and η_R is very small, the interval $(\eta_R s, s)$ is much larger than the interval $[0, \eta_R s)$. On average, it is optimal to over-estimate the signal precision. To make it less appealing to over-estimate, the price sensitivity must adjust upwards. Therefore, an uninformative signal gives rise to an overreaction of market price.

Part (ii) of Proposition 9 requires λ_1 to be large relative to λ_0 for price overreaction. A precise sufficient condition on λ_1 and λ_0 is stated in the proof (see the appendix). The intuition for the requirement is that an increase in $\hat{\alpha}_\varepsilon$ beyond α_ε has two effects: (i) it lowers the perceived variance of the public signal realization from the perspective of period 0; and (ii) it raises the perceived gain from trade in period 1. The former effect lowers the anticipatory utility for period 0, whereas the latter effect raises the anticipatory utilities for both periods. In order that over-estimating the signal precision is optimal, it is necessary that the latter effect has a large enough impact relative to the former effect on the agent's well-being function.

The results reported above apply only in the limiting cases of very informative and uninformative signals. General analytical result is difficult to obtain because the optimum of the well-being function does not have a tractable solution. We can resort to numerical methods to trace how equilibrium price sensitivity responds to signal informativeness. The following figure plots the excess return (i.e., $\frac{\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}-\eta}{\eta}$) against the normalized signal precision (i.e., $\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}$) for the case $\lambda_1 = 1.2$ and $\lambda_0 = 0$.

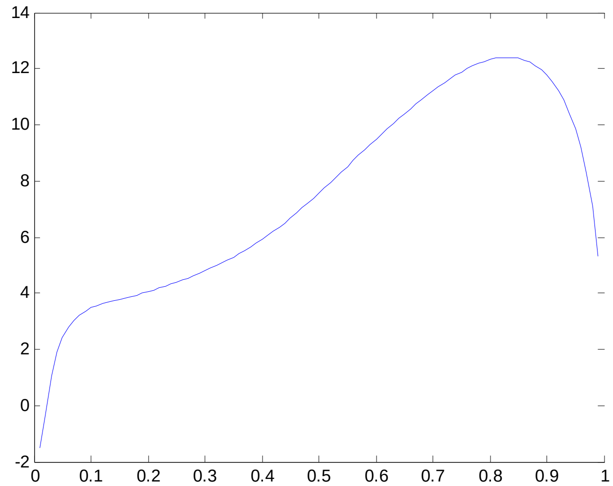

Figure 2: Excess Return versus Normalized Signal Precision

The figure shows that if the public signal is very uninformative, i.e., $\frac{\alpha_\varepsilon}{\alpha_\varepsilon + \alpha_\omega}$ is close to 0, then the excess return of investing in the risky asset is negative. There is a cutoff informativeness, around 0.02, beyond which the excess return becomes positive. This numerical example shows that the message of Proposition 9 is likely to hold beyond the limiting cases considered there.

Our final result concerns the volume of trade:

Corollary 10 *Suppose λ_0 and λ_1 are large so that equilibrium disagreement arises. The volume of trade is increasing in "signal content": i.e., an increase in the realization of $|s|$ increases the volume of trade.*

The result follows from Bayesian updating: the equilibrium magnitude of disagreement is proportional to the magnitude of the signal. Thus, a high value of $|s|$ results in a stronger disagreement, and consequently a larger position (long or short).

6 Conclusion

In this paper, we propose a theory of disagreement formation over the interpretation of public news events, based on the optimal expectation model proposed by Brunnermeier and Parker (2005). By imposing a reasonable structure on their model, we are able to derive interesting comparative statics results concerning the price and volume reaction to the informativeness of the public news events. There are a couple of natural extensions worth exploring. First, a dynamic model with multiple signal arrivals can generate novel insight on the pricing and volume dynamics. Second, considering multiple assets with correlated payoffs and signals may shed light on why some assets have a high price volatility, whereas others have relatively stable prices.

A Appendix

Proof of Proposition 3: We first compute the explicit analytical expression for the agent's well-being function $V\left(\hat{\sigma}^i,\eta\right)$, as defined in (9). In a symmetric equilibrium, the price vector P is fully characterized by the price sensitivity η. Therefore, with a slight abuse of notation, we replace the dependence of various functions on P with η. Moreover, we drop the superscript i to simplify notations.

Lemma 11 *The well-being of agent i with belief $\hat{\sigma}$ at price sensitivity η is given by $V\left(\hat{\sigma};\eta\right)\equiv\Gamma\left(\hat{\sigma};\eta\right)+\lambda_0\Psi_0\left(\hat{\sigma};\eta\right)+\lambda_1\Psi_1\left(\hat{\sigma};\eta\right)$, where*

$$\begin{aligned}
\Psi_0\left(\hat{\sigma};\eta\right) &= -\exp\left(-1\right)\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega}+\frac{1}{\hat{\alpha}_\varepsilon}\right)\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)^2+\eta^2\alpha_\omega\right)}}\exp\left(-\frac{\hat{\mu}_\varepsilon^2\hat{\alpha}_\varepsilon}{2}\right);\\
\Psi_1\left(\hat{\sigma};\eta\right) &= -\exp\left(-1\right)\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)\left(\alpha_\omega+\hat{\alpha}_\varepsilon\right)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}-\eta\right)^2+1}}\\
&\quad\times\exp\left(\frac{1}{2}\left(\frac{\left(\frac{\hat{\alpha}_\varepsilon\alpha_\omega}{\alpha_\omega+\hat{\alpha}_\varepsilon}\right)^2}{\left(\alpha_\omega+\hat{\alpha}_\varepsilon\right)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}-\eta\right)^2+\frac{\alpha_\omega\alpha_\varepsilon}{\alpha_\omega+\alpha_\varepsilon}}-\frac{\left(\hat{\alpha}_\varepsilon\right)^2}{\alpha_\omega+\hat{\alpha}_\varepsilon}\right)\left(\hat{\mu}_\varepsilon\right)^2\right);
\end{aligned}$$

and

$$\Gamma\left(\hat{\sigma};\eta\right)=\begin{cases}\begin{array}{l}-\exp\left(-1\right)\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{\left(\alpha_\varepsilon+\alpha_\omega\right)\left[\alpha_\varepsilon-2\eta\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)-\eta\alpha_\omega\right)\right]-\left[\alpha_\varepsilon-\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)-\eta\alpha_\omega\right)\right]^2}}\\ \exp\left(\frac{\left(\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\right)^2}{2}\frac{\left(1-\eta\right)^2\left(2\hat{\alpha}_\varepsilon+\alpha_\omega\right)+\eta^2\alpha_\omega}{\left(\alpha_\varepsilon+\alpha_\omega\right)\left[\alpha_\varepsilon-2\eta\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)-\eta\alpha_\omega\right)\right]-\left[\alpha_\varepsilon-\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)-\eta\alpha_\omega\right)\right]^2}\right)\end{array} & \text{if } \hat{\alpha}_\varepsilon<B\left(\eta\right)\\ -\infty & \text{otherwise}\end{cases}.$$

Here, the function $B\left(\eta\right)$ is defined in Lemma 6.

Proof. We first simplify the objective function in (6).

$$\begin{aligned}
&\int U\left(W_s\left(L;P\right),\hat{\sigma};P,s\right)d\hat{F}\left(s\right)\\
=&\int-\exp\left(-1-L\left(\eta s\right)-\frac{1}{2}\frac{\left(\widehat{E}_{\omega|s}\left[\omega\right]-P_s\right)^2}{\widehat{V}_{\omega|s}\left[\omega\right]}\right)d\hat{F}\left(s\right)\\
=&-\exp\left(-1\right)\frac{1}{\sqrt{2\pi\left(\frac{1}{\alpha_\omega}+\frac{1}{\hat{\alpha}_\varepsilon}\right)}}\int\exp\left(-L\left(\eta s\right)-\frac{\alpha_\omega+\hat{\alpha}_\varepsilon}{2}\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}\left(s-\hat{\mu}_\varepsilon\right)-\eta s\right)^2-\frac{\left(s-\hat{\mu}_\varepsilon\right)^2}{2\left(\frac{1}{\alpha_\omega}+\frac{1}{\hat{\alpha}_\varepsilon}\right)}\right)ds\\
=&-\exp\left(-1\right)\frac{1}{\sqrt{\left(\frac{1}{\alpha_\omega}+\frac{1}{\hat{\alpha}_\varepsilon}\right)\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)^2+\eta^2\alpha_\omega\right)}}\exp\left(\frac{\left(L\eta-\left(1-\eta\right)\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\right)^2}{2\left(\hat{\alpha}_\varepsilon\left(1-\eta\right)^2+\eta^2\alpha_\omega\right)}-\frac{1}{2}\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon^2\right).
\end{aligned}$$

The first two equalities follow from definition. The final equality makes use of the integration formula $\int\exp\left(-\left(As^2+Bs+C\right)\right)ds=\exp\left(\frac{B^2}{4A}-C\right)\sqrt{\frac{\pi}{A}}$ for constants $A>0$, B and C. Therefore, the optimal holding L in period 0 is given by

$$L\left(\hat{\sigma};\eta\right)=\frac{1-\eta}{\eta}\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon. \tag{12}$$

Using the definition in (6), we have

$$\Psi_0(\hat{\sigma};\eta) = -\exp(-1)\frac{1}{\sqrt{\left(\frac{1}{\alpha_\omega}+\frac{1}{\hat{\alpha}_\varepsilon}\right)\left(\hat{\alpha}_\varepsilon(1-\eta)^2+\eta^2\alpha_\omega\right)}}\exp\left(-\frac{1}{2}\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon^2\right).$$

Next, we compute $\Psi_1(\hat{\sigma};\eta)$.

$$\begin{aligned}
\Psi_1(\hat{\sigma};P) &\equiv \int U\left(W_s\left(L(\hat{\sigma};\eta)\right),\hat{\sigma};P,s\right)dF(s)\\
&= \frac{1}{\sqrt{2\pi\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)}}\int -\exp\left(-1-\eta sL(\hat{\sigma};\eta)-\frac{\hat{\alpha}_\varepsilon+\alpha_\omega}{2}\left(\left(\left(\frac{\hat{\alpha}_\varepsilon}{\hat{\alpha}_\varepsilon+\alpha_\omega}\right)(s-\hat{\mu}_\varepsilon)-\eta s\right)^2-\frac{s^2}{2\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)}\right)ds\\
&= -\frac{\exp\left(-1-\frac{1}{2}\frac{(\hat{\mu}_\varepsilon\hat{\alpha}_\varepsilon)^2}{\alpha_\omega+\hat{\alpha}_\varepsilon}\right)}{\sqrt{2\pi\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)}}\int\exp\left(-\frac{\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\alpha_\omega}{\alpha_\omega+\hat{\alpha}_\varepsilon}s-\frac{1}{2}\left((\alpha_\omega+\hat{\alpha}_\varepsilon)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}-\eta\right)^2+\frac{\alpha_\omega\alpha_\varepsilon}{\alpha_\omega+\alpha_\varepsilon}\right)s^2\right)ds\\
&= -\exp(-1)\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)(\alpha_\omega+\hat{\alpha}_\varepsilon)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}-\eta\right)^2+1}}\\
&\quad\times\exp\left(\frac{1}{2}\left(\frac{\left(\frac{\hat{\alpha}_\varepsilon\alpha_\omega}{\alpha_\omega+\hat{\alpha}_\varepsilon}\right)^2}{(\alpha_\omega+\hat{\alpha}_\varepsilon)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega+\hat{\alpha}_\varepsilon}-\eta\right)^2+\frac{\alpha_\omega\alpha_\varepsilon}{\alpha_\omega+\alpha_\varepsilon}}-\frac{(\hat{\alpha}_\varepsilon)^2}{\alpha_\omega+\hat{\alpha}_\varepsilon}\right)(\hat{\mu}_\varepsilon)^2\right)
\end{aligned}$$

The first equality is the definition in (8). The second equality follows from the definition of U in (2). The third equality follows from (12). The final equality makes use of the integration formula again.

Finally, we compute $\Gamma(\hat{\sigma};\eta)$. Recall the definition from (7). Consider the inner integral:

$$\begin{aligned}
\Gamma(\omega) &\equiv \int u\left(\Omega(\hat{\sigma},\eta;s,\omega)\right)dF(s|\omega)\\
&= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon}{2\pi}}\int\exp\left(-\ell_s(\hat{\sigma};\eta)(\omega-\eta s)-L(\hat{\sigma};\eta)\eta s\right)\exp\left(-\frac{\alpha_\varepsilon}{2}(s-\omega)^2\right)ds\\
&= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon}{2\pi}}\int\exp\left(-\left[\left(\hat{\alpha}_\varepsilon(s-\hat{\mu}_\varepsilon)-(\alpha_\omega+\hat{\alpha}_\varepsilon)\eta s\right)(\omega-\eta s)+(1-\eta)\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon s+\frac{\alpha_\varepsilon}{2}(s-\omega)^2\right]\right)ds\\
&= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon}{2\pi}}\\
&\quad\times\int\exp\left(-\left\{\left[-\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\omega+\frac{\alpha_\varepsilon}{2}\omega^2\right]+\left[\left(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon\right)\omega+\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\right]s+\left[\frac{\alpha_\varepsilon}{2}-\left(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta\right)\eta\right]s^2\right\}\right)ds
\end{aligned}$$

The second equality follow from definitions. The third equality follows from (12) and (3). Assume

$$\widehat{\alpha}_\varepsilon < \frac{1}{1-\eta}\left(\frac{\alpha_\varepsilon}{2\eta}+\eta\alpha_\omega\right), \tag{13}$$

so that the coefficient of s^2 is negative. The integral is then well-defined and $\Gamma(\omega)$ is given by

$$
\begin{aligned}
\Gamma(\omega) &= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon}{2\pi}}\sqrt{\frac{\pi}{\left[\frac{\alpha_\varepsilon}{2}-(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta\right]}}\exp\left(\frac{\left[(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon)\omega+\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\right]^2}{4\left[\frac{\alpha_\varepsilon}{2}-(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta\right]}-\left[-\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\omega+\frac{\alpha_\varepsilon}{2}\omega^2\right]\right) \\
&= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}}\exp\left(\begin{array}{c}\frac{1}{2}\left(\frac{(\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}\right)+\left(\frac{\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+1\right)\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\omega \\ -\frac{1}{2}\left(\alpha_\varepsilon-\frac{(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}\right)\omega^2\end{array}\right).
\end{aligned}
$$

Note that if (13) fails, then the integral and hence $\Gamma(\omega)$ equals $-\infty$. Now we evaluate the outer integral in the definition of $\Gamma(\hat{\sigma};\eta)$:

$$
\begin{aligned}
\Gamma(\hat{\sigma};\eta) &= \sqrt{\frac{\alpha_\omega}{2\pi}}\int\Gamma(\omega)\exp\left(-\frac{\alpha_\omega}{2}\omega^2\right)d\omega \\
&= -\exp(-1)\sqrt{\frac{\alpha_\omega}{2\pi}}\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}} \\
&\quad\times\int\exp\left(\begin{array}{c}\frac{1}{2}\left(\frac{(\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}\right)+\left(\frac{\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+1\right)\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon\omega \\ -\frac{1}{2}\left(\alpha_\varepsilon-\frac{(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+\alpha_\omega\right)\omega^2\end{array}\right)d\omega.
\end{aligned}
$$

The integral is well-defined if and only if the coefficient of ω^2 is negative, i.e.,

$$
\widehat{\alpha}_\varepsilon<\alpha_\varepsilon+\frac{\sqrt{(\alpha_\varepsilon+\alpha_\omega)\left(\eta^2\alpha_\omega+(1-\eta)^2\alpha_\varepsilon\right)}}{1-\eta}\equiv B(\eta). \tag{14}
$$

It is straightforward algebra to check that (14) implies (13). If $\hat{\alpha}_\varepsilon\geq B(\eta)$, then $\Gamma(\hat{\sigma};\eta)=-\infty$. If $\hat{\alpha}_\varepsilon<B(\eta)$, $\Gamma(\hat{\sigma};\eta)$ is given by

$$
\begin{aligned}
\Gamma(\hat{\sigma};\eta) &= -\exp(-1)\sqrt{\frac{\alpha_\omega}{2\pi}}\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}}\sqrt{\frac{\pi}{\alpha_\varepsilon-\frac{(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+\alpha_\omega}} \\
&\quad\exp\left(\frac{\left(\frac{\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+1\right)^2(\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon)^2}{2\left(\alpha_\varepsilon-\frac{(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta-\alpha_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}+\alpha_\omega\right)}+\frac{1}{2}\left(\frac{(\hat{\alpha}_\varepsilon\hat{\mu}_\varepsilon)^2}{\alpha_\varepsilon-2(\hat{\alpha}_\varepsilon(1-\eta)-\alpha_\omega\eta)\eta}\right)\right).
\end{aligned}
$$

Upon rearranging, it is equal to the expression stated in the lemma. ■

Several observations concerning the function $V(\hat{\sigma};\eta)$ is immediate. First, it is continuous in $\hat{\sigma}=(\hat{\mu}_\varepsilon,\hat{\alpha}_\varepsilon)$ on $\mathbb{R}\times[0,B(\eta)]$. Second, it is symmetric in $\hat{\mu}_\varepsilon$ around 0, as it depends on $\hat{\mu}_\varepsilon$ via $(\hat{\mu}_\varepsilon)^2$ only. Third, Γ approaches $-\infty$ as $\hat{\mu}_\varepsilon\to\infty$ or $-\infty$, so does V. Therefore, for each η, there exists an optimal belief, defined by

$$
\hat{\sigma}^*(\eta)\equiv(\hat{\mu}^*_\varepsilon(\eta),\hat{\alpha}^*_\varepsilon(\eta))\equiv\arg\max_{\hat{\sigma}'\in\mathbb{R}\times[0,B(\eta)]}V\left(\hat{\sigma}';\eta\right).
$$

Now we show equilibrium existence. We first show that there exists a η such that either (i) $\eta=\frac{x}{\alpha_\omega+x}$ for some $x\in\hat{\alpha}^*_\varepsilon(\eta)$; or (ii) $\eta\in\left(\frac{x_1}{x_1+\alpha_\omega},\frac{x_2}{x_2+\alpha_\omega}\right)$ for some $x_1,x_2\in\hat{\alpha}^*_\varepsilon(\eta)$. To see this holds, by the theorem of maximum, $\hat{\alpha}^*_\varepsilon(\eta)$ is upper semi-continuous. Convexify $\hat{\alpha}^*_\varepsilon(\eta)$ by $\xi(\eta)\equiv\left\{y:y=\lambda\frac{x_1}{x_1+\alpha_\omega}+(1-\lambda)\frac{x_2}{x_2+\alpha_\omega},\text{for some }\lambda\in[0,1],\text{ and }x_1,x_2\in\hat{\alpha}^*_\varepsilon(\eta)\right\}$.

Clearly, $\xi(\eta)$ is a mapping from $[0,1]$ to $[0,1]$, upper-semicontinuous, and convex-valued. Now we can apply the Kakutani's fixed point theorem to get that there exists a $\eta \in \xi(\eta)$. Either case (i) or (ii) above must hold.

To see the market clearing is achieved, suppose case (i) $\eta = \frac{x}{\alpha_\omega + x}$ for some $x \in \hat{\alpha}_\varepsilon^*(\eta)$ holds. Recall V is symmetric in $\hat{\mu}_\varepsilon$ around 0. Thus there exists a $y \geq 0$ such that $y, -y \in \hat{\mu}_\varepsilon^*(\eta)$. Suppose half of all agents hold belief $\left(y, \frac{\eta}{1-\eta}\alpha_\omega\right)$, while the other half holds belief $\left(-y, \frac{\eta}{1-\eta}\alpha_\omega\right)$. The period-0 position of the former group is $\alpha_\omega y$, whereas that of the latter group is $-\alpha_\omega y$. The period-1 position of the former group is $-\frac{\eta}{1-\eta}\alpha_\omega y$; whereas that of the latter group is $\frac{\eta}{1-\eta}\alpha_\omega y$. Market-clearing clearly holds.

Next consider case (ii): $\eta \in \left(\frac{x_1}{x_1+\alpha_\omega}, \frac{x_2}{x_2+\alpha_\omega}\right)$ for some $x_1, x_2 \in \hat{\alpha}_\varepsilon^*(\eta)$. As there exists a $y \geq 0$ such that $y, -y \in \hat{\mu}_\varepsilon^*(\eta)$. Now the following beliefs are optimal: (a) (y, x_1), (b) $(-y, x_1)$, (c) (y, x_2), and (d) $(-y, x_2)$. Suppose the respective proportions of each group in the populations are given by

$$\begin{aligned}
\beta(y, x_1) &= \beta(-y, x_1) = \frac{1}{2}\frac{1}{x_2 - x_1}\left(x_2 - \frac{\eta}{1-\eta}\alpha_\omega\right); \\
\beta(y, x_2) &= \beta(-y, x_2) = \frac{1}{2}\frac{1}{x_2 - x_1}\left(\frac{\eta}{1-\eta}\alpha_\omega - x_1\right).
\end{aligned}$$

To see the market clear in period 0, note that

$$\begin{aligned}
&\beta(y, x_1) L((y, x_1); \eta) + \beta(-y, x_1) L((-y, x_1); \eta) + \beta(y, x_2) L((y, x_2); \eta) + \beta(-y, x_2) L((-y, x_2); \eta) \\
= &\ \beta(y, x_1)\left(\frac{1-\eta}{\eta}x_1 y - \frac{1-\eta}{\eta}x_1 y\right) + \beta(y, x_2)\left(\frac{1-\eta}{\eta}x_2 y - \frac{1-\eta}{\eta}x_2 y\right) = 0.
\end{aligned}$$

To see the market clear in period 1 following the realization of s, note that

$$\begin{aligned}
&\beta(y, x_1)\ell_s((y, x_1); \eta) + \beta(-y, x_1)\ell_s((-y, x_1); \eta) + \beta(y, x_2)\ell_s((y, x_2); \eta) + \beta(-y, x_2)\ell_s((-y, x_2); \eta) \\
= &\ \frac{1}{2}\frac{1}{x_2 - x_1}\left(x_2 - \frac{\eta}{1-\eta}\alpha_\omega\right)\left(\{-x_1 y + [(1-\eta)x_1 - \eta\alpha_\omega]s\} + \{x_1 y + [(1-\eta)x_1 - \eta\alpha_\omega]s\}\right) \\
&+ \frac{1}{2}\frac{1}{x_2 - x_1}\left(\frac{\eta}{1-\eta}\alpha_\omega - x_1\right)\left(\{-x_2 y + [(1-\eta)x_2 - \eta\alpha_\omega]s\} + \{x_2 y + [(1-\eta)x_2 - \eta\alpha_\omega]s\}\right) \\
= &\ 0.
\end{aligned}$$

Proof of Lemma 4: Substituting $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$ into the formula in Lemma 11 yields:

$$\begin{aligned}
\Psi_0(\hat{\mu}_\varepsilon; \eta) &= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{(\alpha_\varepsilon + \alpha_\omega)\left((1-\eta)^2\alpha_\varepsilon + \eta^2\alpha_\omega\right)}}\exp\left(-\frac{\hat{\mu}_\varepsilon^2\alpha_\varepsilon}{2}\right); \\
\Psi_1(\hat{\mu}_\varepsilon; \eta) &= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{(\alpha_\varepsilon + \alpha_\omega)\left((1-\eta)^2\alpha_\varepsilon + \eta^2\alpha_\omega\right)}}\exp\left(\frac{1}{2}\left(\frac{\alpha_\omega^2}{(\alpha_\omega + \alpha_\varepsilon)^2\left(\frac{\alpha_\varepsilon}{\alpha_\omega + \alpha_\varepsilon} - \eta\right)^2 + \alpha_\omega\alpha_\varepsilon} - 1\right)\frac{\alpha_\varepsilon^2}{\alpha_\omega + \alpha_\varepsilon}(\hat{\mu}_\varepsilon)^2\right); \\
\Gamma(\hat{\mu}_\varepsilon; \eta) &= -\exp(-1)\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{(\alpha_\varepsilon + \alpha_\omega)\left((1-\eta)^2\alpha_\varepsilon + \eta^2\alpha_\omega\right)}}\exp\left(\frac{1}{2}\frac{(1-\eta)^2(2\alpha_\varepsilon + \alpha_\omega) + \eta^2\alpha_\omega}{(\alpha_\varepsilon + \alpha_\omega)\left((1-\eta)^2\alpha_\varepsilon + \eta^2\alpha_\omega\right)}(\alpha_\varepsilon\hat{\mu}_\varepsilon)^2\right).
\end{aligned}$$

Clearly, each component of $V(\hat{\mu}_\varepsilon; \eta)$ is continuous and symmetric in $\hat{\mu}_\varepsilon$ around 0.

Proof of Proposition 5: The reason for the uniqueness of equilibrium price sensitivity $\eta = \eta_R$ is discussed in the text. It remains to show that equilibrium disagreement arises if and only if the weights λ_0 and λ_1 are large enough.

With $\eta = \eta_R \equiv \frac{\alpha_\varepsilon}{\alpha_\varepsilon + \alpha_\omega}$, the expressions from the proof of Lemma 4 simplify to

$$\begin{aligned}
\Gamma(\hat{\mu}_\varepsilon; \eta_R) &= -\exp(-1)\exp\left(\frac{1}{2}\alpha_\varepsilon (\hat{\mu}_\varepsilon)^2\right); \\
\Psi_0(\hat{\mu}_\varepsilon; \eta_R) &= -\exp(-1)\exp\left(-\frac{1}{2}\alpha_\varepsilon (\hat{\mu}_\varepsilon)^2\right); \\
\Psi_1(\hat{\mu}_\varepsilon; \eta_R) &= -\exp(-1)\exp\left(\frac{1}{2}\frac{\alpha_\varepsilon(\alpha_\omega - \alpha_\varepsilon)}{\alpha_\varepsilon + \alpha_\omega}(\hat{\mu}_\varepsilon)^2\right).
\end{aligned}$$

Recall $V(\hat{\mu}_\varepsilon; \eta_R) = \Gamma(\hat{\mu}_\varepsilon; \eta_R) + \lambda_0 \Psi_0(\hat{\mu}_\varepsilon; \eta_R) + \lambda_1 \Psi_1(\hat{\mu}_\varepsilon; \eta_R)$. Clearly, $V(\hat{\mu}_\varepsilon; \eta_R)$ depends on $\hat{\mu}_\varepsilon$ only through the $(\hat{\mu}_\varepsilon)^2$ term. The optimal value of $\hat{\mu}_\varepsilon$ is different from 0 if $\frac{d}{d(\hat{\mu}_\varepsilon)^2} V(\hat{\mu}_\varepsilon; \eta_R)$ is positive evaluated at $\hat{\mu}_\varepsilon = 0$. Straightforward algebra shows this is equivalent to the formula in the proposition statement.

Proof of Lemma 6: Substituting $\hat{\mu}_\varepsilon = 0$ into the formula in Lemma 11 yields:

$$\begin{aligned}
\Gamma(\hat{\alpha}_\varepsilon; \eta) &= \begin{cases} -\exp(-1)\sqrt{\frac{\alpha_\varepsilon \alpha_\omega}{[\alpha_\varepsilon - 2\eta(\hat{\alpha}_\varepsilon - \eta(\alpha_\omega + \hat{\alpha}_\varepsilon))](\alpha_\varepsilon + \alpha_\omega) - [\alpha_\varepsilon - (\hat{\alpha}_\varepsilon - \eta(\alpha_\omega + \hat{\alpha}_\varepsilon))]^2}} & \text{if } \hat{\alpha}_\varepsilon < B(\eta) \\ -\infty & \text{otherwise} \end{cases}; \\
\Psi_0(\hat{\alpha}_\varepsilon; \eta) &= -\exp(-1)\sqrt{\frac{1}{\frac{1}{\alpha_\omega \hat{\alpha}_\varepsilon}(\hat{\alpha}_\varepsilon - \eta(\alpha_\omega + \hat{\alpha}_\varepsilon))^2 + 1}}; \\
\Psi_1(\hat{\alpha}_\varepsilon; \eta) &= -\exp(-1)\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega} + \frac{1}{\alpha_\varepsilon}\right)(\alpha_\omega + \hat{\alpha}_\varepsilon)\left(\frac{\hat{\alpha}_\varepsilon}{\alpha_\omega + \hat{\alpha}_\varepsilon} - \eta\right)^2 + 1}}.
\end{aligned}$$

Clearly, the well-being, defined by $V(\hat{\alpha}_\varepsilon; \eta) \equiv \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta)$ is continuous is $\hat{\alpha}_\varepsilon$.

Proof of Lemma 7: First, we define

$$\tilde{V}(\hat{\alpha}_\varepsilon; \eta; \lambda_0, \lambda_1) \equiv \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta).$$

Straightforward computation shows that $\Gamma(\hat{\alpha}_\varepsilon; \eta)$ has a single peak at $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$; whereas $\Psi_0(\hat{\alpha}_\varepsilon; \eta)$ and $\Psi_1(\hat{\alpha}_\varepsilon; \eta)$ has a single trough at $\hat{\alpha}_\varepsilon = \frac{\eta}{1-\eta}\alpha_\omega$.

Suppose $\eta = \eta_R$ and $\lambda_0 = \lambda_1 = 0$. The strict optimal belief is $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$. In this case, no disagreement arises and no trade occurs. The market clears and there is an equilibrium. By the continuity of $\tilde{V}$ in (λ_0, λ_1), the optimality of objective belief α_ε remains true as long as (λ_0, λ_1) is close to $(0, 0)$.

Next, $\tilde{V}(\hat{\alpha}_\varepsilon; \eta_R; \lambda_0, \lambda_1)$ does NOT peak at some $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$ whenever

$$\begin{aligned}
&\tilde{V}(\alpha_\varepsilon; \eta_R; \lambda_0, \lambda_1) < \tilde{V}(0; \eta_R; \lambda_0, \lambda_1) \\
\Leftrightarrow \quad &\lambda_0 + \lambda_1\left(1 - \sqrt{\frac{\alpha_\varepsilon + \alpha_\omega}{2\alpha_\varepsilon + \alpha_\omega}}\right) > \sqrt{\frac{(\alpha_\varepsilon + \alpha_\omega)^2}{\alpha_\varepsilon^2 + \alpha_\varepsilon \alpha_\omega + \alpha_\omega^2}} - 1.
\end{aligned}$$

Now suppose (λ_0, λ_1) is such that $\tilde{V}(\hat{\alpha}_\varepsilon; \eta_R; \lambda_0, \lambda_1)$ peaks at some $\hat{\alpha}_\varepsilon \neq \alpha_\varepsilon$. In this case, equilibrium disagreement necessarily arises. Suppose not. Market-clearing in period 1 requires all agents hold belief equal to $\frac{\eta}{1-\eta}\alpha_\omega$, where η is the equilibrium price sensitivity.[11] Consequently, by assumption, $\eta \neq \eta_R$ and $\frac{\eta}{1-\eta}\alpha_\omega \neq \alpha_\varepsilon$. However, the belief $\frac{\eta}{1-\eta}\alpha_\omega$ is dominated by the belief α_ε; all of $\Gamma(\hat{\alpha}_\varepsilon; \eta)$, $\Psi_0(\hat{\alpha}_\varepsilon; \eta)$, and $\Psi_1(\hat{\alpha}_\varepsilon; \eta)$ are strictly higher with $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$ than $\hat{\alpha}_\varepsilon = \frac{\eta}{1-\eta}\alpha_\omega$. A contradiction.

Finally, observe that if (λ_0, λ_1) is such that $\tilde{V}(\hat{\alpha}_\varepsilon; \eta_R; \lambda_0, \lambda_1)$ peaks at some $\hat{\alpha}_\varepsilon \neq \alpha_\varepsilon$, then so does all $(\lambda_0', \lambda_1') \geq (\lambda_0, \lambda_1)$.

Proof of Lemma 8: Part (i) is immediate. Part (iii) is symmetric to part (ii), so its proof is omitted. Define

$$\begin{aligned}
\tilde{V}_h(\eta; \lambda_0, \lambda_1) &\equiv \max_{\hat{\alpha}_\varepsilon \in \left[\frac{\eta}{1-\eta}\alpha_\omega, B(\eta)\right]} \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta); \\
\tilde{V}_l(\eta; \lambda_0, \lambda_1) &\equiv \max_{\hat{\alpha}_\varepsilon \in \left[0, \frac{\eta}{1-\eta}\alpha_\omega\right]} \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta).
\end{aligned}$$

Intuitively, $\tilde{V}_h$ ($\tilde{V}_l$) is the optimal well-being of over-estimating (under-estimating) the signal precision (with respect to the level implied by the market price $\frac{\eta}{1-\eta}\alpha_\omega$). Note that each of Γ, Ψ_0 and Ψ_1 are continuous in $(\hat{\alpha}_\varepsilon, \eta)$ (recall the expressions in the proof of Lemma 6). By the theorem of maximum, $\tilde{V}_h$ and $\tilde{V}_l$ are well-defined and continuous in η.

Next, note also that η is an equilibrium price sensitivity if and only if $\tilde{V}_h(\eta; \lambda_0, \lambda_1) = \tilde{V}_l(\eta; \lambda_0, \lambda_1)$. To see this, suppose $\tilde{V}_h(\eta; \lambda_0, \lambda_1) = \tilde{V}_l(\eta; \lambda_0, \lambda_1)$ and select a pair $\left(\hat{\alpha}^*_{\varepsilon,h}, \hat{\alpha}^*_{\varepsilon,l}\right)$ such that

$$\begin{aligned}
\hat{\alpha}^*_{\varepsilon,h} &\in \arg\max_{\hat{\alpha}_\varepsilon \in \left[\frac{\eta}{1-\eta}\alpha_\omega, B(\eta)\right]} \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta); \\
\hat{\alpha}^*_{\varepsilon,l} &\in \arg\max_{\hat{\alpha}_\varepsilon \in \left[0, \frac{\eta}{1-\eta}\alpha_\omega\right]} \Gamma(\hat{\alpha}_\varepsilon; \eta) + \lambda_0 \Psi_0(\hat{\alpha}_\varepsilon; \eta) + \lambda_1 \Psi_1(\hat{\alpha}_\varepsilon; \eta).
\end{aligned}$$

Define the measure of agents with belief $\hat{\alpha}^*_{\varepsilon,h}$ by

$$\beta\left(\hat{\alpha}^*_{\varepsilon,h}\right) = -\frac{(1-\eta)\hat{\alpha}^*_{\varepsilon,l} - \eta\alpha_\omega}{(1-\eta)\left(\hat{\alpha}^*_{\varepsilon,h} - \hat{\alpha}^*_{\varepsilon,l}\right)} \text{ and } \beta\left(\hat{\alpha}^*_{\varepsilon,l}\right) = 1 - \beta\left(\hat{\alpha}^*_{\varepsilon,h}\right).$$

It is straightforward to check that the market clearing condition is satisfied.

Now suppose case (ii) arises: $\hat{\alpha}_\varepsilon > \alpha_\varepsilon$ for all $\hat{\alpha}_\varepsilon \in \alpha^*(\eta_R)$, or equivalently $\tilde{V}_h(\eta_R; \lambda_0, \lambda_1) > \tilde{V}_l(\eta_R; \lambda_0, \lambda_1)$. To see that there exists a $\eta \in (\eta_R, 1]$ such that $\tilde{V}_h(\eta; \lambda_0, \lambda_1) = \tilde{V}_l(\eta; \lambda_0, \lambda_1)$, it suffices to show that $\tilde{V}_h(1; \lambda_0, \lambda_1) < \tilde{V}_l(1; \lambda_0, \lambda_1)$.[12] Direct computation shows that both $\Psi_0(\hat{\alpha}_\varepsilon; 1)$ and $\Psi_1(\hat{\alpha}_\varepsilon; 1)$ are decreasing in $\hat{\alpha}_\varepsilon$. Moreover, $\Gamma(\hat{\alpha}_\varepsilon; 1)$ peaks at $\hat{\alpha}_\varepsilon = \alpha_\varepsilon$. Therefore, $\lim_{\hat{\alpha}_\varepsilon \to \infty} V(\hat{\alpha}_\varepsilon; 1) < V(\alpha_\varepsilon; 1)$ which immediately implies $\tilde{V}_h(1; \lambda_0, \lambda_1) < \tilde{V}_l(1; \lambda_0, \lambda_1)$.

[11] This ensures no trade occurs in equilibrium.

[12] The result then follows from the intermediate value theorem.

Proof of Proposition 9: (i) First, we show that if α_ε is large enough, we have $\tilde{V}_h(\eta_R;\lambda_0,\lambda_1) < \tilde{V}_l(\eta_R;\lambda_0,\lambda_1)$. A lower bound for $\tilde{V}_l(\eta_R;\lambda_0,\lambda_1)$ is given by

$$\Gamma(0;\eta_R) + \lambda_0\Psi_0(0;\eta_R) + \lambda_1\Psi_1(0;\eta_R) = -\exp(-1)\left(\sqrt{\frac{(\alpha_\varepsilon+\alpha_\omega)^2}{\alpha_\varepsilon^2+\alpha_\varepsilon\alpha_\omega+\alpha_\omega^2}} + \lambda_1\sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}}\right). \tag{15}$$

An upper bound for $\tilde{V}_h(\eta_R;\lambda_0,\lambda_1)$ is given by

$$\begin{aligned}
&\Gamma(\eta_R;\eta_R) + \lambda_0\Psi_0(B(\eta_R);\eta_R) + \lambda_1\Psi_1(B(\eta_R);\eta_R)\\
= &-\exp(-1)\left(1+\lambda_0\sqrt{\frac{\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}} + \lambda_1\sqrt{\frac{1+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}}\right).
\end{aligned} \tag{16}$$

A sufficient condition for $\tilde{V}_h(\eta_R;\lambda_0,\lambda_1) < \tilde{V}_l(\eta_R;\lambda_0,\lambda_1)$ is thus

$$\begin{aligned}
&\sqrt{\frac{(\alpha_\varepsilon+\alpha_\omega)^2}{\alpha_\varepsilon^2+\alpha_\varepsilon\alpha_\omega+\alpha_\omega^2}} + \lambda_1\sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}} < 1+\lambda_0\sqrt{\frac{\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}} + \lambda_1\sqrt{\frac{1+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}}\\
\Leftrightarrow\ &\lambda_1\left(\sqrt{\frac{1+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2+\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}} - \sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}}\right) + \lambda_0\sqrt{\frac{\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}{2\alpha_\varepsilon+(\alpha_\varepsilon+\alpha_\omega)\sqrt{\frac{\alpha_\varepsilon}{\alpha_\omega}}}} > \sqrt{\frac{(\alpha_\varepsilon+\alpha_\omega)^2}{\alpha_\varepsilon^2+\alpha_\varepsilon\alpha_\omega+\alpha_\omega^2}} - 1.
\end{aligned} \tag{17}$$

As $\alpha_\varepsilon \to \infty$, the left-hand side of the inequality approaches $\lambda_1\left(1-\sqrt{\frac{1}{2}}\right) + \lambda_0 > 0$, whereas the right-hand side approaches 0. Therefore, there exists a $\underline{\alpha}_\varepsilon$ such that for all $\alpha_\varepsilon > \underline{\alpha}_\varepsilon$, inequality (17) holds. By Lemma 8, there exists an equilibrium price sensitivity $\eta < \eta_R$.

Next, we show that if α_ε is large enough, there is no $\eta > \eta_R$ such that $\tilde{V}_h(\eta;\lambda_0,\lambda_1) > \tilde{V}_l(\eta;\lambda_0,\lambda_1)$. A uniform lower bound for $\tilde{V}_l(\eta;\lambda_0,\lambda_1)$ (over $\eta \geq \eta_R$) is given by

$$\begin{aligned}
&\min_{\eta\geq\eta_R}\left[\Gamma(0;\eta) + \lambda_0\Psi_0(0;\eta) + \lambda_1\Psi_1(0;\eta)\right]\\
= &-\exp(-1)\max_{\eta\geq\eta_R}\left[\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon(1-\eta)^2+\eta^2(\alpha_\varepsilon+\alpha_\omega)}} + \lambda_1\sqrt{\frac{1}{\left(1+\frac{\alpha_\omega}{\alpha_\varepsilon}\right)\eta^2+1}}\right]\\
= &-\exp(-1)\left[\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon(1-\eta_R)^2+\eta_R^2(\alpha_\varepsilon+\alpha_\omega)}} + \lambda_1\sqrt{\frac{1}{\left(1+\frac{\alpha_\omega}{\alpha_\varepsilon}\right)\eta_R^2+1}}\right]\\
= &-\exp(-1)\left(\sqrt{\frac{(\alpha_\varepsilon+\alpha_\omega)^2}{\alpha_\varepsilon^2+\alpha_\varepsilon\alpha_\omega+\alpha_\omega^2}} + \lambda_1\sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}}\right).
\end{aligned}$$

Note that this coincide with (15). A uniform upper bound for $\tilde{V}_h(\eta;\lambda_0,\lambda_1)$ (over $\eta \geq \eta_R$) is given by

$$
\begin{aligned}
&\max_{\eta \geq \eta_R}\left[\Gamma(\eta;\eta)+\lambda_0\Psi_0(B(\eta);\eta)+\lambda_1\Psi_1(B(\eta);\eta)\right]\\
=\;& -\exp(-1)\min_{\eta\geq\eta_R}\left[1+\lambda_0\sqrt{\frac{1}{\frac{1}{\alpha_\omega B(\eta)}\left[B(\eta)-(\alpha_\omega+B(\eta))\eta\right]^2+1}}+\lambda_1\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)\frac{[B(\eta)-(\alpha_\omega+B(\eta))\eta]^2}{(\alpha_\omega+B(\eta))}+1}}\right].
\end{aligned}
$$

We claim that the term in the square bracket is minimized at η_R. Observe that $B(\eta)$ is increasing in η. Thus, it suffices to show that $B(\eta)-(\alpha_\omega+B(\eta))\eta$ is decreasing in η. To see this, note that

$$
\begin{aligned}
&B(\eta)-(\alpha_\omega+B(\eta))\eta\\
=\;&(1-\eta)\left(\alpha_\varepsilon+\frac{\sqrt{(\alpha_\varepsilon+\alpha_\omega)\left(\eta^2\alpha_\omega+(1-\eta)^2\alpha_\varepsilon\right)}}{1-\eta}\right)-\alpha_\omega\eta\\
=\;&\left(\sqrt{\left(\eta^2\alpha_\omega+(1-\eta)^2\alpha_\varepsilon\right)}-\sqrt{\left(\eta-\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}\right)^2(\alpha_\varepsilon+\alpha_\omega)}\right)\sqrt{(\alpha_\varepsilon+\alpha_\omega)}\\
=\;&\left(\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{\alpha_\varepsilon+\alpha_\omega}+\left(\eta-\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}\right)^2(\alpha_\varepsilon+\alpha_\omega)}-\sqrt{\left(\eta-\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}\right)^2(\alpha_\varepsilon+\alpha_\omega)}\right)\sqrt{(\alpha_\varepsilon+\alpha_\omega)}.
\end{aligned}
$$

As $\eta \geq \frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}$ and $\sqrt{x}$ is a concave function, the final line is decreasing in η. Consequently, a uniform upper bound for $\tilde{V}_h(\eta;\lambda_0,\lambda_1)$ is exactly (16). Recall whenever $\alpha_\varepsilon > \underline{\alpha}_\varepsilon$, inequality (17) holds. In this case, for all $\eta > \eta_R$, $\tilde{V}_l(\eta;\lambda_0,\lambda_1) > \tilde{V}_h(\eta;\lambda_0,\lambda_1)$, and there is no $\eta > \eta_R$ that constitutes an equilibrium.

(ii) Suppose $\lambda_1 > 1$ and λ_0 small relative to λ_1. We first show that if α_ε is sufficiently small, then we have $\tilde{V}_l(\eta_R;\lambda_0,\lambda_1) < \tilde{V}_h(\eta_R;\lambda_0,\lambda_1)$. An upper bound for $\tilde{V}_l(\eta_R;\lambda_0,\lambda_1)$ is given by

$$
\Gamma(\alpha_\varepsilon;\eta_R)+\lambda_0\Psi_0(0;\eta_R)+\lambda_1\Psi_1(0;\eta_R) = -\exp(-1)\left(1+\lambda_1\sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}}\right).
$$

A lower bound for $\tilde{V}_h(\eta_R;\lambda_0,\lambda_1)$ is given by

$$
\begin{aligned}
&\Gamma\left(\frac{\alpha_\omega\left(\eta_R+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta_R};\eta_R\right)+\lambda_0\Psi_0\left(\frac{\alpha_\omega\left(\eta_R+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta_R};\eta_R\right)+\lambda_1\Psi_1\left(\frac{\alpha_\omega\left(\eta_R+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta_R};\eta_R\right)\\
=\;&-\sqrt{\frac{1}{1-\kappa}}-\lambda_0\sqrt{\frac{1}{\frac{1}{1+\left(\frac{\alpha_\omega}{\alpha_\varepsilon}+1\right)\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\kappa+1}}-\lambda_1\sqrt{\frac{1}{\left(\frac{1}{1+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\right)\kappa+1}}. \qquad (18)
\end{aligned}
$$

A sufficient condition for $\tilde{V}_l(\eta_R;\lambda_0,\lambda_1) < \tilde{V}_h(\eta_R;\lambda_0,\lambda_1)$ is thus

$$-1-\lambda_1\sqrt{\frac{\alpha_\varepsilon+\alpha_\omega}{2\alpha_\varepsilon+\alpha_\omega}} < -\sqrt{\frac{1}{1-\kappa}}-\lambda_0\sqrt{\frac{1}{\frac{1}{1+\left(\frac{\alpha_\omega}{\alpha_\varepsilon}+1\right)\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\kappa+1}}-\lambda_1\sqrt{\frac{1}{\left(\frac{1}{1+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\right)\kappa+1}}$$

$$\Leftrightarrow \lambda_1\left(\sqrt{\frac{1+\frac{\alpha_\omega}{\alpha_\varepsilon}}{2+\frac{\alpha_\omega}{\alpha_\varepsilon}}}-\sqrt{\frac{1}{\left(\frac{1}{1+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\right)\kappa+1}}\right)-\lambda_0\sqrt{\frac{1}{\frac{1}{1+\left(\frac{\alpha_\omega}{\alpha_\varepsilon}+1\right)\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\kappa+1}} > \sqrt{\frac{1}{1-\kappa}}-1.$$

Taking limit as $\alpha_\varepsilon\to 0$, the left-hand side of the inequality is

$$\lambda_1\left(1-\sqrt{\frac{1}{\kappa+1}}\right)-\lambda_0.$$

Thus, if (λ_0,λ_1) are such that

$$\max_{k\in[0,1]}\lambda_1\left(1-\sqrt{\frac{1}{\kappa+1}}\right)-\left(\sqrt{\frac{1}{1-\kappa}}-1\right) > \lambda_0, \tag{19}$$

then there exists a $\bar{\alpha}_\varepsilon$ such that for all $\alpha_\varepsilon<\bar{\alpha}_\varepsilon$, we have $\tilde{V}_l(\eta_R;\lambda_0,\lambda_1) < \tilde{V}_h(\eta_R;\lambda_0,\lambda_1)$. By Lemma 8, there exists an equilibrium price sensitivity $\eta>\eta_R$.

Next, we show that whenever (19) holds, then there exists a $\bar{\alpha}'_\varepsilon$ such that for all $\alpha_\varepsilon<\bar{\alpha}'_\varepsilon$, we have $\tilde{V}_l(\eta;\lambda_0,\lambda_1) < \tilde{V}_h(\eta;\lambda_0,\lambda_1)$ for all $\eta\le\eta_R$.

A uniform upper bound for $\tilde{V}_l(\eta;\lambda_0,\lambda_1)$ (over $\eta\le\eta_R$) is given by

$$\Gamma(\alpha_\varepsilon;\eta)+\lambda_0\Psi_0(0;\eta)+\lambda_1\Psi_1(0;\eta) = -\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{(\alpha_\varepsilon+\alpha_\omega)\left((1-\eta)^2\alpha_\varepsilon+\eta^2\alpha_\omega\right)}}-\lambda_1\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\eta^2(\alpha_\varepsilon+\alpha_\omega)}}$$

$$\le\quad -\sqrt{\frac{\alpha_\varepsilon\alpha_\omega}{(\alpha_\varepsilon+\alpha_\omega)(\alpha_\varepsilon)}}-\lambda_1\sqrt{\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\left(\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}\right)^2(\alpha_\varepsilon+\alpha_\omega)}} = -\sqrt{\frac{\alpha_\omega}{\alpha_\varepsilon+\alpha_\omega}}-\lambda_1\sqrt{\frac{1}{1+\frac{\alpha_\varepsilon}{\alpha_\varepsilon+\alpha_\omega}}}.$$

The inequality above follows because the first term $\Gamma(\alpha_\varepsilon;\eta)$ is decreasing in η whereas the second term $\Psi_1(0;\eta)$ is increasing in η.

A uniform lower bound for $\tilde{V}_h(\eta;\lambda_0,\lambda_1)$ (over $\eta\le\eta_R$) is given by

$$\min_{\eta\le\eta_R}\Gamma\left(\frac{\alpha_\omega\left(\eta+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta};\eta\right)+\lambda_0\Psi_0\left(\frac{\alpha_\omega\left(\eta+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta};\eta\right)+\lambda_1\Psi_1\left(\frac{\alpha_\omega\left(\eta+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}\right)}{1-\eta};\eta\right)$$

$$\begin{aligned}=\quad&\min_{\eta\le\eta_R}-\sqrt{\frac{\sqrt{\alpha_\varepsilon}\alpha_\omega}{\left(\sqrt{\alpha_\varepsilon}-2\eta\sqrt{\alpha_\omega\kappa}\right)(\alpha_\varepsilon+\alpha_\omega)-\sqrt{\alpha_\varepsilon}\left(\sqrt{\alpha_\varepsilon}-\sqrt{\alpha_\omega\kappa}\right)^2}}\\&-\lambda_0\sqrt{\frac{1}{\frac{1-\eta}{\eta+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}+1}}-\lambda_1\sqrt{\frac{1}{\left(\frac{1}{\alpha_\omega}+\frac{1}{\alpha_\varepsilon}\right)\alpha_\varepsilon\kappa\frac{1-\eta}{1+\sqrt{\frac{\alpha_\varepsilon\kappa}{\alpha_\omega}}}+1}}.\end{aligned}$$

It is clear that the objective function is decreasing in η. Thus, the uniform lower bound is (18). A sufficient condition for $\tilde{V}_l(\eta; \lambda_0, \lambda_1) < \tilde{V}_h(\eta; \lambda_0, \lambda_1)$ for all $\eta \le \eta_R$ is thus

$$-\sqrt{\frac{\alpha_\omega}{\alpha_\varepsilon + \alpha_\omega}} - \lambda_1 \sqrt{\frac{1}{1 + \frac{\alpha_\varepsilon}{\alpha_\varepsilon + \alpha_\omega}}} < -\sqrt{\frac{1}{1-\kappa}} - \lambda_0 \sqrt{\frac{1}{\frac{1}{1 + \left(\frac{\alpha_\omega}{\alpha_\varepsilon} + 1\right)\sqrt{\frac{\alpha_\varepsilon \kappa}{\alpha_\omega}}}\kappa + 1}} - \lambda_1 \sqrt{\frac{1}{\left(\frac{1}{1 + \sqrt{\frac{\alpha_\varepsilon \kappa}{\alpha_\omega}}}\right)\kappa + 1}}.$$

Taking limit as $\alpha_\varepsilon \to 0$ and rearranging, the inequality coincides with (19).

Proof for Corollary 10: Applying (3) to the current setting, the holding of an agent with belief $\hat{\alpha}_\varepsilon$ following signal realization s is given by:

$$\ell_s(\hat{\alpha}_\varepsilon; \eta) = [\hat{\alpha}_\varepsilon - \eta(\alpha_\omega + \hat{\alpha}_\varepsilon)]\, s.$$

In an equilibrium (η, β) following signal s, the volume of trade is given by

$$\left| \int_{\hat{\alpha}_\varepsilon \ge \frac{\eta}{1-\eta}\alpha_\omega} \ell_s(\hat{\alpha}_\varepsilon; \eta)\, d\beta(\hat{\alpha}_\varepsilon) \right| = |s| \int |\hat{\alpha}_\varepsilon - \eta(\alpha_\omega + \hat{\alpha}_\varepsilon)|\, d\beta(\hat{\alpha}_\varepsilon),$$

which increases proportionally in $|s|$.

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