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Working Paper Series

'Bayesian School Choice: Welfare Comparison of Immediate Acceptance and Deferred Acceptance Mechanism'

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No. 03 / 2023 August 2023

BAYESIAN SCHOOL CHOICE: WELFARE COMPARISON OF IMMEDIATE ACCEPTANCE AND DEFERRED ACCEPTANCE MECHANISMS

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Abstract. Under incomplete information, we compare the welfare of two widely used school choice mechanisms, Deferred Acceptance (DA) and Immediate Acceptance (IA). Our main model involves three students and two schools. Each student's value vector for the two schools is independently drawn, and schools do not have priorities over students. We show that there is no general interim welfare domination in *any* asymmetric case; thus, the previous results in the literature are fragile. In fact, DA might interim-dominate BM in environments that are arbitrarily close to cases explored in the literature. Nonetheless, we establish that IA outperforms DA in terms of ex-ante welfare when each student's values are independently drawn across schools, regardless of the value distributions. Additionally, we investigate the case when values are interdependent across schools, analyze the effects of different tie-breaking methods on our main results, and explore a continuum model in a unitarian setting.

1. Introduction

In the two-sided matching literature, a "school choice mechanism" refers to a set of rules that govern how students are assigned to schools. Different school choice mechanisms may involve various rules for students and schools. The most commonly studied school choice mechanisms in this literature include the Deferred Acceptance (DA) mechanism, the Boston (also known as the "Immediate Acceptance (IA)") mechanism, and their variants.

Date: August 7, 2023.

Keywords: Deferred Acceptance, Immediate Acceptance, Boston Mechanism, Welfare comparison.

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The DA mechanism involves students submitting their preferences over schools, followed by a process in which schools provisionally (tentatively) offer spots to their most preferred students. A student accepted by a school in one step may be rejected later if another student with higher priority applies to the same school. As a result, assignments under DA only become permanent in the last step, which is why it is referred to as the "deferred" acceptance mechanism. Similarly, the IA mechanism requires students to submit their preferences over schools, followed by schools offering spots to the students they most prefer. However, in the IA mechanism, each acceptance during the process is immediate and final. Unlike the DA mechanism, the IA mechanism does not allow a student who has been accepted in a particular step to be rejected later.

This paper aims to compare the DA and IA mechanisms in terms of students' welfare within an incomplete information setting. The DA mechanism is well-known to be "strategy-proof", meaning that truthfully reporting school rankings is a weakly dominant strategy for students under this mechanism.[1] In contrast, students may benefit by misrepresenting their school rankings in the IA mechanism (see, for example, Abdulkadiroğlu and Sönmez (2003)). As a result, to calculate welfare under the IA mechanism, one must determine the equilibrium reports, which can be challenging in general setups.

We examine a simple model with two schools, each with a capacity of one, and three students. We add a realistic feature to the model by assuming that the two schools are ex-ante different, with students' private valuations for schools being drawn from distributions that differ across schools. This allows for a scenario where students are more likely to prefer school 1 over school 2 ex-ante, but not all students prefer school 1 over school 2 ex-post (in contrast to Abdulkadiroğlu et al. (2011)'s model). We assume that schools do not have priorities over students, and in our main model, tie-breaking is via a single lottery drawn uniformly at random.

To conduct a welfare comparison, we start by solving a symmetric equilibrium of the IA mechanism. We show that in the unique symmetric equilibrium,

[1]See Dubins and Freedman (1981) or Roth (1982), for example.

students use cutoff strategies, where reporting depends on the ratio of the valuations for two schools (Proposition 1). We then compare students' interim and ex-ante welfare under the IA mechanism with that under the DA mechanism. [2]

Our first surprising finding is that, except for the extreme cases studied in the literature, specifically when students' ordinal rankings are either perfectly uncorrelated or perfectly correlated, there is no interim dominance relation between the two mechanisms in general (Proposition 4). As we discuss in the literature review below, this result highlights that earlier findings of interim welfare domination of the IA mechanism over the DA mechanism are knife-edge results.

Given that there cannot be a general interim welfare dominance relation between the two mechanisms, we compare the ex-ante welfare of the IA mechanism with that of the DA mechanism. We show that the ex-ante welfare under the IA mechanism is *always* larger than that under the DA mechanism, regardless of the value distributions (Theorem 1).

We examine several modifications to the aforementioned model for the sake of robustness. First, we allow for values to be interdependent across schools. Similar to the independent-values setting, there is no general interim welfare dominance relation between IA and DA unless students' ordinal rankings over schools are equally likely. However, we show that there is always a type space in which the IA mechanism's interim welfare dominates that of the DA mechanism. Surprisingly, we also find that when the probability that a school is preferred to the other school is high enough, there is a type space for which the DA mechanism's interim welfare dominates that of the IA mechanism. This refutes the intuitive yet incorrect idea that IA performs better than DA when students have nearly perfectly correlated preferences.

Next, we examine an alternative tie-breaking rule known as "multiple tie-breaking," which has also been studied in the school choice literature. With multiple tie-breaking, each school's priority order over students is independently drawn from a uniform distribution. As a result, one school's priority order may differ from another school's, in contrast to the single tie-breaking approach. We establish two results. First, DA with single-tie breaking interim welfare dominates DA with multiple tie-breaking (Proposition 7) in our setup. Moreover, the

[2]The "interim" stage refers to the stage when each student knows their own type but not others' types, while the "ex-ante" stage refers to the stage when no uncertainties regarding student types are resolved.

tie-breaking method does not alter the random outcomes under the IA mechanism. Thus, our strong result of ex-ante welfare superiority of IA over DA in Theorem 1 continues to hold with this alternative tie-breaking method as well. Second, we show that the IA mechanism is ex-ante welfare superior to DA with multiple tie-breaking in terms of ex-ante welfare in any unitary setting, where the type space is such that the maximum value from a school is normalized to 1 (Proposition 9).

Finally, we explore a large market setup with a continuum of agents and slots. In this case, for any distribution of cardinal preferences, IA yields higher ex-ante welfare in any unitary setting.[3]

1.1. Related Literature. There is a literature that compares these mechanisms in terms of welfare under incomplete information. However, the comparison is restricted to two extreme cases. In one extreme, all ordinal preferences are equally likely—perfectly uncorrelated preferences. In the other extreme, all ordinal preferences are perfectly aligned across students—perfectly correlated preferences. The current paper tries to ascertain welfare differences as soon as we depart from these two extreme scenarios.

In a context with no prioritized students, Abdulkadiroğlu et al. (2011) show that, when *ordinal preferences over schools are perfectly aligned*, any Immediate Acceptance Bayes-Nash equilibrium outcome interim dominates the dominant-strategy allocation under Deferred Acceptance. As we discussed above, we show that this finding is indeed a knife-edge result. Our result Proposition 4 shows that even an arbitrarily small deviation from perfectly correlated preferences fails interim dominance. Additionally, Miralles (2008) goes further in stating that the Deferred Acceptance allocation, which in this case collapses to Even Randomization, is weakly interim dominated by every Bayes-Nash equilibrium outcome of any other mechanism. Miralles states that Deferred Acceptance is *abysmal*, or pessimal, when ordinal preferences are perfectly correlated among individuals.

Troyan (2012) has shown that Abdulkadiroğlu et al. (2011)'s dominance result is not robust to the introduction of coarse priority structures that are common

[3]For a normative and axiomatic justification of the unitarian approach, see Dhillon (1998), Dhillon and Mertens (1999), and, more recently, Borgers and Choo (2017).

to school choice in practice. However, he shows that, from an ex-ante perspective, that is, before knowing the priority-preference type, IA still dominates DA, restoring the positive result of IA.

Instead of introducing priorities, we focus on deviations from the assumption of perfect correlation of ordinal preferences. A reasonable hypothesis is that, under some continuity argument, IA would keep dominating DA, if not in the interim sense, at least in ex-ante terms. As in Troyan (2012), our Proposition 4 highlights the lack of robustness. Surprisingly, one could design distributions of vNM valuations arbitrarily close to a perfect alignment of ordinal preferences such that the outcome of DA interim dominates that of IA. In order to restore the positive result of IA, one possibility–not the only one–is to utilize independence, with some similarity to the result of Troyan (2012). In our case, we render the distribution of valuations for each school independent from each other, and we evaluate welfare ex-ante, before types are learned. This leads to our main result in Theorem 1.

On an opposite assumption on preferences, one could consider the case in which, ex-ante, *all ordinal preferences are equally possible*. Featherstone and Niederle (2016) show that the game induced by IA admits truth-telling as the unique Nash equilibrium. More related to our study, although they do not provide a formal proof, they present a discussion that demonstrates that the IA mechanism can interim dominate DA in an example, which they refer to as the "art and science schools example" , with three students and two one-seat schools (an art school and a science school) when each student is equally likely to prefer the art school or the science school and school priorities are determined via a common random lottery.[4] Our Proposition 4 highlights again that this observation is yet another knife-edge case. The interim dominance result fails when the distribution of valuations is not identical between schools.

In a similar setting to Featherstone and Niederle (2016), Akyol (2022a) shows that when each student's value for each school is independently drawn from an identical distribution, the Boston mechanism is ex-ante welfare superior to the DA mechanism for a large class of value distributions in the case of *three* schools (each with an equal number of available seats). On a different approach, Akyol

[4]Note that we also consider the same setting, except that ordinal rankings are not equally likely. This way, we formally prove that in Featherstone and Niederle (2016)'s setting, IA interim dominates DA.

(2022b) analyzes a growing market with n individuals and n one-slot schools, when n grows large. He finds that IA delivers higher welfare than DA in the limit.

The rest of the paper is organized as follows. First, we present our main model. Then, we analyze the IA mechanism and DA mechanisms in Section 3 and Section 4, respectively. In Section 5, we compare the two mechanisms in terms of welfare and present our main results. We then present discussions by changing some of the features of our main model in Section 6. Finally, we conclude.

2. Model

We consider an environment where there are two schools $\{c_1, c_2\}$, each with one seat; and three students $\{i_1, i_2, i_3\}$.[5] We denote the random variable that represents the value for c_1 by X and the random variable that represents the value for c_2 by Y. Generic realizations are represented by x and y. Furthermore, $X, Y \geq 0$ and an unassigned student receives a zero payoff. We consider a setting where each student's value for each school is independently drawn over a type space V and $\Pr(X = Y) = 0$.

For convenience, let $p \in (0, 1)$ denote the ex-ante probability that c_1 is ranked above c_2. That is, $p = \Pr(X \geq Y)$. Without loss of generality, we assume that $p \geq \frac{1}{2}$. That is, from an *ex-ante* perspective, c_1 is weakly more desirable than c_2.

In our main model, we assume that X is independently drawn from a continuous and increasing cumulative distribution function $F(\cdot)$ with the density function $f(\cdot) \equiv F'(\cdot)$ over $[0, 1]$, and Y is independently drawn from a continuous and increasing cumulative distribution function $G(\cdot)$ with the density function $g(\cdot) \equiv G'(\cdot)$ over $[0, 1]$.[6]

Furthermore, we assume that schools do not have priorities over students. In our main model, tie-breaking is via a single lottery drawn uniformly at random, referred to as "single tie-breaking."[7]

[5]In order to guarantee that each student is placed into some school, one may assume that there is a third school, say c_3, from which each student derives a value of 0. All of our results would go through for this case.

[6]Restricting the values in $[0, 1]$ interval is just a normalization. We allow for interdependent values in Section 6.1.

[7]In Section 6.2, we consider an alternative tie-breaking method, the so-called "multiple tie-breaking."

3. Immediate Acceptance (IA) Mechanism

Under the IA mechanism, the seats of each school are assigned according to the rank of students assigned to that school. That is, students who rank a school first are accepted first, followed by those who rank it second only when seats are available, and so on. Those who rank a school the same are assigned in the order of their priorities at that school. Each acceptance is final, i.e., a student who is accepted to a school at any step, is permanently assigned to that school under IA. The algorithm stops when all seats of each school are filled, or there are no unassigned students.

Consider the preference revelation game induced by the IA mechanism: each student learns their type, i.e. values for each school, and reports a strict ranking over schools. Formally, a student's strategy under the IA mechanism is a function $\beta : V \to \mathcal{R}$, where $\mathcal{R}$ is the set of all possible strict rankings over schools.[8] Then, according to the pre-determined school priorities based on a single tie-breaking lottery and student reports, students are assigned to schools by using the IA mechanism.

We solve for the symmetric Bayesian Nash equilibrium under the IA mechanism. First, we establish that each student's best response to any strategy of other students must be of the "cut-off" form:

Lemma 1. *Student i's best response to any strategy of other students is of the following form: Student i with type (x, y) reports c_1 as a first choice iff $x \geq k_i y$ for some constant k_i.*

Proof. See Appendix A.1.1. □

We next establish that there is a unique symmetric equilibrium and give a characterization of the unique equilibrium.

Proposition 1. *There is a unique symmetric equilibrium under the IA mechanism. In the unique equilibrium, a student with type (x, y) reports c_1 as a first choice iff $x \geq ky$, where $k = \frac{1+q}{2-q}$ and q is the probability of ranking c_1 as a top choice in equilibrium and it satisfies*

$$q = \Pr\left(X \geq \frac{1+q}{2-q} Y\right). \tag{1}$$

[8]When there are two schools, $\mathcal{R}$ consists of two rankings: (c_1, c_2) and (c_2, c_1), where (c, c') represents that c is ranked first, and c' is ranked second.

Proof. See Appendix A.1.2. □

In the proof, we show that there is always a unique $q \in [\frac{1}{2}, 1)$ that satisfies the equilibrium condition in (1). Furthermore, we also show in the proof that $q \leq p$, and equality holds iff $q = \frac{1}{2}$. Hence, we can characterize an equilibrium by q (or cutoff level k) that satisfies the equilibrium condition.

Let $P^{IA}(x,y) = \left(P_1^{IA}(x,y), P_2^{IA}(x,y)\right)$, where $P_i^{IA}(x,y)$ denotes the interim probability that a student with type (x,y) is assigned to c_i under the unique symmetric equilibrium in the IA mechanism. Similarly, let $u^{IA}(x,y)$ denote the (interim) expected utility of a student with type (x,y) under the unique symmetric equilibrium of the IA mechanism.[9]

Proposition 2. *In the unique symmetric equilibrium of the IA mechanism, we have*

$$P^{IA}(x,y) = \begin{cases} \left(\frac{q^2-3q+3}{3}, \frac{q^2}{3}\right) & \textit{if} \quad x \geq ky \\ \left(\frac{q^2-2q+1}{3}, \frac{q^2+q+1}{3}\right) & \textit{if} \quad ky > x \end{cases}.$$

Hence,

$$u^{IA}(x,y) = \begin{cases} \frac{q^2-3q+3}{3}x + \frac{q^2}{3}y & \textit{if} \quad x \geq ky \\ \frac{q^2-2q+1}{3}x + \frac{q^2+q+1}{3}y & \textit{if} \quad ky > x \end{cases}.$$

Proof. See Appendix A.1.3 □

4. Deferred Acceptance (DA) Mechanism

In the first step of the DA, students apply to their first choice, and each school *tentatively* accepts students up to its capacity following its priority order over students, and reject the others. Subsequently, in each successive round, the applicants who were not accepted in the previous step apply to their next highest-ranked school. Each school considers both the applicants tentatively accepted in the prior step and new applicants. Once again, each school tentatively accepts students up to its capacity following its priority order over students, and reject the others. This iterative process continues until no student is rejected. At that point, assignments become final.

Because the assignment is solely based on the school's priorities over students and does not depend on the rank of students assigned to that school, the DA

[9] We suppress (x,y) when there is no danger of confusion.

mechanism is strategy-proof. That is, truthful reporting of rankings is a weakly dominant strategy for students under DA.

Let $P^{DA}(x,y) = \left(P_1^{DA}(x,y), P_2^{DA}(x,y)\right)$ and $P_i^{DA}(x,y)$ denote the interim probability that a student with type (x,y) is assigned to c_i under DA. Again, let $u^{DA}(x,y)$ denote the (interim) expected utility of a student with type (x,y) under the DA mechanism.[10]

Since students report truthfully under the DA mechanism, a student with type (x,y) reports c_1 as a first choice under DA iff $x \geq y$. By this observation, we obtain the following result:

Proposition 3.

$$P^{DA}(x,y) = \begin{cases} \left(\frac{2-p}{3}, \frac{p}{3}\right) & \text{if} \quad x \geq y \\ \left(\frac{1-p}{3}, \frac{1+p}{3}\right) & \text{if} \quad y > x \end{cases}.$$

Hence,

$$u^{DA}(x,y) = \begin{cases} \frac{2-p}{3}x + \frac{p}{3}y & \text{if} \quad x \geq y \\ \frac{1-p}{3}x + \frac{1+p}{3}y & \text{if} \quad y > x \end{cases}.$$

Proof. See Appendix A.2. □

5. Welfare Comparison

Initially, we present a set of definitions that are employed for the purpose of welfare comparison between IA and DA mechanisms. Let us denote the interim utility of type (x,y) under mechanism μ by $u^{\mu}(x,y)$.

Definition 1. *We say that a mechanism φ interim welfare dominates another mechanism ϕ in type space V iff $u^{\varphi}(\mathbf{v}) \geq u^{\phi}(\mathbf{v})$ for all $\mathbf{v} \in V$ and $u^{\varphi}(\mathbf{v}') > u^{\phi}(\mathbf{v}')$ for some $\mathbf{v}' \in V$.*

Definition 2. *For any mechanism φ, the ex-ante welfare under φ, denoted by EW^{φ}, is defined to be the expected utility under φ, where the expectation is over the type space V. That is,*

$$EW^{\varphi} = E[u^{\varphi}(X,Y)],$$

where the expectation is over the type space V.

Definition 3. *We say that a mechanism φ ex-ante welfare dominates another mechanism ϕ in type space V iff*

$$EW^{\varphi} > EW^{\phi}.$$

[10]Similarly, we suppress (x,y) when there is no danger of confusion.

Trivially, the interim dominance implies the ex-ante welfare dominance, but not the other way around.

First, we show that when $p = \frac{1}{2}$, the IA mechanism interim dominates DA in *any* type space V, including $[0, 1] \times [0, 1]$.[11] More importantly, we also show that this is the only case when there can be interim dominance relation between the two mechanisms in general.

Proposition 4. *The IA mechanism interim welfare dominates DA for any type space V if and only if $p = \frac{1}{2}$. Moreover, the DA mechanism does not interim dominate the IA mechanism in general for any $p \in [\frac{1}{2}, 1)$.*

Proof. See Appendix A.3.1. □

The first part of Proposition 4 shows that when $p = \frac{1}{2}$, we have a very strong superiority of the IA mechanism over the DA mechanism. Irrespective of the type space, any type has a higher interim utility under IA than that under DA. On the other hand, when $p \neq \frac{1}{2}$, it is always possible to identify a type space containing a type that has a higher interim utility under DA than under IA. Moreover, there exists a type space including a type which has a higher interim utility under IA than under DA, indicating that the DA mechanism cannot generally interim dominate the IA mechanism.

Proposition 4 has an important implication regarding existing results in the literature on the interim dominance of the IA mechanism over DA mechanism. First, it sheds light on the result presented by Abdulkadiroğlu et al. (2011) (ACY, hereafter) when students' ordinal rankings over schools are identical, revealing it to be a highly specific scenario. ACY's main finding, based on a type space where students' ordinal rankings over schools are identical, indicates that at any symmetric equilibrium of the IA mechanism, each type of student has a higher interim payoff than under DA. In contrast, Proposition 4 highlights that whenever students' ordinal rankings over schools are different with positive probability, even if that probability is very small, there exists a type space for which the interim dominance of the IA mechanism over DA does not hold. Second, similarly, as soon as we move away from the perfectly uncorrelated preference

[11] Note that this situation corresponds to Featherstone and Niederle (2016)'s "art and science" example. As discussed in the Introduction, although Featherstone and Niederle (2016) do not present a formal proof of the interim dominance relation, they present a discussion regarding this.

case where each possible ranking over schools is equally likely, which is studied in the earlier literature, the general interim dominance relation between the two mechanisms cannot be established.

Given that there is no interim dominance relation between the two mechanisms when $p \neq \frac{1}{2}$, we compare these two mechanisms in terms of ex-ante welfare. In our main result below, we show that although there is no interim welfare dominance relation, the IA mechanism is *ex-ante* welfare superior to the DA mechanism for *any* continuous and increasing value distributions F and G over $[0, 1]$.

Theorem 1. *Assume that each student's value for school c_1 is independently drawn from a continuous and increasing distribution $F(.)$ over $[0, 1]$. Similarly, each student's value for school c_2 is independently drawn from a continuous and increasing distribution $G(.)$ over $[0, 1]$. The IA mechanism ex-ante welfare dominates the DA mechanism for any F and G.*

Proof. See Appendix A.3.2. □

6. Discussions

6.1. Interdependent Values. In the primary model, we examined a scenario with "independent values" and demonstrated that the IA mechanism outperforms the DA mechanism in terms of ex-ante welfare. Now, we extend our analysis to include interdependence among values across schools. To be more precise, consider a setting where each student's values (X, Y) are independently drawn over a type space V such that $\Pr(X = Y) = 0$. Let again $p = \Pr(X \geq Y) \in (0, 1)$ and without loss of generality assume that $p \geq \frac{1}{2}$.

All the findings presented in Section 3, Section 4, and Proposition 4 hold in this setting as well. This is because none of the arguments made in those results relied on the assumption of independence. In particular, in an interdependent setting, there exists no general interim welfare dominance relation between IA and DA unless $p = \frac{1}{2}$. That is, when $p \in \left(\frac{1}{2}, 1\right)$, there is a type space for which IA does *not* interim welfare dominate DA and one for which DA does *not* interim welfare dominate IA.

Nevertheless, the following result shows that for any $p \in \left(\frac{1}{2}, 1\right)$, there is *some* type space and a distribution over it that is consistent with p and under which IA interim welfare dominates DA.

Proposition 5. *Take any $p \in \left(\frac{1}{2}, 1\right)$. Then, there is a type space V and a distribution over V such that $\Pr(X \geq Y) = p$ and the IA mechanism interim welfare dominates the DA mechanism in V.*

Proof. See Appendix A.4. □

Perhaps a more interesting result is the following. In contrast to the existing findings in the literature, which commonly find evidence on the dominance of IA over DA in terms of welfare, we below show that for any $p \in \left(\frac{3}{4}, 1\right)$, we can always find a type space under which DA interim welfare dominates IA.

Proposition 6. *Take any $p \in \left(\frac{3}{4}, 1\right)$. Then, there is a type space V and a distribution over V such that $\Pr(X \geq Y) = p$ and the DA mechanism interim welfare dominates the IA mechanism in V.*

Proof. See Appendix A.4. □

We provide a full *characterization* of the set of type spaces under which DA interim welfare dominates IA and similarly under which IA interim welfare dominates DA in the proofs of Proposition 5 and Proposition 6.

For an illustration of Proposition 6, we present the following example:

Example 1. Let $p = 0.975$. Assume that each student's values are independently drawn from a type space $\{(1, 0.8749), (1, 0.8751), (0, 1)\}$ with a probability distribution such that $(1, 0.8749)$ occurs with probability 0.6, $(1, 0.8751)$ occurs with probability 0.375, and $(0, 1)$ occurs with probability 0.025. Note that

$$p = \Pr(X \geq Y) = 0.975.$$

Furthermore, it is easy to show that, in equilibrium of Boston, we have $k = \frac{8}{7}$. To see this, note that we know that there is a unique symmetric equilibrium under the IA mechanism the strategy that a student with type (x, y) reports c_1 as a top choice iff $x \geq ky$ and let $q = \Pr(X \geq kY)$ with $k = \frac{1+q}{2-q}$. Hence, we require

$$\Pr\left(\frac{2-q}{1+q} \geq Y\right) = q. \tag{2}$$

Now,

$$\Pr(Y \le y) = \begin{cases} 0 & \text{if} \quad y < 0.8749 \\ 0.6 & \text{if} \quad 0.8749 \le y < 0.8751 \\ 0.975 & \text{if} \quad 0.8751 \le y < 1 \\ 1 & \text{if} \quad y \ge 1 \end{cases}.$$

Hence, we must have $q = 0.6$ for (2) to be satisfied, and hence $k = \frac{1+q}{2-q} = \frac{8}{7}$. Hence, type $(1, 0.8749)$ reports c_1 as a first choice and types $(1, 0.8751)$ and $(0, 1)$ report c_2 as a first choice under the IA mechanism.

First, consider type $(1, 0.8749)$. If he reports c_1 as a first choice under Boston, his utility under Boston is

$$\frac{q^2 - 3q + 3}{3} + \frac{q^2}{3} \times 0.8749 = 0.62499$$

His expected utility under DA is

$$\frac{2-p}{3} + \frac{p}{3} \times 0.8749 = 0.62601$$

Expected utility of type $(1, 0.8751)$: He reports c_2 as a first choice and hence his utility under Boston is

$$\frac{q^2 - 2q + 1}{3} + \frac{q^2 + q + 1}{3} \times 0.8751 = 0.62507$$

His expected utility under DA

$$\frac{2-p}{3} + \frac{p}{3} \times 0.8751 = 0.62607$$

Expected utility of type $(0, 1)$: He reports c_2 as a first choice and hence his utility under Boston is

$$\frac{q^2 - 2q + 1}{3} \times 0 + \frac{q^2 + q + 1}{3} \times 1 = 0.65333$$

His expected utility under DA is

$$\frac{1-p}{3} \times 0 + \frac{1+p}{3} \times 1 = 0.65833.$$

Hence, for each type of student DA yields a higher expected utility than IA.

6.2. Tie-Breaking. In our main model, we employ the "single tie-breaking" method for resolving ties in school priorities. However, there may be other tie-breaking methods that can be considered. Nevertheless, it's worth exploring other tie-breaking approaches that could be considered. For instance, an alternative method involves each school conducting a separate lottery, drawn uniformly and randomly, commonly known as "multiple tie-breaking".

Some papers have compared the DA mechanism under these two tie-breaking methods. For instance, Ashlagi et al. (2019) analyze the likelihood of students being assigned to one of their top choices under single-tie breaking and multiple-tie breaking as the numbers of students and schools grow large when students have randomly drawn preferences over schools. Another study by Ashlagi and Nikzad (2020) reveals that the trade-offs between the tie-breaking rules disappear when attention is restricted to the assignments to "popular" schools; within the set of popular schools a single lottery is found to be preferable over independent lotteries.[12]

We below show that each type of student has a higher interim utility under DA with single-tie breaking than that under DA with multiple tie-breaking in our setting.

Proposition 7. *In our model with two schools and three students, DA with single-tie breaking interim dominates DA with multiple tie-breaking in any type space V.*

Proof. See Appendix A.5. □

These tie-breaking methods induce equivalent interim allocations under the IA mechanism. This outcome is primarily due to the nature of the competition among students, which is limited to those who find themselves in precisely the same circumstances, under IA. To be more specific, under IA, students only compete with others who have ranked the school in the exact same position.

Hence, the ex-ante welfare superiority of IA over DA in Theorem 1, continues to hold under both tie-breaking rules.[13] It also follows trivially from Proposition 5 and Proposition 7 that there is a type space in which IA interim dominates DA with multiple tie-breaking. However, the following result shows that DA with

[12]For additional insights and discussions on these alternative tie-breaking methods, see also Abdulkadiroğlu et al. (2009) and De Haan et al. (2023).

[13]Alternatively, although not previously considered in the literature, ties may be resolved via a random lottery at each step of Boston and DA (random tie-breaking). We also show that DA with single-tie breaking induces higher ex-ante welfare than DA with random tie-breaking when values are independent across schools as in the main model. The proof is available upon request from the authors.

multiple tie-breaking cannot interim dominate IA in any type space. That is, the conclusion reached in Proposition 6, which shows that DA (with single-tie breaking) can interim dominate IA in some type space, does not hold true when multiple tie-breaking is employed.

Proposition 8. *There is no type space and a distribution over it such that DA with multiple tie-breaking interim dominates IA in that type space.*

Proof. See Appendix A.5.2. □

6.3. Unitarian Setting. In the subsequent analysis, we aim to derive additional welfare comparison outcomes between IA and DA mechanisms. To achieve this, we consider a *unitarian* setting where the type space is structured in a way that the maximum valuation of the two schools is normalized to 1. The result presented below shows that, in any unitarian setting, IA mechanism ex-ante welfare dominates the DA mechanism (with multiple tie-breaking) even without the independence of values across schools.

Proposition 9. *The IA mechanism is ex-ante welfare superior to the DA* (*with multiple tie-breaking*) *mechanism in any unitarian setting.*

Proof. See Appendix A.5.3. □

6.4. Large Market in a Unitarian Setting. Until this point, our primary focus in this paper has been on investigating a small finite allocation problem, with two schools and one slot per school, and three students. It is worth exploring the applicability of the findings to an extended allocation problem with replicas. At the limit, we could treat each individual as a continuum of individuals with mass 1, and each slot as a continuum of slots with mass 1. We consider a unitarian setting as in the later part of Section 6.2, where the type space is such that the maximum of the two school valuations is normalized to 1.

We now have to consider two scenarios in the analysis of Bayesian equilibria under IA. In the first scenario, both schools are overdemanded ($q \in [1/2, 2/3)$). In the second scenario ($q \in (2/3, 1)$), school 2 is underdemanded in the first round of the allocation mechanism, with the remaining slots to be allocated in the second round.

Equilibrium is characterized by $k = \frac{1-q}{q}$ in the first case, and by $k = 1/2$ in the second scenario.[14] In this continuum model, we show that DA can never interim dominate IA in the following result.

Proposition 10. *In the continuum model with the unitarian setting, IA ex-ante welfare dominates DA.*

Proof. See Appendix A.6. □

7. Conclusion

Our paper undertakes a comparison of two prevalent school choice mechanisms–the Immediate Acceptance (IA) and Deferred Acceptance (DA) mechanisms–focusing on the effects on welfare in conditions of incomplete information. We venture beyond the existing research (which generally only contemplates two extremes of preferences; those that are perfectly correlated and those that are perfectly uncorrelated) by considering arbitrary value distributions (and, therefore, preference correlations) of the students. In our attempt to better comprehend the impact of the most general distributional setting, we take into account a simple model that involves two schools and three students.

Our findings unveil that established welfare comparison findings are delicate. Indeed, the formerly identified "interim welfare dominance" results no longer stand valid as soon as we deviate from extreme preferences. However, we manage to demonstrate that within a scenario of independent values–where the values of each student to each school are independently distributed from arbitrary distributions–the IA mechanism outperforms the DA mechanism in terms of ex-ante welfare.

In addition to our main analysis, we have also explored several extensions to our base model. These include the possibility of values that are mutually dependent across schools, the utilization of an alternative tie-breaking method, and the examination of a continuum model.

We hope that our findings will pave the way to enhance our understanding of the welfare comparison between these two widely-implemented school choice mechanisms.

[14]In the first scenario, the indifferent type is characterized by $\frac{1}{3q} = \frac{k}{3(1-q)}$. In the second scenario, we have $\frac{1}{3q} + \left(1 - \frac{1}{3q}\right) \frac{1-3(1-q)}{3q-\left(1-\frac{1}{3q}\right)} k = k$, leading to $k = 1/2$.

Appendix A. Appendix

A.1. Missing Proofs in Section 3.

A.1.1. *Proof of Lemma 1.* Suppose that the student i_2 and student i_3 follows strategies such that the rank distribution of student $i \in \{i_2, i_3\}$ is such that student i ranks c_1 as top choice with probability $q_i \in [0,1]$ and c_2 as top choice with probability $1-q_i$. Consider student i_1 with type (x_1, y_1). This student ranks c_1 as a top choice iff

$$q_2 q_3 \left(\frac{x_1+y_1}{3}\right) + \left[q_2\left(1-q_3\right)+q_3\left(1-q_2\right)\right]\frac{x_1}{2} + \left(1-q_2\right)\left(1-q_3\right)x_1$$
$$\geq q_2 q_3 y_1 + \left[q_2\left(1-q_3\right)+q_3\left(1-q_2\right)\right]\frac{y_1}{2} + \left(1-q_2\right)\left(1-q_3\right)\left(\frac{x_1+y_1}{3}\right)$$
$$\Longleftrightarrow$$
$$\frac{x_1}{y_1} \geq \frac{2+\left(q_2+q_3\right)}{4-\left(q_2+q_3\right)}.$$

Let's call $k_1 = \frac{2+(q_2+q_3)}{4-(q_2+q_3)}$. Hence, student i_1 with type (x_1, y_1) ranks c_1 as top choice iff $x_1 \geq k_1 y_1$. Similarly, student i_2 with type (x_2, y_2) ranks c_1 as top choice iff $x_2 \geq k_2 y_2$ where $k_2 = \frac{2+(q_1+q_3)}{4-(q_1+q_3)}$; and student i_3 with type (x_3, y_3) ranks c_1 as top choice iff $x_3 \geq k_2 y_3$ where $k_3 = \frac{2+(q_1+q_2)}{4-(q_1+q_2)}$.

A.1.2. *Proof of Proposition 1.* By the proof of Lemma 1, at equilibrium, $q_1, q_2, q_3 \in [0,1]$ satisfy

$$q_i = \Pr\left(X \geq k_i Y\right) \tag{3}$$

and

$$k_i = \frac{2+\sum\limits_{j\neq i} q_j}{4-\sum\limits_{j\neq i} q_j}.$$

We look for a symmetric equilibrium such that $k_1 = k_2 = k_3 = k$ and hence $q_1 = q_2 = q_3 = q$ with $k = \frac{1+q}{2-q}$. From the arguments of the proof of Lemma 1 above, a student with type (x, y) reports c_1 as a first choice iff $x \geq ky$ where $k = \frac{1+q}{2-q}$. Hence, the probability that this student ranks c_1 as a first choice is given by

$$q = \Pr\left(X \geq \frac{1+q}{2-q} Y\right).$$

Thus, if there is a symmetric equilibrium, we must have

$$q - \Pr\left(X \geq \frac{1+q}{2-q}Y\right) = 0. \tag{4}$$

We claim that there is such $q \in [\frac{1}{2}, 1)$. Note that $\frac{1+q}{2-q}$ is increasing in q and hence $\Pr\left(X \geq \frac{1+q}{2-q}Y\right)$ is decreasing in q. Thus, denoting left hand side (LHS) of (4) as $H\left(q\right)$, H is a strictly increasing function of q. When $q = \frac{1}{2}$,

$$H\left(\frac{1}{2}\right) = \frac{1}{2} - \Pr\left(X \geq Y\right) = \frac{1}{2} - p.$$

Note that since $p \geq \frac{1}{2}$, we have $H\left(\frac{1}{2}\right) \leq 0$. Moreover, when $q = 1$,

$$\begin{aligned} H\left(1\right) &= 1 - \Pr\left(X \geq 2Y\right) \\ &\geq 1 - \Pr\left(X \geq Y\right) \\ &= 1 - p \\ &> 0. \end{aligned}$$

Hence, there is $q \in [\frac{1}{2}, 1)$ that solves (4). Furthermore, since H is strictly increasing, there is a unique such q; and hence there is a unique symmetric equilibrium.

A.1.3. *Proof of Proposition 2.* Consider a student with type $v_1 \geq kv_2$. In this case, he reports c_1 as a first choice. In this case, the probabilities of obtaining c_1 and c_2 are as follows:

$$P_1^B = q^2\frac{1}{3} + 2q\left(1-q\right)\frac{1}{2} + \left(1-q\right)^2 = \frac{1}{3}q^2 + \left(1-q\right) = \frac{q^2}{3} - q + 1$$

and

$$P_2^B = \frac{q^2}{3}$$

Consider a student with type $v_1 \leq kv_2$: In this case, he reports c_2 as a first choice. In this case, the probabilities of obtaining $c_1$and c_2 are as follows:

$$P_1^B = \frac{\left(1-q\right)^2}{3}$$

and

$$P_2^B = q^2 + 2q\left(1-q\right)\frac{1}{2} + \frac{1}{3}\left(1-q\right)^2 = \frac{1}{3}\left(1-q\right)^2 + q = \frac{q^2+q+1}{3}.$$

A.2. Proof of Proposition 3. Assume that each school's priority order over students is identical, and randomly and uniformly determined. Then, a student's interim probability of receiving each school is as follows:

$$P^{DA} = \begin{cases} \left(\frac{2-p}{3}, \frac{p}{3}\right) & \text{if} \quad x \geq y \\ \left(\frac{1-p}{3}, \frac{1+p}{3}\right) & \text{if} \quad x \leq y \end{cases}$$

where the first component is the probability of getting into c_1 and the second component is the probability of getting into c_2. These can be calculated easily by computing probabilities under Random Serial Dictatorship (RSD) since DA is equivalent to RSD when school have identical priority list over students[15]: Consider a student, say s_1, with type $x \geq y$. With probability $\frac{1}{3}$, s_1 ranks first in the random order and in that case he chooses c_1. With probability $\frac{1}{3}$, s_1 ranks second in the random order. In that case, he chooses c_1 if c_1 was not picked by the student who ranks first. This happens with probability $(1-p)$. Otherwise, he picks c_2. If s_1 ranks third in the order, he remains unassigned for sure. Symmetrically, consider s_1 with type $x \leq y$. He picks c_2 if he is the first in the order or he is the second and the first student did not pick c_2. He picks c_1 if he is the second and the first student picked c_2.

A.3. Missing Proofs in Section 5.

A.3.1. *Proof of Proposition 4.* Consider a student with type (x, y). Then,

$$u^{IA}(x,y) - u^{DA}(x,y) = \begin{cases} \frac{q^2-3q+1+p}{3}x + \frac{q^2-p}{3}y & \text{if} \quad x \geq ky \\ \left(\frac{q^2-2q-1+p}{3}\right)x + \left(\frac{q^2+q+1-p}{3}\right)y & \text{if} \quad ky \geq x \geq y \\ \left(\frac{q^2-2q+p}{3}\right)x + \left(\frac{q^2+q-p}{3}\right)y & \text{if} \quad y \geq x \end{cases}$$

where $k = \frac{1+q}{2-q}$.

Assume that $p = \frac{1}{2}$. Note that in equilibrium of IA mechanism

$$q = \Pr\left(X \geq \frac{1+q}{2-q}Y\right) \tag{5}$$

Now, when $q = \frac{1}{2}$, right hand side (RHS) of (5) becomes $\Pr(X \geq Y) = p = \frac{1}{2}$. Hence, $q = p = \frac{1}{2}$. Hence, (5) holds when $q = \frac{1}{2}$. We know that there is a unique equilibrium, and hence $q = \frac{1}{2}$, $k = 1$. That is, each agent reports truthfully under

[15]See Balinski and Sönmez (1999), for example.

IA mechanism. Then,

$$u^{IA}(x,y)-u^{DA}(x,y)=\begin{cases}\frac{q^2-3q+1+p}{3}x+\frac{q^2-p}{3}y & \text{if } x\geq y\\ \left(\frac{q^2-2q-1+p}{3}\right)x+\left(\frac{q^2+q+1-p}{3}\right)y & \text{if } y\geq x\end{cases}$$

and hence

$$u^{IA}(x,y)-u^{DA}(x,y)=\begin{cases}\frac{(x-y)}{12} & \text{if } x\geq y\\ \frac{5(y-x)}{12} & \text{if } y\geq x\end{cases}.$$

Thus, for any (x,y), $u^{IA}(x,y)\geq u^{DA}(x,y)$, which is strict when $x\neq y$.

Suppose that $p\neq\frac{1}{2}$. In that case, $p>\frac{1}{2}$, $q>\frac{1}{2}$, and $k>1$. Consider a student with type (x,y) where $x=ky$. In that case,

$$\begin{aligned}&u^{IA}(x,y)-u^{DA}(x,y)\\&=\frac{q^2-3q+1+p}{3}x+\frac{q^2-p}{3}y\\&=\frac{q^2-3q+1+p}{3}ky+\frac{q^2-p}{3}y\\&=\left(\frac{q^2-3q+1+p}{3}\frac{1+q}{2-q}+\frac{q^2-p}{3}\right)y\\&=-\frac{1}{3}(2q-1)\frac{1-p}{2-q}y\\&<0,\end{aligned}$$

for any $y\in(0,\frac{1}{k}]$. Hence, student with type (ky,y) has a strictly higher payoff under DA than under Boston.

Next, consider a student with type (x,y) where $x=y$. In that case,

$$\begin{aligned}&u^{IA}(x,x)-u^{DA}(x,x)\\&=\left(\frac{q^2-2q+p}{3}+\frac{q^2+q-p}{3}\right)x\\&=\frac{1}{3}q(2q-1)x\\&>0\end{aligned}$$

for any $x\in(0,1]$. Hence, there cannot be any interim dominance relation when $p\in\left(\frac{1}{2},1\right)$.

A.3.2. *Proof of Theorem 1.* Note that

$$q = \int_0^1 G\left(\frac{x}{k}\right) f(x)\, dx$$

where

$$k = \frac{1+q}{2-q}$$

and

$$p = \int_0^1 G(x) f(x)\, dx$$

Denote

$$A = \int_0^1 x f(x)\, dx = 1 - \int_0^1 F(x)\, dx$$
$$B = \int_0^1 x^2 f(x)\, dx = 1 - 2\int_0^1 xF(x)\, dx$$

We first establish the following lemma:

Lemma 2. $B \geq 2A - 1$

Proof. This immediately follows from the fact that $x^2 \geq (2x-1)$. □

The ex-ante welfare under the IA mechanism is then given by

$$EW^{IA}$$
$$= \int_0^1 \left(\left(\int_0^{\frac{x}{k}} \left(\frac{q^2-3q+3}{3}x + \frac{q^2}{3}y \right) dy \right) + \left(\int_{\frac{x}{k}}^1 \left(\frac{q^2-2q+1}{3}x + \frac{q^2+q+1}{3}y \right) dy \right) f(x)\, dx. \right.$$

We can rewrite this as

$$EW^{IA} = \frac{q+q^2+1}{6} + \frac{2(1-q)^2}{6}A + \frac{(q^2-4q+4)}{6(1+q)}B.$$

On the other hand, the ex-ante welfare under the DA mechanism is given by

$$EW^{DA} = \int_0^1 \left(\int_0^x \left(\frac{2-p}{3}x + \frac{p}{3}y \right) dy + \int_x^1 \left(\frac{1-p}{3}x + \frac{1+p}{3}y \right) dy \right) f(x)\, dx.$$

We can rewrite this as

$$EW^{DA} = \frac{1+p}{6} + \frac{2(1-p)}{6}A + \frac{1}{6}B.$$

We need to show that

$$q+q^2+2\left(1-q\right)^2A+\frac{q^2-4q+4}{1+q}B\geq p+2(1-p)A+B. \tag{6}$$

We prove this result in three cases, depending on the comparison of A and $\frac{1}{2}$.

A.3.3. *First Case:* $A=\frac{1}{2}$. First, consider the case that $A=\frac{1}{2}$, then we have

$$p+2(1-p)A+B=1+B,$$

and

$$q+q^2+2\left(1-q\right)^2A+\frac{q^2-4q+4}{1+q}B=2q^2-q+1+\frac{q^2-4q+4}{1+q}B.$$

Hence, we need to show that

$$2q^2-q+\left(\frac{q^2-4q+4}{1+q}-1\right)B\geq 0.$$

Since $2q^2-q\geq 0$, whenever $\frac{q^2-4q+4}{1+q}\geq 1$, this inequality is satisfied. For the cases where $\frac{q^2-4q+4}{1+q}<1$, the higher the B is, the lower the LHS is, and since $B\leq A=\frac{1}{2}$, we have

$$\begin{aligned}&2q^2-q+\left(\frac{q^2-4q+4}{1+q}-1\right)B\\&\geq 2q^2-q+\left(\frac{q^2-4q+4}{1+q}-1\right)\frac{1}{2}\\&=\frac{4q^3+3q^2-7q+3}{2\left(q+1\right)}.\end{aligned}$$

We claim that $(4q^3+3q^2-7q+3)$ is positive for $q\in\left[\frac{1}{2},1\right]$. The derivative with respect to q is $12q^2+6q-7$, which is decreasing over $q\in\left[\frac{1}{2},\overline{q}\right]$ and increasing over $[\overline{q},1]$ where $\overline{q}=\frac{1}{12}\sqrt{3}\sqrt{31}-\frac{1}{4}\approx 0.553\,64$. The value of $12q^2+6q-7$ at $\overline{q}$ is $3\left(\frac{1}{12}\sqrt{3}\sqrt{31}-\frac{1}{4}\right)^2+4\left(\frac{1}{12}\sqrt{3}\sqrt{31}-\frac{1}{4}\right)^3-\frac{7}{12}\sqrt{3}\sqrt{31}+\frac{19}{4}\approx 0.722\,87>0$. Hence, $(4q^3+3q^2-7q+3)$ is positive for $q\in[0.5,1]$.

A.3.4. *Second Case:* $A>\frac{1}{2}$. Next, consider the case where $A>\frac{1}{2}$. For this case, note that $p+2(1-p)A+B$ is decreasing in p. Therefore, it would be maximized at its minimum possible value. And we know that $p\geq q$. Therefore, (by plugging in $p = q$ in 6) it suffices to show that

$$q + q^2 + 2(1-q)^2 A + \frac{q^2 - 4q + 4}{1+q} B \geq q + 2(1-q)A + B$$

or

$$q^2 - 2q(1-q)A - \left(1 - \frac{q^2 - 4q + 4}{1+q}\right) B \geq 0.$$

Note that $A, B \leq 1$; and for $q \in \left[\frac{1}{2}, 1\right]$, $q^2 - 2q(1-q)$ is positive if and only if $q \geq \frac{2}{3}$ and $1 - \frac{q^2-4q+4}{1+q}$ is positive if and only if $q \geq \frac{5}{2} - \frac{1}{2}\sqrt{13} \approx 0.697\,22$. We analyze 3 subcases below.

(1) For $q > \frac{5}{2} - \frac{1}{2}\sqrt{13}$, $q^2 - 2q(1-q)A - \left(1 - \frac{q^2-4q+4}{1+q}\right) B$ is minimized when $A = B = 1$, and for this case, we can confirm that

$$q^2 - 2q(1-q) - \left(1 - \frac{q^2 - 4q + 4}{1+q}\right)$$

is always positive.

(2) For $q \in \left[\frac{2}{3}, \frac{5}{2} - \frac{1}{2}\sqrt{13}\right]$, the lowest value $q^2 - 2q(1-q)A - \left(1 - \frac{q^2-4q+4}{1+q}\right) B$ can take is given when $A = 1$ and $B = 0$. In this interval, $q^2 - 2q(1-q)$ is always positive. Hence, the conclusion follows.

(3) For $q < \frac{2}{3}$, in order to minimize $q^2 - 2q(1-q)A - \left(1 - \frac{q^2-4q+4}{1+q}\right) B$, given any value of A, we would need to choose the minimum value of B, and from the lemma above, we know that $B \geq 2A - 1$. Hence,

$$\begin{aligned}
&q^2 - 2q(1-q)A - \left(1 - \frac{q^2 - 4q + 4}{1+q}\right) B \\
&\geq q^2 - 2q(1-q)A - \left(1 - \frac{q^2 - 4q + 4}{1+q}\right)(2A - 1) \\
&= \left(q^2 + 1 - \frac{q^2 - 4q + 4}{1+q}\right) - \left(2q(1-q) + 2\left(1 - \frac{q^2 - 4q + 4}{1+q}\right)\right) A \\
&= \frac{q^3 + 5q - 3}{q+1} + \frac{2\left(q^3 + q^2 - 6q + 3\right)}{q+1} A \\
&= \frac{1}{q+1}\left[\left(q^3 + 5q - 3\right) + 2\left(q^3 + q^2 - 6q + 3\right) A\right].
\end{aligned}$$

Note that $(q^3 + q^2 - 6q + 3) > 0$ for $q \in [\frac{1}{2}, \overline{q})$ and $(q^3 + q^2 - 6q + 3) < 0$ for $q \in (\overline{q}, 1]$ where $\overline{q} \approx 0.593\,58$ solves $q^3 + q^2 - 6q + 3 = 0$.

When $q \in [\frac{1}{2}, \overline{q})$,

$$\begin{aligned}
&\left(q^3 + 5q - 3\right) + 2\left(q^3 + q^2 - 6q + 3\right) A \\
&\geq \left(q^3 + 5q - 3\right) + \left(q^3 + q^2 - 6q + 3\right) \\
&= 2q^3 + q^2 - q \\
&\geq 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \frac{1}{2} \\
&= \frac{1}{4},
\end{aligned}$$

where the last inequality is due to the fact that $(2q^3 + q^2 - q)$ is increasing.

When $q \in (\overline{q}, 1]$,

$$\begin{aligned}
&\left(q^3 + 5q - 3\right) + 2\left(q^3 + q^2 - 6q + 3\right) A \\
&\geq \left(q^3 + 5q - 3\right) + 2\left(q^3 + q^2 - 6q + 3\right) \\
&= 3q^3 + 2q^2 - 7q + 3
\end{aligned}$$

Now, the derivative of $(3q^3 + 2q^2 - 7q + 3)$ is $(6q^2 + 4q - 7)$, which is increasing and $a = \frac{1}{6}\sqrt{2}\sqrt{23} - \frac{1}{3} \approx 0.797\,05$ solves

$$6q^2 + 4q - 7 = 0.$$

Hence, the minimum value of $(3q^3 + 2q^2 - 7q + 3)$ is

$$\begin{aligned}
&3a^3 + 2a^2 - 7a + 3 \\
&= 2\left(\frac{1}{6}\sqrt{2}\sqrt{23} - \frac{1}{3}\right)^2 + 3\left(\frac{1}{6}\sqrt{2}\sqrt{23} - \frac{1}{3}\right)^3 - \frac{7}{6}\sqrt{2}\sqrt{23} + \frac{16}{3} \\
&= \frac{25}{6} - \frac{7}{12}\sqrt{2}\sqrt{23} \\
&= \frac{1}{12}\left(50 - 7\sqrt{46}\right) \\
&> \frac{1}{12}\left(50 - 7\sqrt{49}\right) > 0.
\end{aligned}$$

A.3.5. *Third case:* $A < \frac{1}{2}$. Consider the case where $A < \frac{1}{2}$. For this case, note that $p + 2(1-p)A + B$ is increasing in p. Therefore, it would be maximized at its maximum possible value. And we know that $p \leq 1$. Therefore, (by plugging in

$p=1$ in 6) it suffices to show that

$$q+q^2+2\left(1-q\right)^2A+\frac{q^2-4q+4}{1+q}B\geq 1+B$$

or

$$q+q^2-1+\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)B\geq 0.$$

Note that $A\geq 0$, and for $q\in\left[0,\frac{1}{2}\right]$, $q+q^2-1$ is positive if and only if $q\geq\frac{1}{2}\sqrt{5}-\frac{1}{2}\approx 0.618\,03$ and $\frac{q^2-4q+4}{1+q}-1$ is positive if and only if $q\leq\frac{5}{2}-\frac{1}{2}\sqrt{13}\approx 0.697\,22$.
If $q\geq\frac{1}{2}\sqrt{5}-\frac{1}{2}$:. For $q>\frac{1}{2}\sqrt{5}-\frac{1}{2}$, for a given q and B, $q+q^2-1+2\left(1-q\right)^2A+\left(\frac{q^2-4q+4}{1+q}-1\right)B$ is minimized when A takes the lowest value, and we know that $B\leq A$. Hence, it suffices to show that

$$q+q^2-1+\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)B\geq 0.$$

Note that

$$\frac{\partial}{\partial q}\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)=\frac{4q^3+5q^2-2q-12}{\left(q+1\right)^2},$$

Furthermore, $4q^3+5q^2-2q-12$ is increasing and takes negative values for $q\leq 1$. Hence,

$$2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\begin{cases}>0 & \text{if} \quad q<b\\ =0 & \text{if} \quad q=b\\ <0 & \text{if} \quad q>b\end{cases},$$

where $b\approx 0.75612$

When $q\leq b$:

$$\begin{aligned}&q+q^2-1+\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)B\\ &\geq q+q^2-1+\frac{1}{2}\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)\\ &=\frac{4q^3+3q^2-7q+3}{2\left(q+1\right)}.\end{aligned}$$

Consider $4q^3+3q^2-7q+3$, whose derivative is $12q^2+6q-7$, which is increasing and

$$\begin{aligned}
&12\left(\frac{\sqrt{5}-1}{2}\right)^2+6\left(\frac{\sqrt{5}-1}{2}\right)-7\\
&=12\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2+3\sqrt{5}-10\\
&=8-3\sqrt{5}>0
\end{aligned}$$

Hence, $4q^3+3q^2-7q+3$ is increasing for $q\geq\frac{\sqrt{5}-1}{2}$ and hence for $q\geq\frac{\sqrt{5}-1}{2}$

$$\begin{aligned}
&4q^3+3q^2-7q+3\\
&\geq 4\left(\frac{\sqrt{5}-1}{2}\right)^3+3\left(\frac{\sqrt{5}-1}{2}\right)^2-7\left(\frac{\sqrt{5}-1}{2}\right)+3\\
&=3-\sqrt{5}>0,
\end{aligned}$$

hence we have the desired inequality.

When $q>b:$

$$\begin{aligned}
&q+q^2-1+\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)B\\
&\geq q+q^2-1+\left(2\left(1-q\right)^2+\frac{q^2-4q+4}{1+q}-1\right)\\
&=\frac{3q^3+q^2-7q+4}{q+1}.
\end{aligned}$$

Consider $3q^3+q^2-7q+4$. The derivative is $6q^2+2q-7$, which is increasing. Furthermore,

$$6q^2+2q-7>0$$
$$\Leftrightarrow$$
$$q>\frac{1}{6}\sqrt{43}-\frac{1}{6}\approx 0.926\,24.$$

Hence, $3q^3+q^2-7q+4$ takes its minimum value at $\frac{\sqrt{43}-1}{6}$ and

$$\begin{aligned}&3\left(\frac{\sqrt{43}-1}{6}\right)^3+\left(\frac{\sqrt{43}-1}{6}\right)^2-7\left(\frac{\sqrt{43}-1}{6}\right)+4\\&=\frac{55}{12}-\frac{7}{12}\sqrt{43}\\&>0,\end{aligned}$$

hence we have the desired inequality.
If $q<\frac{1}{2}\sqrt{5}-\frac{1}{2}$:. We want to show

$$q+q^2+2\left(1-q\right)^2A+\left(\frac{q^2-4q+4}{1+q}-1\right)B\geq p+2(1-p)A$$

$\implies$

$$\left(q+q^2-p\right)+2\left(\left(1-q\right)^2-\left(1-p\right)\right)A+\left(\frac{q^2-4q+4}{1+q}-1\right)B\geq 0$$

$\implies$

$$\left(q^2+q-p\right)+2\left(q^2-2q+p\right)A+\frac{q^2-5q+3}{1+q}B\geq 0.$$

The welfare difference is:

$$\begin{aligned}&\left(q^2+q-p\right)+2\left(q^2-2q+p\right)A+\frac{q^2-5q+3}{1+q}B\\&=\left(q^2+q-p\right)+2\left(q^2-2q+p\right)\left(1-\int_0^1F\left(x\right)dx\right)+\frac{q^2-5q+3}{1+q}B\\&=\left(q^2+q-p+2\left(q^2-2q+p\right)\right)-2\left(q^2-2q+p\right)\int_0^1F\left(x\right)dx+\frac{q^2-5q+3}{1+q}B\\&=\left(3q^2-3q+p-2\left(q^2-2q+p\right)\int_0^1F\left(x\right)dx\right)+\frac{q^2-5q+3}{1+q}B\end{aligned}$$

When $q^2-2q+p<0$. Assume first that $q^2-2q+p<0$. Then,

$$\begin{aligned}
&\left(3q^2-3q+p-2\left(q^2-2q+p\right)\int_0^1 F(x)\,dx\right)+\frac{q^2-5q+3}{1+q}B\\
&>\left(3q^2-3q+p-\left(q^2-2q+p\right)\right)\\
&=2q^2-q\\
&>0
\end{aligned}$$

as $\int_0^1 F(x)\,dx = 1-A>\frac{1}{2}$ and $\frac{q^2-5q+3}{1+q}, B>0$.

When $q^2-2q+p\geq 0$. Next, assume that $q^2-2q+p\geq 0$.(iff $q\leq 1-\sqrt{1-p}$).

First note that when $q+q^2-p\geq 0$, we are done as

$$\left(q+q^2-p\right)+2\left(q^2-2q+p\right)A+\left(\frac{q^2-4q+4}{1+q}-1\right)B>0$$

Suppose finally that $q+q^2-p<0$ (iff $q<\frac{\sqrt{1+4p}-1}{2}$).

$$\begin{aligned}
&\left(3q^2-3q+p-2\left(q^2-2q+p\right)\int_0^1 F(x)\,dx\right)+\frac{q^2-5q+3}{1+q}B\\
&=\left(3q^2-3q+p-2\left(q^2-2q+p\right)\int_0^1 F(x)\,dx\right)+\frac{q^2-5q+3}{1+q}\left(1-2\int_0^1 xF(x)\,dx\right)\\
&=\left(3q^2-3q+\frac{q^2-5q+3}{1+q}+p\right)-2\left(\left(q^2-2q+p\right)\int_0^1 F(x)\,dx+\frac{q^2-5q+3}{1+q}\int_0^1 xF(x)\,dx\right)\\
&>\left(3q^2-3q+\frac{q^2-5q+3}{1+q}+p\right)-2\left(q^2-2q+\frac{q^2-5q+3}{1+q}+p\right)\int_0^1 F(x)\,dx\\
&=\left(3q^2-3q+\frac{q^2-5q+3}{1+q}\right)-2\left(q^2-2q+\frac{q^2-5q+3}{1+q}\right)\int_0^1 F(x)\,dx+p\left(1-2\int_0^1 F(x)\,dx\right)\\
&=\left(3q^2-3q+\frac{q^2-5q+3}{1+q}\right)-2\left(q^2-2q+\frac{q^2-5q+3}{1+q}\right)(1-A)+pB.
\end{aligned}$$

We want to show

$$\left(3q^2-3q+\frac{q^2-5q+3}{1+q}\right)-2\left(q^2-2q+\frac{q^2-5q+3}{1+q}\right)(1-A)+pB\geq 0$$

when $q^2 + q - p < 0$.

Let

$$h(x) = 3x^2 - 3x + \frac{x^2 - 5x + 3}{1 + x}.$$

Note that h is decreasing on $\left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$ since

$$\frac{\partial}{\partial x}(h(x)) = \frac{1}{(x+1)^2}\left(6x^3 + 10x^2 + 2x - 11\right),$$

where $6x^3 + 10x^2 + 2x - 11$ is increasing and

$$6\left(\frac{\sqrt{5}-1}{2}\right)^3 + 10\left(\frac{\sqrt{5}-1}{2}\right)^2 + 2\left(\frac{\sqrt{5}-1}{2}\right) - 11 < 0.$$

Furthermore,

$$h\left(\frac{1}{2}\right) = -0.25$$

and

$$h\left(\frac{\sqrt{5}-1}{2}\right) \approx -0.527\,86.$$

Let

$$t(x) = x^2 - 2x + \frac{x^2 - 5x + 3}{1 + x}.$$

Note that t is also decreasing over $\left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$ and

$$t\left(\frac{1}{2}\right) = -0.25$$

and

$$t\left(\frac{\sqrt{5}-1}{2}\right) \approx -0.673\,76.$$

We want to show that

$$h(q) - 2t(q)(1 - A) + pB \geq 0.$$

Since $t(q)$ is negative for $q \in \left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$, this reduces to

$$1 - A \geq \frac{h(q) + pB}{2t(q)},$$

or

$$A \leq 1 - \frac{h(q) + pB}{2t(q)}.$$

As $A \geq B$, we have

$$\frac{h(q) + pB}{2t(q)} \leq \frac{h(q) + pA}{2t(q)},$$

and hence

$$1 - \frac{h(q) + pB}{2t(q)} \geq 1 - \frac{h(q) + pA}{2t(q)}.$$

Thus, if we show that

$$A \leq 1 - \frac{h(q) + pA}{2t(q)}, \tag{7}$$

we are done. (7) reduces to

$$2t(q)\, A \geq 2t(q) - h(q) - pA,$$

or

$$(2t(q) + p)\, A \geq 2t(q) - h(q)\,. \tag{8}$$

Note that we consider the case where $p > q + q^2$. Thus, if we show that $(2t(q) + q + q^2)\, A \geq 2t(q) - h(q)$, we are done.
First, consider $2t(q) - h(q)$. Now,

$$\begin{aligned}
&2t(q) - h(q) \\
&= 2\left(q^2 - 2q + \frac{q^2 - 5q + 3}{1 + q}\right) - \left(3q^2 - 3q + \frac{q^2 - 5q + 3}{1 + q}\right) \\
&= -q^2 - q + \frac{q^2 - 5q + 3}{1 + q} < 0
\end{aligned}$$

when $q \in \left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$. Now, over $\left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$

$$\begin{gathered}
2t(q) + q + q^2 \geq 0 \\
\Longleftrightarrow \\
q \leq \overline{q}
\end{gathered}$$

where $\overline{q} \approx 0.544\,35$. Assume first that $2t(q) + q + q^2 \geq 0$. In that case (8) trivially holds since LHS is negative.

Second consider the case $2t(q) + q + q^2 < 0$. Thus, $q > \overline{q}$ and we want to show

$$A \leq \frac{2t(q) - h(q)}{2t(q) + q + q^2}.$$

Note that

$$\begin{aligned}
&\frac{2t(q) - h(q)}{2t(q) + q + q^2} \\
&= \frac{-q^2 - q + \frac{q^2-5q+3}{1+q}}{2\left(q^2 - 2q + \frac{q^2-5q+3}{1+q}\right) + q + q^2} \\
&= \frac{-q^2 - q + \frac{q^2-5q+3}{1+q}}{3q^2 - 3q + \frac{2(q^2-5q+3)}{1+q}}.
\end{aligned}$$

Now,

$$\frac{\partial}{\partial q}\left(\frac{-q^2 - q + \frac{q^2-5q+3}{1+q}}{3q^2 - 3q + \frac{2(q^2-5q+3)}{1+q}}\right) = \frac{(q^4 + 62q^3 - 20q^2 - 24q + 3)}{(3q^3 + 2q^2 - 13q + 6)^2}$$

Consider $(q^4 + 62q^3 - 20q^2 - 24q + 3)$. The derivative is $4q^3 + 186q^2 - 40q - 24$ whose derivative is $12q^2 + 372q - 40$, which is increasing and positive on $\left[\frac{1}{2}, 1\right]$. Hence, $4q^3 + 186q^2 - 40q - 24$ is increasing and takes its minimum value at $q = \frac{1}{2}$, which is

$$4\left(\frac{1}{2}\right)^3 + 186\left(\frac{1}{2}\right)^2 - 40\left(\frac{1}{2}\right) - 24 = 3 > 0.$$

Thus, $(q^4 + 62q^3 - 20q^2 - 24q + 3)$ is increasing over $q \in \left[\frac{1}{2}, 1\right]$. Thus, it takes its maximum over value at $q \in \left[\overline{q}, \frac{\sqrt{5}-1}{2}\right]$ at $\frac{\sqrt{5}-1}{2}$, which is

$$\begin{aligned}
&\left(\frac{\sqrt{5}-1}{2}\right)^4 + 62\left(\frac{\sqrt{5}-1}{2}\right)^3 - 20\left(\frac{\sqrt{5}-1}{2}\right)^2 - 24\left(\frac{\sqrt{5}-1}{2}\right) + 3 \\
&= \frac{117}{2}\sqrt{5} - \frac{271}{2} < 0
\end{aligned}$$

Thus, $q^4+62q^3-20q^2-24q+3<0$ and hence $\frac{2t(q)-h(q)}{2t(q)+q+q^2}$ is decreasing over $\left[\overline{q},\frac{\sqrt{5}-1}{2}\right]$. Thus,

$$\begin{aligned}&\frac{2t\left(q\right)-h\left(q\right)}{2t\left(q\right)+q+q^2}\\&\geq\frac{2t\left(\frac{\sqrt{5}-1}{2}\right)-h\left(\frac{\sqrt{5}-1}{2}\right)}{2t\left(\frac{\sqrt{5}-1}{2}\right)+\frac{\sqrt{5}-1}{2}+\left(\frac{\sqrt{5}-1}{2}\right)^2}\\&=\frac{4}{11}\sqrt{5}+\frac{17}{11}\\&>\frac{1}{2}.\end{aligned}$$

Since $A<\frac{1}{2}$, (8) holds and we are done.

A.4. Proofs of Proposition 5 and Proposition 6. We first present the following observations that will be helpful to prove Proposition 5 and Proposition 6.

Lemma 3. *Consider some type (x,y). Assume that $q=\frac{1}{2}$. Then,*

$$u^{IA}\left(x,y\right)-u^{DA}\left(x,y\right)\begin{cases}=0 & \textit{if}\quad x=y\\ >0 & \textit{if}\quad x\neq y\end{cases}.$$

Lemma 4. *Consider some type (x,y). Assume that $q>\frac{1}{2}$.First,*

$$u^{IA}\left(x,y\right)-u^{DA}\left(x,y\right)\leq 0$$

iff (i) $\frac{q^2-3q+p+1}{p-q^2}x\leq y\leq\frac{2q-q^2+1-p}{q^2+q-p+1}x$, or (ii) $q^2+q-p\leq 0$ and $x\leq-\frac{q^2+q-p}{q^2-2q+p}$, $y\in\left[-\frac{q^2-2q+p}{q^2+q-p}x,1\right]$.[16]

Second,

$$u^{IA}\left(x,y\right)-u^{DA}\left(x,y\right)\geq 0$$

iff (i) $0\leq y\leq\frac{q^2-3q+p+1}{p-q^2}x$, or (ii) $\frac{2q-q^2+1-p}{q^2+q-p+1}x\leq y\leq x$, or, (iii) $q^2+q-p\geq 0$ and $y\geq x$, or, (iv) $q^2+q-p<0$ and $\left(y\geq x\geq-\frac{q^2+q-p}{q^2-2q+p}\text{ or }\left(x\leq-\frac{q^2+q-p}{q^2-2q+p},y\leq-\frac{q^2-2q+p}{q^2+q-p}x\right)\right)$.

Proofs (of Lemma 3 and Lemma 4). Recall that

$$P^{DA}=\begin{cases}\left(\frac{2-p}{3},\frac{p}{3}\right) & \text{if}\quad x\geq y\\ \left(\frac{1-p}{3},\frac{1+p}{3}\right) & \text{if}\quad y\leq x\end{cases}$$

[16]Note that $q^2+q-p\leq 0$ can occur only when $p>\frac{3}{4}$.

and hence
$$u^{DA}(x,y)=\begin{cases}\frac{2-p}{3}x+\frac{p}{3}y & \text{if} \quad x\geq y\\ \frac{1-p}{3}x+\frac{1+p}{3}y & \text{if} \quad x\leq y\end{cases}$$
Furthermore,
$$P^{IA}=\begin{cases}\left(\frac{q^2-3q+3}{3},\frac{q^2}{3}\right) & \text{if} \quad x\geq ky\\ \left(\frac{q^2-2q+1}{3},\frac{q^2+q+1}{3}\right) & \text{if} \quad ky\leq x\end{cases}$$
$$u^{IA}(x,y)=\begin{cases}\frac{q^2-3q+3}{3}x+\frac{q^2}{3}y & \text{if} \quad x\geq ky\\ \frac{q^2-2q+1}{3}x+\frac{q^2+q+1}{3}y & \text{if} \quad ky\geq x\end{cases}$$
Thus,
$$u^{IA}(x,y)-u^{DA}(x,y)=\begin{cases}\frac{q^2-3q+p+1}{3}x+\frac{q^2-p}{3}y & \text{if} \quad x\geq ky\\ \frac{q^2-2q+p-1}{3}x+\frac{q^2+q-p+1}{3}y & \text{if} \quad ky\geq x\geq y\\ \frac{q^2-2q+p}{3}x+\frac{q^2+q-p}{3}y & \text{if} \quad y\geq x\end{cases}$$
First, when $q=\frac{1}{2}$, we have $p=\frac{1}{2}$ and $k=1$.[17] Hence,
$$u^{IA}(x,y)-u^{DA}(x,y)=\begin{cases}\frac{1}{12}x-\frac{1}{12}y & \text{if} \quad x\geq y\\ -\frac{1}{12}x+\frac{1}{12}y & \text{if} \quad y\geq x\end{cases},$$
which proves Lemma 3.

Next, assume that $q>\frac{1}{2}$. For a given $\widehat{x}\in[0,1]$, consider $u^{IA}(\widehat{x},y)-u^{DA}(\widehat{x},y)$. For convenience, define
$$\Delta(y)=\begin{cases}\frac{q^2-3q+p+1}{3}\widehat{x}+\frac{q^2-p}{3}y & \text{if} \quad \frac{1}{k}\widehat{x}\geq y\\ \frac{q^2-2q+p-1}{3}\widehat{x}+\frac{q^2+q-p+1}{3}y & \text{if} \quad \frac{1}{k}\widehat{x}\leq y\leq\widehat{x}\end{cases}$$
and
$$\delta(y)=\frac{q^2-2q+p}{3}\widehat{x}+\frac{q^2+q-p}{3}y \text{ for } y\geq\widehat{x}.$$
Now,
$$\Delta'(y)=\begin{cases}\frac{q^2-p}{3}y & \text{if} \quad \frac{1}{k}\widehat{x}\geq y\\ \frac{q^2+q-p+1}{3}y & \text{if} \quad \frac{1}{k}\widehat{x}\leq y\leq\widehat{x}\end{cases},$$

[17] Recall that $p=q$ iff $p=q=\frac{1}{2}$.

and as $\frac{q^2-p}{3} < \frac{q-p}{3} \leq 0$ and $\frac{q^2+q-p+1}{3} > 0$, $\Delta'(y) < 0$ if $\frac{1}{k}\widehat{x} \geq y$ and $\Delta'(y) > 0$ if $\frac{1}{k}\widehat{x} \leq y \leq \widehat{x}$. Thus, Δ gets its minimum value at $\frac{\widehat{x}}{k}$ over $[0, \widehat{x}]$. Furthermore,

$$\begin{aligned}
& \Delta\left(\frac{\widehat{x}}{k}\right) \\
= & \left(\frac{q^2-3q+p+1}{3} + \frac{q^2-p}{3}\frac{2-q}{1+q}\right)\widehat{x} \\
= & \frac{(2q-1)(p-1)}{3(q+1)}\widehat{x} \\
\leq & \; 0,
\end{aligned}$$

where the inequality holds with equality iff $q = \frac{1}{2}$ or $p = 1$ or $\widehat{x} = 0$. Note also that

$$\begin{aligned}
\Delta(0) &= \frac{q^2-3q+p+1}{3}\widehat{x} \\
&\geq \frac{q^2-2q+1}{3}\widehat{x} \geq 0
\end{aligned}$$

and thus

$$\Delta(0) \geq 0,$$

where the inequality is equality iff $q = \frac{1}{2}$ or $\widehat{x} = 0$. Moreover,

$$\begin{aligned}
\Delta(\widehat{x}) &= \frac{q^2-2q+p-1}{3}\widehat{x} + \frac{q^2+q-p+1}{3}\widehat{x} \\
&= \left(\frac{q^2-2q+p-1}{3} + \frac{q^2+q-p+1}{3}\right)\widehat{x} \\
&= \frac{1}{3}q(2q-1)\widehat{x}.
\end{aligned}$$

Thus,

$$\Delta(\widehat{x}) \geq 0,$$

where the inequality is equality iff $q = \frac{1}{2}$ or $\widehat{x} = 0$. Now, when $ky \leq \widehat{x}$,

$$\Delta(y) = \frac{q^2-3q+p+1}{3}\widehat{x} + \frac{q^2-p}{3}y \begin{cases} > 0 & \text{iff} \quad y < \frac{q^2-3q+p+1}{p-q^2}\widehat{x} \\ = 0 & \text{iff} \quad y = \frac{q^2-3q+p+1}{p-q^2}\widehat{x} \\ < 0 & \text{iff} \quad \frac{1}{k}\widehat{x} \geq y > \frac{q^2-3q+p+1}{p-q^2}\widehat{x} \end{cases}$$

Note that

$$\begin{aligned} & q^2 - 3q + p + 1 \\ \geq \; & p^2 - 3p + p + 1 \\ \geq \; & 0, \end{aligned}$$

where the first inequality is due to the fact that $q^2 - 3q$ is decreasing in q and $q \leq p$. furthermore,

$$1 - \frac{q^2 - 3q + p + 1}{p - q^2} = -\frac{(2q-1)(q-1)}{p - q^2} \geq 0$$

and

$$\frac{q^2 - 3q + p + 1}{p - q^2} - \frac{1}{k} = \frac{q^2 - 3q + p + 1}{p - q^2} - \frac{2-q}{1+q} = \frac{2q-1}{p-q^2}\frac{p-1}{q+1} \leq 0.$$

When $\frac{1}{k}\widehat{x} \leq y \leq \widehat{x}$,

$$\Delta(y) = \frac{q^2 - 2q + p - 1}{3}\widehat{x} + \frac{q^2 + q - p + 1}{3}y \begin{cases} > 0 & \text{iff} \quad y > \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1}\widehat{x} \\ = 0 & \text{iff} \quad y = \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1}\widehat{x} \\ < 0 & \text{iff} \quad y < \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1}\widehat{x} \end{cases}$$

Note that

$$\begin{aligned} & \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1} - 1 \\ = \; & q\frac{2q - 1}{p - q - q^2 - 1} \leq 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1} - \frac{1}{k} \\ = \; & \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1} - \frac{2 - q}{1 + q} \\ = \; & \frac{(2q-1)(p-1)}{(q+1)(p - q - q^2 - 1)} \geq 0 \end{aligned}$$

Thus,

$$\Delta(y) \begin{cases} \geq 0 & \text{iff} \quad \widehat{x} \geq y > \frac{\widehat{x}}{k} \\ \geq 0 & \text{iff} \quad \frac{\widehat{x}}{k} \geq y > \frac{q^2 - 3q + p + 1}{p - q^2}\widehat{x} \\ \leq 0 & \text{iff} \quad \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1}\widehat{x} < y < \frac{q^2 - 3q + p + 1}{p - q^2}\widehat{x} \\ \geq 0 & \text{iff} \quad 0 \leq y < \frac{2q - q^2 + 1 - p}{q^2 + q - p + 1}\widehat{x} \end{cases}$$

Next, $\delta'(y) = \frac{q^2+q-p}{3}$ for $y \geq \widehat{x}$. Note that

$$\delta(\widehat{x}) = \frac{q^2-2q+p}{3}\widehat{x} + \frac{q^2+q-p}{3}\widehat{x} = \frac{1}{3}q(2q-1)\widehat{x} \geq 0$$

and

$$\delta(1) = \frac{q^2-2q+p}{3}\widehat{x} + \frac{q^2+q-p}{3}.$$

If $q^2+q-p>0$ (note that this is always the case when $p<\frac{3}{4}$), $\delta'(y)>0$ and hence $u^{IA}(\widehat{x},y) - u^{DA}(\widehat{x},y) > 0$ for any $y > \widehat{x} \geq 0$. If $q^2+q=p$,

$$\delta(y) = \frac{q^2-2q+p}{3}\widehat{x} = \frac{2q^2-q}{3}\widehat{x} \geq 0$$

for $y \geq \widehat{x}$ (strict inequality when $q > \frac{1}{2}$ and $\widehat{x} > 0$). If $q^2+q-p<0$, then $q^2-2q+p > 2q^2-q \geq 0$. Thus, $\delta(1) > 0$ if $\widehat{x} > -\frac{q^2+q-p}{q^2-2q+p}$ and $\delta(1) < 0$ if $\widehat{x} < -\frac{q^2+q-p}{q^2-2q+p}$. (Note that $\frac{q^2+q-p}{q^2-2q+p} > -1$ and hence $-\frac{q^2+q-p}{q^2-2q+p} < 1$.) Now, assume that $\widehat{x} < -\frac{q^2+q-p}{q^2-2q+p}$ and $q^2+q-p<0$, (and hence $q^2-2q+p>0$). Then,

$$\delta(y) = \frac{q^2-2q+p}{3}\widehat{x} + \frac{q^2+q-p}{3}y \begin{cases} >0 & \text{if} \quad \widehat{x} \leq y < -\frac{q^2-2q+p}{q^2+q-p}\widehat{x} \\ <0 & \text{if} \quad y > -\frac{q^2-2q+p}{q^2+q-p}\widehat{x} \end{cases}$$

Note also that when $\widehat{x}=0$,

$$\delta(y) = \frac{q^2+q-p}{3}y \text{ for } y \geq 0.$$

and hence $u^{IA}(0,y) - u^{DA}(0,y) \leq 0$ iff $q^2+q-p \leq 0$. □

Remark 1. $\frac{q^2-3q+p+1}{p-q^2} \leq \frac{1}{k} \leq \frac{2q-q^2+1-p}{q^2+q-p+1} \leq 1$, where $k = \frac{1+q}{2-q}$.

Remark 2. (**An illustration: Special case:** $q^2+q=p$) Let $\widehat{x} \in [0,1]$. For $y \geq \widehat{x}$,

$$u^{IA}(\widehat{x},y) - u^{DA}(\widehat{x},y) = \delta(y) = \frac{q^2-2q+p}{3}\widehat{x} = \frac{2q^2-q}{3}\widehat{x} \geq 0,$$

strict inequality when $q > \frac{1}{2}$ and $\widehat{x} > 0$. For $y \leq \widehat{x}$,

$$u^{IA}(\widehat{x},y) - u^{DA}(\widehat{x},y) = \Delta(y) = \begin{cases} \frac{q^2-3q+p+1}{3}\widehat{x} + \frac{q^2-p}{3}y & \text{if} \quad \frac{1}{k}\widehat{x} \geq y \\ \frac{q^2-2q+p-1}{3}\widehat{x} + \frac{q^2+q-p+1}{3}y & \text{if} \quad \frac{1}{k}\widehat{x} \leq y \leq \widehat{x} \end{cases}$$

and hence

$$\Delta(y) = \begin{cases} \frac{2q^2-2q+1}{3}\widehat{x} - \frac{q}{3}y & \text{if} \quad \frac{1}{k}\widehat{x} \geq y \\ \frac{2q^2-q-1}{3}\widehat{x} + \frac{1}{3}y & \text{if} \quad \frac{1}{k}\widehat{x} \leq y \leq \widehat{x} \end{cases}$$

$$\begin{aligned}\Delta(y) &= \frac{q^2-2q+p-1}{3}\widehat{x}+\frac{q^2+q-p+1}{3}y \\ &= \frac{2q^2-q-1}{3}\widehat{x}+\frac{1}{3}y\end{aligned}$$

and hence

$$\Delta(y)\le 0 \text{ iff } y\in\left[\frac{2q^2-2q+1}{q}\widehat{x},\frac{1}{k}\widehat{x}\right]\cup\left[\frac{1}{k}\widehat{x},\left(1+q-2q^2\right)\widehat{x}\right]$$

Recall that $\frac{1}{k}=\frac{2-q}{1+q}$.

Proof of Proposition 5. The proof is in divided into two. We first show it for $p<\frac{3}{4}$ and then for $p\ge\frac{3}{4}$. First, consider the case that $p\in(\frac{1}{2},\frac{3}{4})$. Take some $q\in\left(\frac{1}{2},p\right)$.[18] Consider any distribution H_q over the type space

$$V\subset([0,1]\times[0,1])\setminus\left\{(x,y):\frac{q^2-3q+p+1}{p-q^2}x<y<\frac{2q-q^2+1-p}{q^2+q-p+1}x\right\}$$

such that

$$\Pr\left(\{\mathbf{v}\in(x,y):y\ge x\}\right)=1-p,$$
$$\Pr\left(\left\{(x,y):0\le y<\frac{q^2-3q+p+1}{p-q^2}x\right\}\right)=q$$

and

$$\Pr\left(\left\{(x,y):\frac{2q-q^2+1-p}{q^2+q-p+1}x<y\le x\right\}\right)=p-q.$$

Then, $\Pr(X\ge Y)=p$ and $\Pr(X\ge kY)=q$, where $k=\frac{1+q}{2-q}$. Moreover, by Lemma 4, for each type in V, $u^{IA}(x,y)-u^{DA}(x,y)\ge 0$. Next, consider the case that $p\ge\frac{3}{4}$. Consider any distribution H_q over the type space

$$V\subset([0,1]\times[0,1])\setminus\left(\begin{array}{c}\left\{(x,y):\frac{q^2-3q+p+1}{p-q^2}x<y<\frac{2q-q^2+1-p}{q^2+q-p+1}x\right\}\\ \cup\left\{(x,y):x<-\frac{q^2+q-p}{q^2-2q+p},y\in\left[-\frac{q^2-2q+p}{q^2+q-p}x,1\right]\right\}\end{array}\right)$$

such that

$$\Pr\left(\{\mathbf{v}\in(x,y):y\ge x\}\right)=1-p,$$
$$\Pr\left(\left\{(x,y):0\le y\le\frac{q^2-3q+p+1}{p-q^2}x\right\}\right)=q$$

and

$$\Pr\left(\left\{(x,y):\frac{2q-q^2+1-p}{q^2+q-p+1}x\le y\le x\right\}\right)=p-q.$$

[18]Note that it must be $q^2+q>p$ since $p<\frac{3}{4}$.

Then, $\Pr(X \geq Y) = p$ and $\Pr(X \geq kY) = q$, where $k = \frac{1+q}{2-q}$. Moreover, by Lemma 4, for each type in V, $u^{IA}(x,y) - u^{DA}(x,y) \geq 0$. □

Proof of Proposition 6. Let $q \in \left(\frac{1}{2}, p\right)$ be such that $q(1+q) \leq p$. Consider any distribution H_q over the type space

$$V \subset \left\{(x,y) : \frac{q^2-3q+p+1}{p-q^2}x \leq y \leq \frac{2q-q^2+1-p}{q^2+q-p+1}x\right\} \cup \left\{(x,y) : x \leq -\frac{q^2+q-p}{q^2-2q+p}, y \in \left[-\frac{q^2-2q+p}{q^2+q-p}x, 1\right]\right\}$$

such that

$$\Pr\left(\mathbf{v} \in \left\{(x,y) : x \leq -\frac{q^2+q-p}{q^2-2q+p}, y \in \left[-\frac{q^2-2q+p}{q^2+q-p}x, 1\right]\right\}\right) = 1-p,$$

$$\Pr\left(\mathbf{v} \in \left\{(x,y) : \frac{q^2-3q+p+1}{p-q^2}x \leq y \leq \frac{2-q}{1+q}x\right\}\right) = q$$

and

$$\Pr\left(\mathbf{v} \in \left\{(x,y) : \frac{2-q}{1+q}x \leq y \leq \frac{2q-q^2+1-p}{q^2+q-p+1}x\right\}\right) = p-q.$$

Then, by Lemma 4, we have that for any $(x,y) \in V$ $u^{IA}(x,y) - u^{DA}(x,y) \leq 0$. □

A.5. Proof of Proposition 7.

A.5.1. *Proof of Proposition 7.* We compare welfare under single tie-breaking to that under multiple tie-breaking. Recall that under single tie-breaking,

$$P^{sDA} = \begin{cases} \left(\frac{2-p}{3}, \frac{p}{3}\right) & \text{if} \quad x \geq y \\ \left(\frac{1-p}{3}, \frac{1+p}{3}\right) & \text{if} \quad x \leq y \end{cases}$$

Consider DA with multiple tie-breaking. For each school, priority order over students is randomly and uniformly determined. Hence, each school may have a different priority order. Let's compute the probabilities.

Consider student s_1 with $x \geq y$. There are three possible cases: s_2 and s_3 rank c_1 above c_2, or, they both rank c_2 above c_1, or one ranks c_1 as a top choice and the other ranks c_2 as a first choice.

First, assume that s_2 and s_3 rank c_1 as a first choice which happens with probability p^2. In that case, s_1 gets into c_1 with probability $\frac{1}{3}$ in Step 1 and rejected with probability $\frac{2}{3}$. If accepted in Step 1, he gets into c_1 for sure. If rejected in Step 1, s_1 applies to c_2 in Step 2. Note that there is another student rejected by c_1

in step 1 and will apply to c_2 is Step 2. Hence, s_1 gets into c_2 with probability $\frac{1}{2}$ if he is rejected in step 1. Thus, probabilities for s_1:

$$\left(\frac{1}{3}, \frac{1}{3}\right).$$

Second, assume that s_2 and s_3 rank c_2 as a first choice which happens with probability $(1-p)^2$. In that case, s_1 is tentatively accepted in Step 1. However, one of the other students applied to c_2 is step 1 will be rejected and will apply to c_1 in step 2. In this case, s_1 is accepted by c_1 in step 2 with probability $\frac{1}{2}$. In that case, he is assigned to c_1 for sure. If rejected in Step 2, s_1 applies to c_2 in Step 3. Note that there is another student, say s_2, accepted by c_2 in step 1. Hence, s_1 gets into c_2 with probability $\frac{1}{3}$ if he is rejected in step 2. This is because s_2 ranks above s_3 for c_2 since s_2 is accepted in step 1. Hence, the priority order of c_2 can be one of the three: $s_1 - s_2 - s_3$, $s_2 - s_1 - s_3$, or $s_2 - s_3 - s_1$. Thus, probabilities for s_1:

$$\left(\frac{1}{2}, \frac{1}{6}\right).$$

Finally, assume that one of s_2 and s_3 ranks c_1 as a top choice and the other ranks c_2 as a first choice which happens with probability $2p(1-p)$. First, assume that s_1 is tentatively accepted in Step 1, which happens with probability $\frac{1}{2}$. In that case, one student, say s_2, is rejected by c_1 is step 1 and will apply to c_2 in step 2. s_2 will be rejected by c_2 in step 2 as well with probability $\frac{1}{2}$. In this case, s_1 gets into c_1 for sure. With probability $\frac{1}{2}$, s_2 will be tentatively accepted by c_2. In that case, s_3 is rejected by c_2 and will apply to c_1 in step 3. s_1 will be tentatively accepted by c_1 with probability $\frac{2}{3}$ and gets into c_1 permanently. s_1 will be rejected by c_1 with probability $\frac{1}{3}$. In that case, he will apply to c_2 in step 4. He will be accepted by c_2 with probability $\frac{1}{3}$ and rejected with probability $\frac{2}{3}$. If accepted, s_1 gets into c_2, and if rejected s_1 remains unassigned.

Second, assume that s_1 is rejected in step 1. In that case, he will apply to c_2 in step 2. He will be tentatively accepted by c_2 with probability $\frac{1}{2}$ and rejected with probability $\frac{1}{2}$. If rejected, s_1 remains unassigned. If s_1 is tentatively accepted by c_2 is step 2, the student rejected by c_2 in step 2, say s_2, will apply to c_1 in step 3. If s_2 is rejected by c_1, which happens with probability $\frac{2}{3}$, s_1 is assigned to c_2 for sure. On the other hand, if s_2 is accepted by c_1, the rejected student s_3 will apply to c_2 in the next step. In that case, s_1 will be accepted by c_2 with probability $\frac{1}{3}$ and gets into c_2. If s_1 is rejected by c_2, s_1 remains unassigned..

To summarize, the probability that s_1 gets into c_1 is

$$\frac{1}{2} \times \left(\frac{1}{2} + \frac{1}{2} \times \frac{2}{3}\right) = \frac{5}{12}$$

and the probability that s_1 gets into c_2 is

$$\frac{1}{2} \times \left(\frac{1}{2} \times \left(\frac{1}{3} \times \frac{1}{3}\right)\right) + \frac{1}{2} \times \frac{1}{2} \times \left(\frac{2}{3} + \frac{1}{3} \times \frac{2}{3}\right) = \frac{1}{4}$$

Thus, the probabilities for s_1:

$$\left(\frac{5}{12}, \frac{1}{4}\right)$$

To sum up, the probabilities for s_1 when $x \geq y$ under multiple tie-breaking is:

$$\begin{aligned} &\left(\frac{p^2}{3} + \frac{(1-p)^2}{2} + \frac{5p\,(1-p)}{6}, \frac{p^2}{3} + \frac{(1-p)^2}{6} + \frac{p\,(1-p)}{2}\right) \\ &= \left(\frac{1}{6}\,(3-p)\,, \frac{1}{6}\,(1+p)\right) \end{aligned}$$

Next, consider student s_1 with $x \leq y$. Again, there are three possible cases: s_2 and s_3 rank c_1 above c_2, or, they both rank c_2 above c_1, or one ranks c_1 as a top choice and the other ranks c_2 as a first choice. Due to symmetry, we obtain the following probabilities.

First, assume that s_2 and s_3 rank c_1 as a first choice which happens with probability p^2. The probabilities for s_1:

$$\left(\frac{1}{6}, \frac{1}{2}\right).$$

Second, assume that s_2 and s_3 rank c_2 as a first choice which happens with probability $(1-p)^2$. The probabilities for s_1:

$$\left(\frac{1}{3}, \frac{1}{3}\right).$$

Finally, assume that one of s_2 and s_3 ranks c_1 as a top choice and the other ranks c_2 as a first choice which happens with probability $2p\,(1-p)$. The probabilities for s_1:

$$\left(\frac{1}{4}, \frac{5}{12}\right).$$

To sum up, the probabilities for s_1 when $x \leq y$ under multiple tie-breaking is:

$$\left(\frac{p^2}{6}+\frac{(1-p)^2}{3}+\frac{p(1-p)}{2},\frac{p^2}{2}+\frac{(1-p)^2}{3}+\frac{5p(1-p)}{6}\right)$$
$$=\left(\frac{1}{6}(2-p),\frac{1}{6}(2+p)\right)$$

Thus,

$$P^{mDA}=\begin{cases}\left(\frac{1}{6}(3-p),\frac{1}{6}(1+p)\right) & \text{if} \quad x\geq y\\ \left(\frac{1}{6}(2-p),\frac{1}{6}(2+p)\right) & \text{if} \quad x\leq y\end{cases}$$

Now,

$$P^{sDA}-P^{mDA}=\begin{cases}\left(\frac{2-p}{3},\frac{p}{3}\right)-\left(\frac{1}{6}(3-p),\frac{1}{6}(1+p)\right) & \text{if} \quad x\geq y\\ \left(\frac{1-p}{3},\frac{1+p}{3}\right)-\left(\frac{1}{6}(2-p),\frac{1}{6}(2+p)\right) & \text{if} \quad y\leq x\end{cases}$$

$$P^{sDA}-P^{mDA}=\begin{cases}\left(\frac{1-p}{6},-\frac{1-p}{6}\right) & \text{if} \quad x\geq y\\ \left(-\frac{p}{6},\frac{p}{6}\right) & \text{if} \quad y\leq x\end{cases}$$

Note that P^{sDA} first-order stochastically dominates P^{mDA}.

A.5.2. *Proof of Proposition 8.* Recall that

$$P^{mDA}=\begin{cases}\left(\frac{1}{6}(3-p),\frac{1}{6}(1+p)\right) & \text{if} \quad x\geq y\\ \left(\frac{1}{6}(2-p),\frac{1}{6}(2+p)\right) & \text{if} \quad x\leq y\end{cases}$$

and hence

$$u^{mDA}(x,y)=\begin{cases}\frac{3-p}{6}x+\frac{1+p}{6}y & \text{if} \quad x\geq y\\ \frac{2-p}{6}x+\frac{2+p}{6}y & \text{if} \quad x\leq y\end{cases}$$

Furthermore,

$$u^{IA}(x,y)=\begin{cases}\frac{q^2-3q+3}{3}x+\frac{q^2}{3}y & \text{if} \quad x\geq ky\\ \frac{q^2-2q+1}{3}x+\frac{q^2+q+1}{3}y & \text{if} \quad ky\geq x\end{cases},$$

and hence

$$u^{IA}(x,y)-u^{mDA}(x,y)=\begin{cases}\left(\frac{2q^2-6q+p+3}{6}\right)x+\left(\frac{2q^2-1-p}{6}\right)y & \text{if} \quad x\geq ky\\ \left(\frac{2q^2-4q-1+p}{6}\right)x+\left(\frac{2q^2+2q+1-p}{6}\right)y & \text{if} \quad ky\geq x\geq y\\ \left(\frac{2q^2-4q+p}{6}\right)x+\left(\frac{2q^2+2q-p}{6}\right)y & \text{if} \quad y\geq x\end{cases}.$$

For a given $\widehat{x}$, define

$$\Delta(y)=\begin{cases}\left(\frac{2q^2-6q+p+3}{6}\right)\widehat{x}+\left(\frac{2q^2-1-p}{6}\right)y & \text{if} \quad \widehat{x}\geq ky\\ \left(\frac{2q^2-4q-1+p}{6}\right)\widehat{x}+\left(\frac{2q^2+2q+1-p}{6}\right)y & \text{if} \quad ky\geq\widehat{x}\geq y\end{cases}$$

and

$$\delta(y)=\left(\frac{2q^2-4q+p}{6}\right)\widehat{x}+\left(\frac{2q^2+2q-p}{6}\right)y.$$

When $\widehat{x}\geq ky$,

$$\Delta'(y)=\frac{2q^2-1-p}{6}<\frac{2q-1-p}{6}\leq\frac{2p-1-p}{6}<0$$

and when $ky\geq\widehat{x}$,

$$\Delta'(y)=\frac{1}{6}\left(2q^2+2q+1-p\right)>0.$$

Hence, Δ achieves its minimum at $y=\frac{\widehat{x}}{k}$ over $[0,\widehat{x}]$. Note also that

$$\begin{aligned}&\Delta\left(\frac{\widehat{x}}{k}\right)\\ =\;&\left[\left(\frac{2q^2-6q+p+3}{6}\right)+\left(\frac{2q^2-1-p}{6}\right)\frac{2-q}{1+q}\right]\widehat{x}\\ =\;&\frac{1}{6}(2q-1)\frac{p-1}{q+1}\widehat{x}\leq 0.\end{aligned}$$

$$\Delta(0)=\frac{2q^2-6q+p+3}{6}\widehat{x}$$

Note that

$$\begin{aligned}&2q^2-6q+p+3\\ \geq\;&2p^2-6p+p+3\\ =\;&2p^2-5p+3\\ \geq\;&0\end{aligned}$$

and hence $\Delta(0)\geq 0$. Note also that

$$\begin{aligned}\Delta(\widehat{x}) &= \left(\frac{2q^2-4q-1+p}{6}\right)\widehat{x}+\left(\frac{2q^2+2q+1-p}{6}\right)\widehat{x}\\ &= \left(\frac{2q^2-4q-1+p}{6}+\frac{2q^2+2q+1-p}{6}\right)\widehat{x}\\ &= \frac{1}{3}q(2q-1)\widehat{x}\geq 0\end{aligned}$$

Furthermore,

$$\delta'(y) = \frac{2q^2+2q-p}{6} > 0$$

and hence δ takes its minimum value at $y = \widehat{x}$ over $[\widehat{x}, 1]$. Furthermore,

$$\delta(\widehat{x}) = \left(\frac{2q^2-4q+p}{6}\right)\widehat{x} + \left(\frac{2q^2+2q-p}{6}\right)\widehat{x} = \frac{1}{3}q(2q-1)\widehat{x} \geq 0$$

and

$$\delta(1) = \left(\frac{2q^2-4q+p}{6}\right)\widehat{x} + \left(\frac{2q^2+2q-p}{6}\right) > 0.$$

Thus, for DA with multiple tie-breaking to interim dominate IA, there should be no type with $x < y$. That is, we must have $p = 1$, a contradiction.

A.5.3. *Proof of Proposition 9.* Under multiple tie-breaking, we have

$$P^{mDA} = \begin{cases} \left(\frac{1}{6}(3-p), \frac{1}{6}(1+p)\right) & \text{if} \quad x \geq y \\ \left(\frac{1}{6}(2-p), \frac{1}{6}(2+p)\right) & \text{if} \quad x \leq y \end{cases}$$

and hence

$$u^{mDA}(x,y) = \begin{cases} \frac{3-p}{6}x + \frac{1+p}{6}y & \text{if} \quad x \geq y \\ \frac{2-p}{6}x + \frac{2+p}{6}y & \text{if} \quad x \leq y \end{cases}$$

Recall that

$$u^{IA}(x,y) = \begin{cases} \frac{q^2-3q+3}{3}x + \frac{q^2}{3}y & \text{if} \quad x \geq ky \\ \frac{q^2-2q+1}{3}x + \frac{q^2+q+1}{3}y & \text{if} \quad ky \geq x \end{cases}$$

Now,

$$u^{IA}(x,y) - u^{mDA}(x,y) = \begin{cases} \left(\frac{2q^2-6q+p+3}{6}\right)x + \left(\frac{2q^2-1-p}{6}\right)y & \text{if} \quad x \geq ky \\ \left(\frac{2q^2-4q-1+p}{6}\right)x + \left(\frac{2q^2+2q+1-p}{6}\right)y & \text{if} \quad ky \geq x \geq y \\ \left(\frac{2q^2-4q+p}{6}\right)x + \left(\frac{2q^2+2q-p}{6}\right)y & \text{if} \quad y \geq x \end{cases}$$

Recall that in the unitarian setting, the type space is $\{(1,y), (x,1) : x, y \leq 1\}$. Define

$$\Delta(y) = \begin{cases} \left(\frac{2q^2-6q+p+3}{6}\right) + \left(\frac{2q^2-1-p}{6}\right)y & \text{if} \quad 1 \geq ky \\ \left(\frac{2q^2-4q-1+p}{6}\right) + \left(\frac{2q^2+2q+1-p}{6}\right)y & \text{if} \quad ky \geq 1 \end{cases}$$

and

$$\delta(x) = \left(\frac{2q^2-4q+p}{6}\right)x + \left(\frac{2q^2+2q-p}{6}\right).$$

Consider type $(1, y)$. When $1 \geq ky$,

$$\Delta'(y) = \frac{2q^2 - 1 - p}{6} < \frac{2q - 1 - p}{6} \leq \frac{2p - 1 - p}{6} < 0$$

and when $ky \geq 1$,

$$\Delta'(y) = \frac{1}{6}\left(2q^2 + 2q + 1 - p\right) > 0$$

Hence, Δ achieves its minimum at $y = \frac{1}{k}$. Note also that

$$\begin{aligned}
& \Delta\left(\frac{1}{k}\right) \\
= & \left(\frac{q^2 - 3q + 3}{3} - \frac{3 - p}{6}\right) + \left(\frac{q^2}{3} - \frac{1 + p}{6}\right)\frac{2 - q}{1 + q} \\
= & \frac{1}{6}(2q - 1)\frac{p - 1}{q + 1} < 0.
\end{aligned}$$

For type $(x, 1)$,

$$\delta'(x) = \frac{2q^2 - 4q + p}{6}.$$

Note that $2q^2 - 4q$ is decreasing in q and takes value $-\frac{3}{2}$ at $q = \frac{1}{2}$ and value -2 at $q = 1$. Hence,

$$\delta'(x) = \frac{2q^2 - 4q + p}{6} \leq \frac{-\frac{3}{2} + p}{6} < 0$$

and

$$\delta(1) = \frac{2q^2 - q}{3} > 0$$

Hence,

$$\begin{aligned}
& EW^{IA} - EW^{mDA} \\
\geq & p\left(\frac{1}{6}(2q - 1)\frac{p - 1}{q + 1}\right) + (1 - p)\left(\frac{2q^2 - q}{3}\right) \\
= & \frac{(2q - 1)(1 - p)}{6(q + 1)}\left(2q^2 + 2q - p\right) \\
\geq & 0
\end{aligned}$$

since $p \leq 1 \leq 2q$.

A.6. Proof (of Proposition 10). There are three areas of the type space to consider: A) $y = 1$, B) $y \in (y^*, 1)$, and C) $y \leq y^*$. We consider two cases, depending on the value $\bar{q}$.

Case 1: $q \geq 2/3$

The difference in interim payoffs between IA and DA is:
For area A, $1 - 2/3 = 1/3$
For area B, $y - \left[\frac{1}{3p} + \left(\frac{2}{3} - \frac{1}{3p}\right) y\right] = y\left(\frac{1}{3} + \frac{1}{3p}\right) - \frac{1}{3p}$
For area C, $\frac{1}{3q} + \frac{q-2/3}{q} y - \left[\frac{1}{3p} + \left(\frac{2}{3} - \frac{1}{3p}\right) y\right] = \frac{1}{3q} - \frac{q-2/3}{q} + y\left[\frac{q-2/3}{q} - \left(\frac{2}{3} - \frac{1}{3p}\right)\right]$
Note that the difference is increasing in y for area B, and decreasing in y for area C.[19] Particularly, for areas B and C, the difference reaches a minimum when $y = y^* = 1/2$.

But then

$$\begin{aligned} EW^{IA} - EW^{DA} &\geq (1-p)1/3 + p\left[\frac{1}{2}\left(\frac{1}{3} + \frac{1}{3p}\right) - \frac{1}{3p}\right] \\ &= \frac{1-p}{6} \geq 0 \end{aligned}$$

Case 2: $q < 2/3$

The difference now between IA and DA is:
For area A, $\frac{1}{3(1-q)} - 2/3$
For area B, $\frac{y}{3(1-q)} - \left[\frac{1}{3p} + \left(\frac{2}{3} - \frac{1}{3p}\right) y\right] = y\left(\frac{1}{3(1-q)} + \frac{1}{3p} - \frac{2}{3}\right) - \frac{1}{3p}$
For area C, $\frac{1}{3q} - \left[\frac{1}{3p} + \left(\frac{2}{3} - \frac{1}{3p}\right) y\right] = \frac{1}{3q} - \frac{1}{3p} - y\left(\frac{2}{3} - \frac{1}{3p}\right)$
Again, the difference is increasing in y for area B, and decreasing in y for area C.[20] Particularly, for areas B and C, the difference reaches a minimum when $y = y^* = \frac{1-q}{q}$.

[19] As for area C, note that $\frac{\bar{q}-2/3}{\bar{q}} - \left(\frac{2}{3} - \frac{1}{3p}\right) \leq \frac{p-2/3}{p} - \left(\frac{2}{3} - \frac{1}{3p}\right) = \frac{1}{3} - \frac{1}{3p} \leq 0.$

[20] As for area C, the result stems from $p \geq 1/2$.

This yields

$$\begin{aligned}
EW^{IA} - EW^{DA} &\geq (1-p)\left(\frac{1}{3(1-q)} - 2/3\right) \\
&\quad + p\left[\frac{1}{3q} - \frac{1}{3p} - \frac{1-q}{q}\left(\frac{2}{3} - \frac{1}{3p}\right)\right] \\
&= (1-p)\left(\frac{1}{3(1-q)} - 2/3\right) \\
&\quad + p\left(\frac{1}{3p} - \frac{1}{3}\right)\left(\frac{1}{q} - 2\right) \\
&= \frac{1-p}{3}\left(\frac{1}{(1-q)} + \frac{1}{q} - 4\right) \geq 0
\end{aligned}$$

since for $q \geq 1/2$ the last expression is increasing in q.

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