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Abstract

In the developed model, without knowing the trading strategies of the other traders in a financial market, traders cannot derive a rational expectations equilibrium. In a dynamic setting, market participants employ learning and adaptation to develop trading strategies to accommodate for this information deficiency. Model-consistent use of market-based information generally improves price performance. It can also produce episodes of extreme sudden mispricing despite model generated historical support for its use. Simulations examine the impact of information constraints and bounded rationality on general price efficiency and sudden market mispricing.

Keywords: Heterogeneous Agents, Efficient Markets, Learning, Dynamics, Computational Economics, Market Failure
(JEL Codes: G14, C62, D82)

1 Introduction

An extraordinary number of traders employing a wide variety of strategies populate financial markets. Many attempt to extract rent through trading. Vigorous trading and extensive market commentary suggests a lack of uniformity among market participants and possible disagreement as to the true price determination process. The disagreement extends to issues of market efficiency and how possible deviations from efficiency can best be exploited. The diversity in trading strategies spans value seeking to extracting profitable information from the markets to exploiting anomalies.[1] Within these categories, individual decisions regarding the processing of information contribute to the large diversity of beliefs and strategies.

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[1] Labels applied to trading strategies such as fundamental trading, speculation, chartism, and technical analysis capture this diversity.

The developed model places traders into an imperfect information environment in which fundamental information is noisy. Market-based strategies have a role in extracting information from market observables but cannot be employed perfectly without unattainable knowledge about other traders' strategies. The analysis explores the extent to which unavoidable and reasonable data-driven adaptation and learning by market participants become the source of market disruptions, moving price substantially away from market fundamentals. The distinguishing feature of this model is that the market-based trading captures information not otherwise available, improving market efficiency when properly employed. Only when used improperly does the market based information undermine price efficiency. The model thus offers insight into market behavior not available from models such as Brock and Hommes (1998) that rely on inherently destabilizing market-based strategy to move the market away from fundamentals or those following Grossman and Stiglitz (1980) in which the market-based trading is simply a low-cost alternative to acquiring the same information known to the informed fundamental traders.

Financial markets have long been recognized as potential feedback systems between market behavior and trader beliefs. Beja and Goldman (1980) formally incorporate such feedback. The complexities associated with devising an optimal trading strategy in the presence of market feedback can lead to considerable uncertainty for the trader. This uncertainty can be incorporated into a model as heterogeneity and bounded rationality. Frankel and Froot (1990) and De Long et al. (1990a) and (1990b) define different groups of traders in order to formally explore the impact of trader heterogeneity on the markets.

Empirically supported learning is one of a number of mechanisms developed to allow boundedly rational agents to adapt to their environment within the limitations of their information or ability. A learning process can generate emergence of rational behavior despite the limits on the agents' information or sophistication, thereby achieving the rational expectations equilibrium (REE) without necessarily relying on fully rational agents. Marcet and Sargent (1989a) and (1989b), and Evans and Honkapohja (2001) include demonstrations of such convergence as examples in the formal development of least-squares learning.[2] Examples of convergence by learning include the models of

[2] Also see Sargent (1993).

Bray (1982) and Townsend (1983). In contrast, Bullard (1994), Bullard and Duffy (2001), and Chiarella and He (2003) highlight the possibility of non-convergence, while Timmermann (1996) has the REE as the asymptotic limit but observes that the path dependent effects of learning can be quite long lived, thus generating persistent non-equilibrium behavior.

Non-learning based adaptations include the genetic algorithm of LeBaron et al. (1999) in which the traders individually evaluate, switch, disassemble and reassemble the trading rules employed. Also prevalent in the literature are more structured models of switching over a fixed set of universal strategic alternatives. Of interest in switching models is how a strategy based on market fundamentals or rational expectations performs in comparison to, and in the presence of, alternative trading strategies. Non-rational strategies are demonstrated to survive and even prosper in such competitive settings. The feedback between the market and beliefs can be particularly important in the earned profits and population adoption of alternatives to fundamental strategies.

The two-choice model often consists of a fundamental strategy and a market-based strategy, such as this example from Gaunersdorfer and Hommes (2007),[3]

$$\begin{aligned} E_{1t}(p_{t+1}) &= p^e_{1,t+1} = p^* + v(p_{t-1} - p^*),\ 0 \le v < 1, \\ E_{2t}(p_{t+1}) &= p^e_{2,t+1} = p_{t-1} + g(p_{t-1} - p_{t-2}),\ g \ge 0. \end{aligned}$$

The fundamental value, p^*, drives expectations in the first equation while the alternate extrapolates the future value from current price innovation and is thus trend-following in nature.[4] The fundamental model generates price convergence to its fundamental value, making it inherently stabilizing. The trend-following strategy tends to cause the market to drift away from fundamental, making it inherently destabilizing. Complex dynamics in the market price can emerge as the consequence of the ebb and flow in the relative popularity of the two strategies. This is particularly true for models in which the population shifts towards adoption of trend-following expectations

[3]This formula is also employed by Frankel and Froot (1987). Variants of this trend following and fundamental reverting system are common in two choice models.

[4]Sometimes a contrarian strategy is incorporated in place of or in addition to the trend-following strategy, for example in Brock and Hommes (1998) and Chiarella and He (2001). Bias in the form of optimism and pessimism is another form of heterogeneity as employed by Lux (1995), Lux (1998), and Kirman (1993). Brock and Hommes (1997) allows for bias and extrapolation.

when the market is near equilibrium and towards fundamental expectations when the market is far from equilibrium. Such population dynamics are explicit in De Grauwe et al. (1993), Giardina and Bouchaud (2003), and Lux (1998) while achieved indirectly as the product of performance chasing in Brock and Hommes (1998) and Brock and LeBaron (1996), among others.[5]

While an individual trader's choice of strategy can be rooted in utility maximization in a discrete choice environment, the strategy options from which the traders must choose typically lack such micro-foundations. In the model developed for this paper, behavioral rules emerge to cope with limited information. The fundamental approach has the trader use public and private fundamental information to minimize mean squared error in forecasting future payoffs. Traders employing the market-based approach seek to accurately extract information from market observables to exploit the price inefficiency. Though they cannot know the exact relationship between price, information, and payoff, they develop the best model supported by the historical data. The market misbehavior that emerges is despite the efforts of the traders to make the best of the information they have and the consequence of inaccessibility of information needed to fully know the environment they populate.

Brock and LeBaron (1996) provides the fundamental traders with useful but idiosyncratically noisy private information. In contrast, Grossman and Stiglitz (1980) provides uniform error-free information and allows the market-based traders access to the current price, providing a mechanism for the trend-followers to extract the private information. The combination of the Brock and LeBaron noisy private signal and the Grossman and Stigliz contemporaneous market information employed in this investigation results in a market in which the price can be the superior source for private fundamental information. Goldbaum (2003) employs such an information environment, eliminating the arbitrary information cost as a necessary mechanism to motivate trader abandonment of the fundamental-based strategy.

Goldbaum (2005) and Goldbaum (2006) substitute a least-squares learning model for the fixed trend-following rule. The result is a market-based trading strategy that is not inherently desta-

[5]Hommes (2006) provides a summary of related dynamic heterogeneous agent models. Past performance can also have an impact on the influence of a strategy on the market through wealth effects, as in Chiarella and He (2001), Farmer and Joshi (2002), Chiarella et al. (2006), and Sciubba (2005).

bilizing, an important differentiating feature when referencing Brock and Hommes (1998). The combination of the dynamics of learning and switching is the alternative generator of financial market features such as clustered volatility and fat-tailed returns. Together, the learning and the use of the replicator dynamic (RD)[6] to capture trader choice create a Grossman and Stiglitz paradox with the accompanying absence of an equilibrium fixed point.

From both Brock and LeBaron (1996) and LeBaron et al. (1999), long memory stabilizes the dynamic system by encouraging a focus on fundamentals while short memory encourages short-run profitable behavior that destabilizes the system. Goldbaum (2006) considers a market model employing convergence-friendly long memory settings to produce an asymptotically stable non-equilibrium environment.

An exploration of learning and adaptation can also be found in Branch and Evans (2006), but their use of the discrete choice dynamics (DCD)[7] with long memory ensures the existence of a fixed point. At the Branch and Evans fixed point, there is no longer interaction between the learning and population processes.[8] This is in contrast to Goldbaum (2006) where the nature of the interaction is the source of the asymptotic stability.

The combination of a noisy signal of private information, an infinitely lived asset, and learning supported information extraction from the current price allows exploration of market efficiency and price dynamics produced by heterogeneity and switching on a platform very different from existing studies. The model contains a tension between a strategy of relying on imperfect fundamental information and that of seeking to optimally exploit the information content of market phenomenon. Unique to this model, neither strategy is inherently superior so that there is no need to introduce an arbitrary cost. Further, the market-based alternative to the fundamental information is not inherently destabilizing but contributes towards market efficiency when used appropriately. The finding is that the market cannot support exclusive use of a single information source in equilibrium. Employing both fundamental and market information is supported either in an equilibrium that

[6]Employed by Sethi and Franke (1995), difference in performance leads to a population shift towards the more successful strategy.

[7]Employed by Brock and Hommes (1997), the population is divided between strategies so that the more successful strategy is employed by a greater proportion of the population.

[8]Learning and adaptation can be considered present in the process of individual trading rule adjustment and discovery produced by the genetic algorithm employed by LeBaron et al. (1999).

tolerates unequal return performance or in a perpetual state of disequilibrium produced by profit-chasing behavior. The latter is found to be capable of producing substantial pricing error, depending on the behavior of the market participants.

Section 2 develops the model and describes the implied sources of market disruptions. Section 3 presents the results of simulations exploring the different sources of disruptions to efficient pricing. Section 4 concludes.

2 The Model

Analysis of the model reveals that equilibrium cannot be achieved with traders employing fundamental information alone. There is a role for market-based information in support of portfolio decisions. The relationship between the price and payoff, though, is found to depend on the unobserved extent to which traders rely on fundamental versus market-based information. In the absence of this information, a dynamic model is developed based on the trader estimating the relationship.

2.1 Information and model development

The market environment consists of a risky dividend-paying asset and a risk-free bond paying R. The risky asset can be purchased at price p_t and is subsequently sold at price p_{t+1} after paying the holder dividend d_{t+1}. The dividend process follows an exogenous AR(1) process

$$d_{t+1} = \phi d_t + \epsilon_{t+1},\ \phi \in (0,1) \tag{1}$$

with innovations distributed $\epsilon_t \sim \text{IIDN}(0, \sigma_\epsilon^2)$. The traders select one of two forecast methods based on the information sets

$$\begin{aligned} Z_{it} &\in \{Z_{it}^F, Z_{it}^M\} \\ Z_{it}^F &= \{s_{it}, d_t, d_{t-1}, \dots\} \\ Z_{it}^M = Z_t^M &= \{p_t, d_t, d_{t-1}, \dots\}. \end{aligned}$$

All traders have access to the dividend history up to time t. The two information types represent two extremes of the trader population. The fundamental method (F) makes use of a noisy signal on next period's dividend,

$$\begin{aligned} s_{it} &= d_{t+1} + e_{it} \\ e_{it} &\sim \text{IIDN}(0, \sigma_e^2). \end{aligned} \tag{2}$$

The market-based approach (M) makes use of endogenous market-generated information. Consistent with the model solution, p_t is the only useful market information available.[9]

Fundamental traders project d_{t+1} on the available information, obtaining the mean squared error minimizing forecast with

$$E(d_{t+1}|Z_{it}^F) = (1-\beta)\phi d_t + \beta s_{it} \tag{3}$$

and $\beta = \sigma_\epsilon^2/(\sigma_\epsilon^2 + \sigma_e^2)$.[10] Consistent with the fundamental traders' model, p_t can be expressed linearly in d_t and d_{t+1} so that

$$E(p_{t+1}|Z_{it}^F) = E\left(\frac{1}{R-\phi}((1-\alpha)\phi d_{t+1} + \alpha d_{t+2}) \middle| Z_{it}^F \right) \tag{4}$$

where α can take any value $\alpha \in [0,1]$.[11] Based on this structure the fundamental information based forecast of the excess payoff to the risky asset is

$$E(p_{t+1} + d_{t+1}|Z_{it}^F) = \left(\frac{R}{R-\phi} \right) E(d_{t+1}|Z_{it}^F). \tag{5}$$

The particular value of α drops out of the fundamental model's forecast of the uncertain payoff. This is convenient from a modeling perspective as the market clearing price will not require specifying the traders' belief about α, contributing to overall model robustness.

[9] Section 4 of Goldbaum (2006) considers a population of traders learning based on $Z_{it} = \{s_{it}, p_t, d_t, d_{t-1}, \dots\}$. The asymptotic behavior of the representative trader market is similar to that produced by this differentiated population driven by the replicator dynamic process.

[10] This value of β also has the interpretation as the signal to noise ratio.

[11] A value $\theta = \beta$ is consistent with a fundamental trader only market clearing price.

The market-based traders employ a forecasting model that is linear in all relevant variables consistent to forecasting the following period's payoff,

$$E(p_{t+1}+d_{t+1}|Z_t^M)=c_{0t}+c_{1t}p_t+c_{2t}d_t. \tag{6}$$

Each trader submits a demand function,

$$q_{it}(p_t) \quad = \quad (E(p_{t+1}+d_{t+1}|Z_{it})-Rp_t)/\gamma\sigma_{kt}^2, \tag{7}$$

that maximizes a CARA utility function with risk aversion coefficient γ and using conditional variance $\sigma_{kt}^2=\mathrm{Var}_{it}(p_{t+1}+d_{t+1}|Z_{it}^k)$. Let q_t^k be the average demand of the population of type k traders, $k=F,M$ based on individual expectations (5) and (6). With portion n_t of traders using the fundamental approach and $1-n_t$ employing the market-based approach, a consistent Walrasian price function is

$$p_t=p_t(n_t,\mathbf{c}_t)=b_{0t}+b_{1t}(n_t,\mathbf{c}_t)d_t+b_{2t}(n_t,\mathbf{c}_t)d_{t+1} \tag{8}$$

in which $\mathbf{c}_t$ represents a vector of the coefficients in (6). The coefficients of (8) solving the market clearing condition, $n_tq_t^F+(1-n_t)q_t^M=0$, are

$$b_0(n_t,\mathbf{c}_t) \quad = \quad \frac{c_{0t}(1-n_t)\frac{\sigma_F^2}{\sigma_M^2}}{n_tR+(1-n_t)(R-c_{1t})\frac{\sigma_F^2}{\sigma_M^2}} \tag{9}$$

$$b_1(n_t,\mathbf{c}_t) \quad = \quad \frac{n_t\frac{R}{R-\phi}(1-\beta)\phi+(1-n_t)c_{2t}\frac{\sigma_F^2}{\sigma_M^2}}{n_tR+(1-n_t)(R-c_{1t})\frac{\sigma_F^2}{\sigma_M^2}} \tag{10}$$

$$b_2(n_t,\mathbf{c}_t) \quad = \quad \frac{n_t\frac{R}{R-\phi}\beta}{n_tR+(1-n_t)(R-c_{1t})\frac{\sigma_F^2}{\sigma_M^2}}. \tag{11}$$

The presence of d_{t+1} in the price equation is a consequence of the market's aggregation of individual trader demand and the law of large numbers, which effectively filters out the individual idiosyncratic error components of the fundamental traders' signals.[12] Fundamental trader uncer-

[12]Formally, $p_t=b_{0t}+b_{1t}(n_t,\mathbf{c}_t)d_t+b_{2t}(n_t,\mathbf{c}_t)(d_{t+1}+\frac{1}{n_tN}\sum e_{it})$ but with large n_tN population of fundamental traders, the last term is of measure zero.

tainty, the consequence of awareness of the idiosyncratic component of their signal, explains the presence of d_t. The extent to which the market clearing price reflects the public d_t or the private d_{t+1} depends on the confidence of the fundamental traders in their signal (β), the beliefs of the market-based traders about the relationship between market observables and future payoffs (c_{0t}, c_{1t}, c_{2t}), the traders' uncertainties in predicting future payoffs ($\sigma^2_{Ft}, \sigma^2_{Mt}$), and the proportion of the market employing the fundamental strategy (n_t). Naturally, also present in the price coefficients is the opportunity cost of investing in the risky asset (R) and the AR(1) coefficient of the dividend process (ϕ).

Let $\hat{\pi}^k_t$ represent the performance measure associated with type k strategy using information up to the beginning of period t. The 2-choice version of the more general K choice Replicator Dynamic (RD) model found in Branch and McGough (2008) generates the transition equation

$$n_{t+1} = \begin{cases} n_t + r(\hat{\pi}^F_t - \hat{\pi}^M_t)(1 - n_t) & \text{for } \hat{\pi}^F_t \geq \hat{\pi}^M_t \\ n_t + r(\hat{\pi}^F_t - \hat{\pi}^M_t)n_t & \text{for } \hat{\pi}^F_t < \hat{\pi}^M_t \end{cases} \tag{12}$$

with

$$r(x) = \tanh(\delta x/2) \tag{13}$$

driving the n_t process. The alternative Discrete Choice Dynamic (DCD) model has as a transition function

$$n_{t+1} = \frac{1}{2}(1 + \tanh(\rho(\hat{\pi}^F_t - \hat{\pi}^M_t)/2)). \tag{14}$$

What distinguishes the two population processes is whether the performance differential determines the innovation in n_t or its level directly. Under the RD process, the more successful strategy attracts adherents from the less successful strategy, consistent with the process described in Grossman and Stiglitz (1980). Under the DCD, employed in Brock and Hommes (1998) and many related papers, $\hat{\pi}^F_t - \hat{\pi}^M_t$ maps directly into n_t with the superior strategy always employed by the majority of the population.

The conditional variance terms associated with the model specific forecast errors are derived using (8) and the appropriate (5) or (6),

$$
\begin{aligned}
p_{t+1} + d_{t+1} - E(p_{t+1} + d_{t+1}|Z_{it}^F) &= \left(1 + b_{1t} + \phi b_{2t} - \frac{R}{R-\phi}\right)(\phi d_t + \epsilon_{t+1}) \\
&\quad + b_{2t}\epsilon_{t+2} + \beta \frac{R}{R-\phi} s_{it} \qquad (15) \\
p_{t+1} + d_{t+1} - E(p_{t+1} + d_{t+1}|Z_t^M) &= (\phi(1 + b_{1t} + \phi b_{2t}) - c_{1t}(b_{1t} + \phi b_{2t}) - c_{2t}) d_t \\
&\quad + (1 + b_{1t} + \phi b_{2t} - c_{1t} b_{2t}))\epsilon_{t+1} + b_{2t}\epsilon_{t+2}. \qquad (16)
\end{aligned}
$$

2.2 A partial solution

Let REE(n_t) represent an n_t-dependent REE according to the definition,

Definition 1. An n_t-dependent Rational Expectations Equilibrium describes a market in which the coefficients of the market-based strategy in (6) are consistent with the actual price coefficients of the market clearing price function in (8). Further, the fundamental strategy employs beliefs about the price function consistent with (4) and forecast dividends according to (3).

The market described by a REE(n_t) is one in which the market-based trader's belief about the price formation process is consistent with the actual price function. Additionally, the fundamental traders forecast price in a manner consistent with its actual formation and forecast divided to minimize their mean-squared error. The equilibrium is described as n_t dependent because the belief-consistent values of $\mathbf{c}$ and $\mathbf{b}$ depend on n_t.

The REE(n_t) solution is the b_2 that solves (19), (22), and (23) of the following so that, for

$n \in (0, 1]$,

$$
\begin{aligned}
p_t^* = p_t^*(n_t) &= b_1^*(n_t)d_t + b_2^*(n_t)d_{t+1} && (17)\\
b_1^*(n_t) &= \frac{n_t(1-\beta)\phi}{(R-\phi)\left(n_t + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}\right)} && (18)\\
b_2^*(n_t) &= \frac{n_t\beta + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}}{(R-\phi)\left(n_t + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}\right)} && (19)\\
c_1^*(n_t) &= \frac{R}{(R-\phi)b_2^*(n_t)} = \frac{R\left(n_t + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}\right)}{n_t\beta + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}} && (20)\\
c_2^*(n_t) &= \frac{\phi}{R-\phi}(R - c_1^*(n_t)) = -\frac{n_t R(1-\beta)\phi}{(R-\phi)\left(n_t\beta + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}\right)} && (21)\\
\sigma_F^*(n_t)^2 &= \left((1-\beta)\left(\frac{R}{R-\phi}\right)^2 + b_2^*(n_t)^2\right)\sigma_\epsilon^2 && (22)\\
\sigma_M^*(n_t)^2 &= b_2^*(n_t)^2\sigma_\epsilon^2. && (23)
\end{aligned}
$$

For $n_t = 0$, $b_1^*(0) = \phi/(R-\phi)$, $b_2^*(0) = 0$ as derived from the consistent solution $c_1^*(0) = 0$ and $c_2^*(0) = R$.

Let p_t^0 and p_t^1 represent the price at the two information extremes based on the accuracy of the private signal. With zero content in the signal, $\beta = 0$, while zero error results in $\beta = 1$. For $n_t \neq 0$,[13]

$$
\begin{aligned}
p_t^0 &\equiv p_t^*(1)|_{\beta=0} = \frac{\phi}{R-\phi}d_t\\
p_t^1 &\equiv p_t^*(n_t)|_{\beta=1} = \frac{1}{R-\phi}d_{t+1}.
\end{aligned}
$$

Let p_t^F represent the price at the extreme of a market populated by only fundamental traders. With $n_t = 1$,

$$
p_t^F \equiv p_t^*(1) = \frac{(1-\beta)\phi}{R-\phi}d_t + \frac{\beta}{R-\phi}d_{t+1}.
$$

The opening for profitable employment of the market-based information is the fact that $p_t^F \in [p_t^0, p_t^1]$,

[13] p_t^0 and p_t^1 also correspond to the Fama (1970) semi-strong and strong form efficient prices.

introducing predictability in the price as a consequence of d_{t+1} contributing to the value of both p_t and p_{t+1}. The presence of the market-based traders moves the market towards the efficient market price, as reflected in $p_t^*(n_t) \in [p_t^F, p_t^1]$ with $\lim_{n_t \to 0} p_t^*(n_t) = p_t^1$. Since $p_t^*(0) = p_t^0$ there is a Grossman and Stiglitz (1980) type discontinuity at $n_t = 0$.

Observe that $b_1^*(n_t) + \phi b_2^*(n_t) = \phi/(R - \phi)$. Let

$$\alpha_t^* = \alpha^*(n_t) = \frac{n_t \beta + (1 - n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}}{n_t + (1 - n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}},$$

allowing the REE(n_t) price to be expressed as $p_t^* = \frac{1}{R-\phi}\left((1 - \alpha_t^*)\phi d_t + \alpha_t^* d_{t+1}\right)$. With $\alpha_t^* \in [\beta, 1]$, the REE($n_t$) price can be interpreted as the present discounted value reflecting the aggregation of the market's forecast of future dividends. The extent to which the REE(n_t) price reflects the public d_t or the private d_{t+1} depends on the traders.

Contributing to the model's robustness, the REE(n_t) solution is consistent with the fundamental trader presumption that $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$. The development of the REE(n_t) does not depend on the trader's knowledge of b_{1t} and b_{2t} individually. This is convenient since the latter would require the fundamental traders be aware of n_t. As a consequence, the fundamental model can rationally employ the correct $p_t^*(n_t)$ price function. Alternatively, without any consequence on the REE(n_t) solution, the traders may mistakenly employ $p_t^*(m_t)$ for $m_t \neq n_t$, or naively employ the p_t^F, p_t^0, p_t^1, or any other price structure consistent with (4) without consequence on the market clearing price.

Useful for non-equilibrium pricing, the condition under which the fundamental traders can rely on $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$ is weaker than the conditions necessary to generate the full REE(n_t). It only requires $c_{2t} = (R - c_{1t})\phi/(R - \phi)$ as evidenced by substituting for c_{2t} in (10) to produce $b_1(n_t, c_{1t}) + \phi b_2(n_t, c_{1t}) = \phi/(R - \phi)$ regardless of the value of c_{1t}. Thus, the condition $c_{2t} = c_2^*(c_{1t})$ implied by (21) is a sufficient condition to support the price structure underpinning the fundamental strategy. That is, in order for the fundamental traders' forecast to conform to the requirements of Definition 1, the market-based traders need only employ a c_{2t} value that is REE(n_t) consistent with c_{1t} without necessarily employing the correct REE(n_t) implied $c_{1t} = c_1^*(n_t)$.

The REE(n_t) depends on the market-based traders correctly employing $\mathbf{c}_t = \mathbf{c}^*(n_t)$ without error. It is appropriate to ascertain whether the traders can deduce $\mathbf{c}^*$ analytically from their knowledge of the market. From (21), c_2^* can be expressed in terms of c_1^*. For a known zero net supply of the risky asset, the traders can determine that $c_0^* = 0$. For market-based traders incorporating these two conditions into their understanding of the market, only c_1^* remains to be derived. From (20), solving for c_1^* requires knowledge of n_t. Reasonably, n_t is not directly observable and thus excluded from Z_{it}. Since n_t is the endogenous product of a dynamic system, the question becomes whether some n^{fp} value can be identified that is consistent with REE(n_t).

Forward looking traders would select a strategy based on a forecast of the performance of the strategy's employment in the current period. Define performance in terms of individual profit,

$$\pi_{it}^k = q_{it}^k (p_{t+1} + d_{t+1} - Rp_t). \tag{24}$$

Using the REE(n_t) consistent $c_0 = 0$ and $c_{2t} = c_2^*(c_{1t})$, and the market clearing condition for b_{2t} from (11), (24) generates, for $n_t \in (0, 1]$,

$$\begin{aligned} E(\pi_t^F) &= (1 - n_t)\Delta_t & (25) \\ E(\pi_t^R) &= -n_t \Delta_t & (26) \end{aligned}$$

so that $E(\pi_t^F - \pi_t^M) = \Delta_t$. Here,

$$\Delta_t = \Delta(c_{1t}, n_t) = \left(\frac{n_t(1-\beta)R + (1-n_t)(R - c_{1t})\frac{\sigma_F^2}{\sigma_M^2}}{\left(n_t R + (1-n_t)(R - c_{1t})\frac{\sigma_F^2}{\sigma_M^2}\right)^2} \right) \left(\frac{R}{R - \phi} \right)^2 \frac{(R - c_{1t})\beta\sigma_\epsilon^2}{\sigma_M^2}. \tag{27}$$

The REE(n_t) expected profit differential, $E(\pi_F^* - \pi_M^*)$, based on $c_{1t} = c_1^*(n_t)$, reduces to

$$\Delta^*(n_t) = -\left(\frac{1-\beta}{n_t + (1-n_t)\frac{\sigma_F^{*2}}{\sigma_M^{*2}}} \right)^2 \left(\frac{R}{R - \phi} \right)^2 \frac{n_t \sigma_\epsilon^2}{\sigma_M^{*2}}. \tag{28}$$

That $\Delta^*(n_t) < 0$ for all $n_t \neq 0$ reveals the benefit to extracting filtered information from the REE

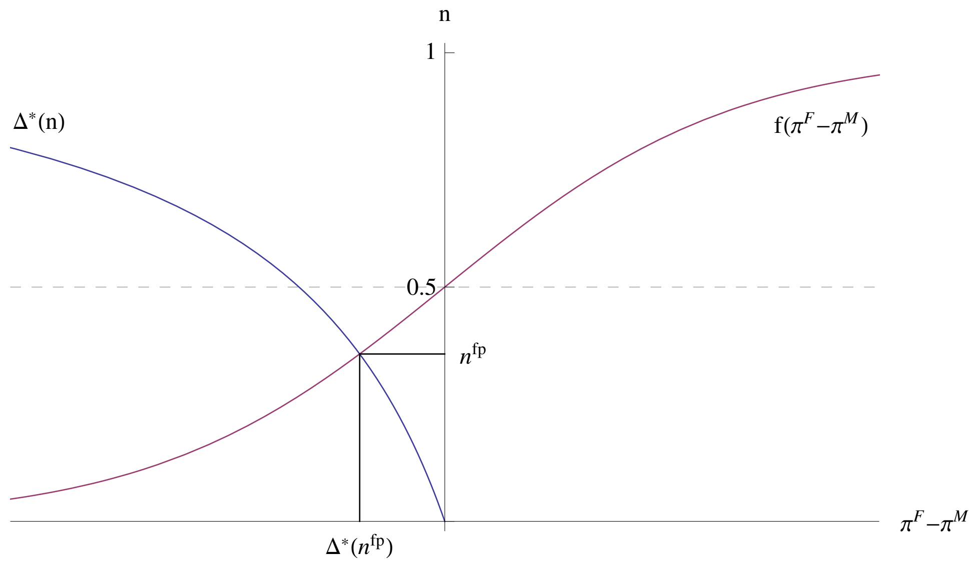


Figure 1: n^{fp} for DCD population dynamics.

market over direct access to noisy information. The fundamental traders only profit in the presence of error in the market-based traders' model, as c_{1t} deviates sufficiently from $c_1^*(n_t)$, allowing Δ_t to be positive.

A fixed point to the entire dynamic system thus requires the REE(n_t) solution combined with a fixed point to the population process. The fixed point condition depends on the population regime.

Proposition 1. *For $\rho \in [0, \infty)$, there exists a unique fixed point n^{fp} to the dynamic system consisting of (1), (14), and (17).*

Proof. Under the DCD population process, $n_{t+1} = f(\hat{\pi}_t^F - \hat{\pi}_t^M)$ according to (14) and at the REE(n_t), $\hat{\pi}_t^F - \hat{\pi}_t^M = \Delta^*(n_t)$. For $\rho < \infty$, $f(x)$ is continuous and monotonically increasing in x. A fixed point solution is n^{fp} such that $n^{fp} = f(\Delta^*(n^{fp}))$. Since $\lim_{n_t \to 0} \Delta^*(n_t) = 0$ and $\Delta^*(n_t)$ is monotonically decreasing as n_t increases to one, a unique n^{fp}, $0 < n^{fp} \leq 1/2$, such that $n^{fp} = f(\Delta^*(n^{fp}))$ exists. □

Figure 1 captures the existence of the fixed point under the DCD population process. Since the slope of $f(0)$ increases with ρ in Figure 1, the value of $n^{fp} \in (0, 1/2]$ decreases with increasing ρ. At the extremes, $\rho = 0$ results in a horizontal $f(\pi^F - \pi^M)$ and $n^{fp} = 1/2$ while $\rho \to \infty$ approaches a step function in $f(\pi^F - \pi^M)$ so that $n^{fp} \to 0$. With $E(\pi_{fp}^F - \pi_{fp}^M) < 0$, the DCD fixed point is inconsistent with the Grossman and Stiglitz (1980) notion of an equilibrium in which the expected performance differential is zero.

Proposition 2. *No fixed point exists for dynamic system consisting of (1), (12), and (17).*

Proof. Under the RD population process, the fixed point condition requires the existence of an n^{fp} such that $n^{fp} = f(\Delta^*(n^{fp}), n^{fp})$, a condition that reduces to simply n^{fp} such that $\Delta^*(n^{fp}) = 0$. Since no such n^{fp} exists, there can be no fixed point to the RD population process. $\square$

The fixed point condition for the population process requires that $\Delta(c_{1t}, n_t) = 0$. A fixed point n^{fp} exists as a function of c_{1t} so that $n_t^{fp} = n^*(c_{1t})$. The existence of an REE, attainable either analytically or through learning, depends on the existence of an (n^{fp}, c_1) combination for which $n^{fp} = n^*(c_1^*(n^{fp}))$. Such a point does not exist since for $c_{1t} = c_1^*(n_t)$, $\Delta^*(n_t) < 0$ for all $n_t \in (0, 1]$ and $E(\pi_t^F - \pi_t^M) > 0$ for $n_t = 0$.

2.3 Learning

With the RD driving the evolution of the population, the dynamic system is characterized by the existence of an REE(n_t) in a belief that depends on knowledge of the unobservable n_t and is the source of instability in n_t. Without a fixed point, the model exists in a permanent state of adjustment. This investigation considers the impact on the market of reasonable boundedly rational trader behavior given the non-equilibrium market condition.

Allow the traders to estimate the value of each of the needed n_t-dependent variables as accommodation to the inability to develop an analytical solution. All traders would need to estimate both π_t^F and π_t^M as inputs to the model adoption decision. Each trader would need an estimate of the σ_{kt}^2 appropriate for the model adopted as an input to demand. The use of the market-based model would need an estimate for $\mathbf{c}_t$. Consider the updating algorithms

$$
\begin{aligned}
\hat{\mathbf{c}}_t &= \hat{\mathbf{c}}_{t-1} + \lambda_t (Q_t^{-1} x_{t-2}(p_{t-1} + d_{t-1} - \hat{\mathbf{c}}_{t-1} x_{t-2}))' && (29)\\
\hat{Q}_t &= \hat{Q}_{t-1} + \lambda_t (x_{t-1} x_{t-1}^{'} - \hat{Q}_{t-1}) && (30)\\
\hat{\sigma}_{kt}^2 &= \hat{\sigma}_{kt-1}^2 + \theta_t ((p_t + d_t - E(p_t + d_t | Z_{t-1}^k))^2 - \hat{\sigma}_{kt-1}^2) && (31)\\
\hat{\pi}_t^k &= \hat{\pi}_{t-1}^k + \mu_t (\pi_{t-1}^k - \hat{\pi}_{t-1}^k),\ k = F, M && (32)
\end{aligned}
$$

with $x_t = \{1, p_t, d_t\}$. For $\lambda_t = 1/t$, the traders update the market-based model consistent with the standard least-squares learning algorithm of Marcet and Sargent (1989b). For $\lambda_t = \lambda$, $0 < \lambda < 1$, the traders update with a constant gain by which the contribution of past observations to the current parameter estimate decays exponentially. Similarly, $\hat{\sigma}_{kt}^2$ and $\hat{\pi}_t^k$ are simple sample averages of all past observations if $\theta_t = \mu_t = 1/t$ but a constant gain emphasizes the more recent observations.

Proposition 3. *For a fixed n with $\lambda_t = 1/t$ and $\sigma_{kt}^2 = \sigma_k^2(n, \mathbf{c}_t)$, the dynamic system consisting of (1), (8), (29), and (30) is locally stable at the REE(n).*

Proof. See Appendix $\square$

By Proposition 3, the DCD population process possibility converges to the REE(n^{fp}). The RD population process requires further analysis to understand the possible system dynamics.

2.4 Evolution without a Fixed Point

The basic system consists of the dividend process according to (1), the Walrasian market clearing price as captured in (8), and the system for updating beliefs described by (29) through (32). In addition, two population processes are considered for setting n_t, the DCD process of (14) and the RD process of (12). Imposing some constraints, much of what transpires in the dynamic system can be understood from the phase space of n_t and $\hat{c}_{1t}$ based on the RD population process in Figure 2. In order to generate the phase space, assume a high degree of rationality in the market-based model so that $c_{0t} = c_0^* = 0$ and $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$, according to (21). This leaves $\hat{c}_{1t}$ as the only unknown parameter of the market-based model. Let $\hat{\sigma}_{tk}^2 = \sigma_k^*(n_t)^2$ for $k \in \{F, M\}$ so that the employed conditional variances are correct expressions reflecting the current n_t. Finally, let $\mu_t = 1$ so that expected relative performance is captured by $\Delta(\hat{c}_{1t}, n_t)$.

The $\hat{c}_{1t}$ process is at a fixed point and the REE(n_t) if $\hat{c}_{1t} = c_1^*(n_t)$. At the REE$(n_t)$, the market-based model correctly reflects the relationship between the observables p_t and d_t and the expected payoff of the following period, $E(p_{t+1} + d_{t+1})$. The function $c_1^*(n_t)$ is monotonically increasing for $0 < n_t \leq 1$ with $c_1^*(n) \to R$ for $n \to 0$ and $c_1^*(1) = R/\beta$.

The population process is at a fixed point if $\Delta(\hat{c}_{1t}, n_t) = 0$. Let $c_1^+(n_t)$ and $c_1^-(n_t)$ represent the two functions capturing combinations of $\hat{c}_{1t}$ and n_t consistent with $\Delta(\hat{c}_{1t}, n_t) = 0$ in (27). For $0 < n_t \leq 1$, the former is monotonically increasing and everywhere above $c_1^*(n_t)$,

$$c_1^+(n_t) = R\left(1 + (1-\beta)\frac{n_t}{1-n_t}\frac{\sigma_M^2}{\sigma_F^2}\right), \tag{33}$$

while the latter is a constant, located below $c_1^*(n_t)$, at $c_1^- = R$. Expected profits are zero at $\hat{c}_{1t} = c_1^+(n_t)$ because the resulting market clearing price is the efficient market price, p_t^1, at which expected profits are zero regardless of the individual trader's position taken in the market. Expected profits are zero at $\hat{c}_{1t} = c_1^-(n_t)$ because the market traders expect the risky asset to offer the same return as the risk-free bond and thus there is no trading at the market clearing price.

A final relevant function included in the phase space is $\tilde{c}_1(n_t)$. The expression $n_t R + (1-n_t)(R - \hat{c}_{1t})\frac{\sigma_F^2}{\sigma_M^2}$ appears in the denominator of the two pricing coefficients, $b_1(\hat{c}_{1t}, n_t)$ and $b_2(\hat{c}_{1t}, n_t)$. The expression's negative is the slope of the aggregate demand function so that when it is zero, the market demand function is horizontal and different from zero, producing an infinite market clearing price (based on a zero net supply). Let $\tilde{c}_1(n_t)$ be the function

$$\tilde{c}_1(n_t) = R\left(1 + \frac{n_t}{(1-n_t)}\frac{\sigma_M^2}{\sigma_F^2}\right), \tag{34}$$

capturing the combinations of $\hat{c}_{1t}$ and n_t such that $n_t R + (1-n_t)(R - \tilde{c}_1(n_t))\frac{\sigma_F^2}{\sigma_M^2} = 0$. For $0 < n_t \leq 1$, $\tilde{c}_1(n_t)$ is monotonically increasing and everywhere above $c_1^+(n_t)$. As the function is approached from below or from the right the $p_t(c_{1t}, n_t) \to \pm\infty$.

Above $\tilde{c}_1(n_t)$, the combination of n_t and $\hat{c}_{1t}$ do not allow for a reasonable market clearing price.

The precarious nature of the market in the vicinity of $\tilde{c}_1(n_t)$ is the consequence of the excessive influence of the market-based traders. As a group, they have an upward sloping demand function in price. From the perspective of the market-based traders, an increase in the price is interpreted as an indication of good news about the underlying d_{t+1}, increasing demand. At $\hat{c}_{1t} = c_1^*(n_t)$, the market-based model correctly accounts for the influence of the market-based trader population on the price. As a consequence, the aggregate demand for the risky asset remains downward sloping in p_t. For $\hat{c}_{1t} > c_1^*(n_t)$, the market-based model projects too large a deviation in d_{t+1} based on the observed p_t. The market-based traders thus take too large a position relative to the underlying reality. For $\hat{c}_{1t} > \tilde{c}_1(n_t)$, the position produces an upward-sloping demand function.[14]

Whether the market can be relied on to behaved reasonably well depends on whether the system can be relied upon to remain well below $\tilde{c}_1(n_t)$. The traders themselves cannot be relied upon to recognize dangerous market conditions introduced by their own belief. For any $\hat{c}_{1t} \in (R, R/\beta]$ there exists n_1 and n_2, $0 < n_1 < n_2 \leq 1$ for which $\hat{c}_{1t} = \tilde{c}_1(n_1)$ and $\hat{c}_{1t} = c_1^*(n_2)$. The market-based traders' belief that $c_{1t} = \hat{c}_{1t}$ is reasonable if the unobserved n_t is near n_2 but disastrously wrong, generating substantial mispricing if n_t is near n_1.

Given n_t, $c_1^*(n_t)$ is an attractor for $\hat{c}_{1t}$. For $\hat{c}_{1t}$ between $c_1^-(n_t)$ and $c_1^+(n_t)$, $E(\Delta(\hat{c}_{1t}, n_t)) < 0$ so that n_t tends to decline. In this range, the market-based model, while not perfectly correct for extracting information from the price, is sufficiently correct so that the market-based forecasts are more accurate than the average fundamental trader relying on a noisy signal. Outside this range, with $\hat{c}_{1t} < c_1^-$ or $c_1^+(n_t) < \hat{c}_{1t} < \tilde{c}_1(n_t)$, the inaccuracy in the market-based model is large enough that the user of the fundamental information expects to earn profits at the expense of the market-based traders and therefore n_t tends to increase in this region.

All four functions of the phase space radiate out from the point $n_t = 0$ and $\hat{c}_{1t} = R$ but because of the discontinuity at $n_t = 0$, none take a value of R at $n_t = 0$. Therefore, though the four functions come arbitrarily close, they never intersect. The failure of $c_1^*(n_t)$ to intersect with either $c_1^-(n_t)$ or $c_1^+(n_t)$ graphically captures the absence of a fixed point to the dynamic system.

[14] As an alternate interpretation, for $\hat{c}_{1t} > c_1^*(n_t)$, the market-based model can be seen as underestimating the influence of the market-based traders on the price since, for $\hat{c}_{1t} < R/\beta$, there exists $n > n_t$ such that $\hat{c}_{1t} = c_1^*(n)$.

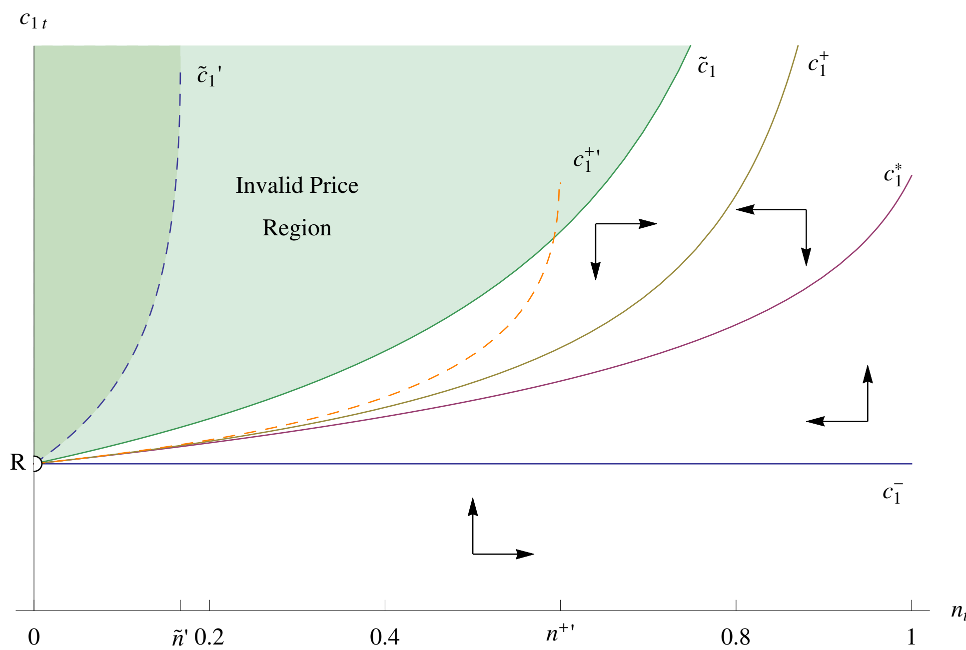


Figure 2: Phase space in n_t and $\hat{c}_{1t}$ for the RD population process. $c_1^*(n_t)$ is the REE(n_t) value of $\hat{c}_{1t}$ and the attractor to the learning process for a given n_t. For $c_1^- < \hat{c}_{1t} < c_1^+(n_t)$ the market-based model is sufficiently accurate to earn profits at the expense of the fundamental strategy leading to a decline in n_t. For $\hat{c}_{1t} < c_1^-$ and for $c_1^+(n_t) < \hat{c}_{1t} < \tilde{c}_1(n_t)$ the fundamental strategy dominates the market-based strategy so that from these regions n_t is increasing. Above $\tilde{c}_1(n_t)$, the aggregate demand curve for the risky security is upward sloping and no positive price exists to clear the market. The boundaries of the phase space are affected by how the traders estimate the conditional variance associated with their forecasts. The dashed lines reflect an alternate specification for which $\hat{c}_{1t} \neq c_1^*(n_t)$ is recognized when calculating the market-based model error.

2.5 Degrees of Bounded Rationality

The market-based traders employ least-squares learning to overcome the inability to form rational expectations about the relationship between market observables and payoffs. Different levels of rationality can be considered. Traders aware of the market and its structure, as previously asserted, can deduce that $c_0 = 0$ regardless of the unobservable n_t. They may also choose to incorporate a feature of the REE(n_t) by imposing $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$ according to (21). At the other extreme, the traders can let the data drive the estimates of all three regression coefficients.

Traders aware of the market structure recognize that the least-squares learning process is not asymptotically consistent with the setting. There is no fixed point to the dynamic system to which the learning process can potentially converge. The least-squares learning performs best for the market-based traders if n_t is relatively stable over time so that the relationship between price and payoff changes slowly relative to the rate of learning. The aware trader recognizes that the more accurate the market-based model, due to consistency in the data across time, the greater the incentive for a performance-induced decline in n_t. As n_t changes over time, the market traders must update their coefficient estimates. The equal weighting of past observations in least-squares learning consistent $\lambda_t = 1/t$ contributes to the stability of the system but individual accommodation to the changing environment may lead traders to employ constant gains, setting $\lambda_t = \lambda$ in (29) and (30), in order to emphasis more recent observations. Similarly, the traders have to choose how to weight realized performance in computing the performance measure in (32).

Another challenge for the traders is how to evaluate their error associated with the forecast of the payoff, a component of the demand equation submitted to the Walrasian auctioneer. The error variance can be derived for each forecast strategy, but only with knowledge of the true n_t-dependent pricing relationship. Equations (22) and (23) capture the REE(n_t) values while (31) is driven by the data. For $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$, but $\hat{c}_{1t} \neq c_1^*(n_t)$, the conditional variances become

$$\sigma_F^2(n_t, \hat{c}_{1t}) = \left((1-\beta) \left(\frac{R}{R-\phi} \right)^2 + b_2^2(n_t, \hat{c}_{1t}) \right) \sigma_\epsilon^2 \tag{35}$$

$$\sigma_M^2(n_t, \hat{c}_{1t}) = \left(\left(\frac{R}{R-\phi} - \hat{c}_{1t} b_2(n_t, \hat{c}_{1t}) \right)^2 + b_2^2(n_t, \hat{c}_{1t}) \right) \sigma_\epsilon^2, \tag{36}$$

reflecting the dependence of the market-based strategy on the accuracy of the employed forecast model parameters. The bounded rational alternative is to have the traders estimate the values.

3 Simulations

Simulations confirm that the model driven by random innovations in dividends can produce the dynamics suggested in the Figure 2 phase space. The simulations also explore the impact of altering aspects of the model, including the parameters δ (or ρ), λ_t, and μ_t. Variations will also include features capturing the rationality or sophistication of the market participants in determining how they estimate $\hat{c}_{2t}$, σ^2_{Mt}, and σ^2_{Ft} as well as comparing the two population processes, RD versus DCD.

3.1 Environment Settings

Let p^1_t, the strong form efficient market price, be the standard against which the market price is evaluated. Let $|p_t - p^1_t|$ be the measure of market inefficiency. In general

$$p_t - p^1_t = (b_{1t} + \phi b_{2t} - \phi/(R-\phi))d_t + (\phi b_{2t} - \phi/(R-\phi))\varepsilon_{t+1}. \tag{37}$$

Equation (37) reveals two sources of deviation from p^1_t. The condition $b_{1t} + \phi b_{2t} = \phi/(R-\phi)$ only requires $\hat{c}_{2t} = c^*_2(\hat{c}_{1t})$, allowing $\hat{c}_{1t}$ to deviate from its REE(n_t) value while still producing zero for the first term. The second term requires the REE(n_t) coefficients and $n_t \to 0$ to generate $\phi b^*_2(n_t) \to \phi/(R-\phi)$. Deviation in the first term from zero indicates market-based trader error induced mispricing of public information. The second term is a reflection of the failure to fully and properly include private d_{t+1} information into the price.

All simulations share the parameter values $R = 1.02$, $\phi = 0.5$, $\sigma_\epsilon = \sigma_e = 1$ so that $\beta = 1/2$, and the starting value, $n_0 = 0.75$. Pre-simulation learning on the market-based model takes place on 200 observations generated using a fixed $n_t = n_0$.

Figures 3 through 8 display the evolution of endogenous parameters typically produced by the simulations. To aid direct comparison, each figure is based on the same underlying randomly

Table 1: Simulation parameter settings

Simulation	n_t process	δ or ρ	λ	μ	$\hat{c}_{2t}$	$\hat{\sigma}_t^k$
1	DCD	10	$1/t$	$1/t$	$c_2^*(\hat{c}_{1t})$	exp
2	RD	0.01	$1/t$	$1/t$	$c_2^*(\hat{c}_{1t})$	exp
3	RD	1	$1/t$	$1/t$	$c_2^*(\hat{c}_{1t})$	exp
4	RD	0.05	$1/t$	1	$c_2^*(\hat{c}_{1t})$	$\sigma^*(n_t)$
5	RD	0.01	0.01	$1/t$	$c_2^*(\hat{c}_{1t})$	exp
6	RD	0.01	$1/t$	$1/t$	estimate	exp
Shared Parameters: $R = 1.02$, $\phi = 0.5$, $\sigma_\epsilon = \sigma_e = 1 \Rightarrow \beta = 1/2$						

generated $\{d_t\}$ series. Each frame plots the time progression of endogenous parameters of the model (green). The top left frame plots $\hat{c}_{1t}$ with a solid line at $\lim_{n\to 0} c_1^*(n) = c_1^- = R$. When useful, the frame also includes $c_1^*(n_t)$ (red), $\hat{c}_{1t}(n_t)$ (cyan), and $\tilde{c}_1(n_t)$ (blue) as determined by $\hat{\sigma}^2_{Mt}$ and $\hat{\sigma}^2_{Ft}$. The top right frame plots ϕb_{2t} with a solid line at $\phi/(R-\phi)$, reflecting the value that sets the second term of (37) to zero. The lower left frame plots n_t and the lower right plots $p_t - p_t^1$. Figure 8 also includes frames for $\hat{c}_{2t}$ and of $b_{1t} + \phi b_{2t}$ since these are not constrained in Simulation 6.

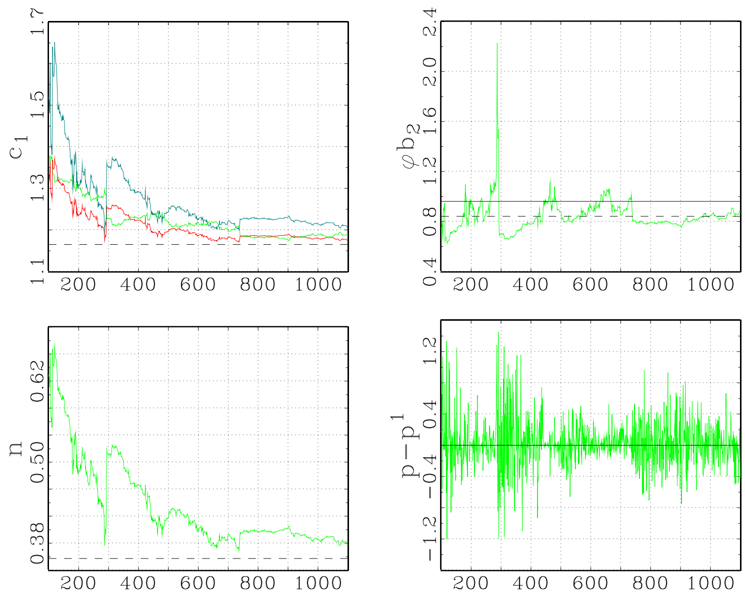
3.2 Discrete Choice Dynamics

The local stability of the fixed point under the DCD is assured if the traders employ $\mu_t = 1/t$ in their performance updating. Figure 3a shows the early convergence of the system towards the fixed point values of the respective parameter, which is indicated with a dashed horizontal line. Figure 3b shows the asymptotic properties of the convergence.

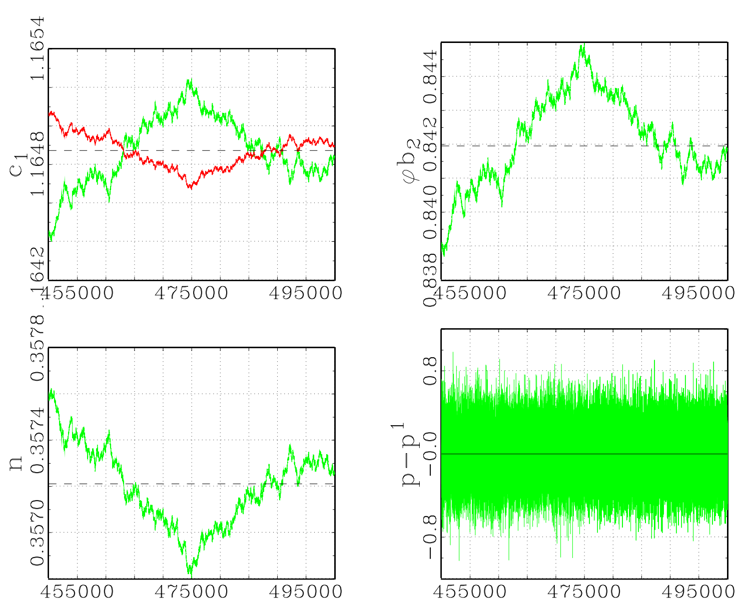
Shorten the memory associated with performance and the random component of realized returns becomes, for $\rho \neq 0$, the source of noise in n_t around n^{fp}. Larger ρ generates a wider distribution of realized n_t around the fixed point n^{fp}. For sufficiently large ρ the lower tail of the distribution of realized n_t enters the invalid price region of Figure 2, halting the simulation. At the default parameters and $\mu_t = 1$, $\rho > 0.5$ is large enough to prevent convergence.

3.3 Replicator Dynamics

The RD process offers a point of attraction at $n_t = 0$ and $\hat{c}_{1t} = R$. When the system is well-behaved, with $n_t \to 0$ and the remaining parameters consistent with the REE(n_t) solution, the



(a) Early convergence towards REE(n^{fp})



(b) Asymptotic properties around REE(n^{fp})

Figure 3: Baseline DCD: Convergence to a REE fixed point at $n^{fp} = 0.3572$.

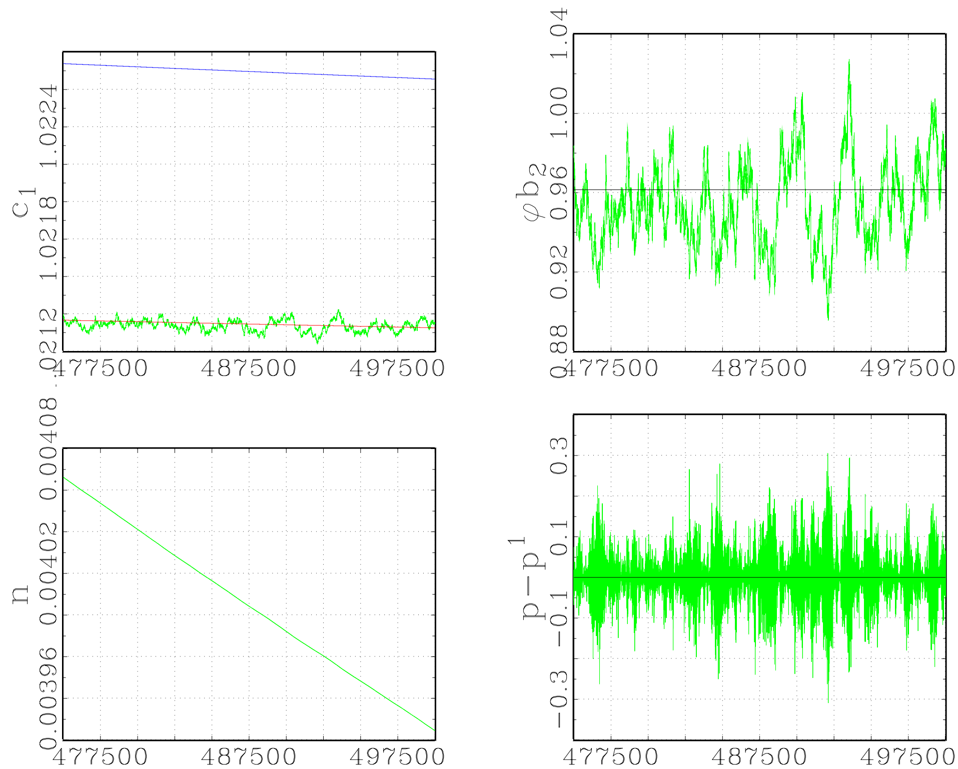


Figure 4: Baseline RD produce smooth convergence $n_t \to 0$ with $\hat{c}_{1t} \to c_1^*(n_t)$.

REE(n_t) solution has $b_{2t} \to \phi/(R-\phi)$ as $n_t \to 0$.

The first RD simulation employs parameters consistent with convergence towards the attractor. The users of the market-based model are highly rational, imposing $c_{0t} = 0$ and $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$. The gain parameters in (29) and (32) are least-squares learning consistent with $\lambda_t = \mu_t = 1/t$ and the traders' estimates of σ_{Ft}^2 and σ_{Mt}^2 are derived from experience and updated according to (31).

Figure 4 is typical of the time series generated by this environment. The main features are that there is convergence in n_t towards zero and progressive updating of $\hat{c}_{1t}$ so that it remains close to $c_1^*(n_t)$ and well below $\tilde{c}_1(n_t)$. As a consequence, $\hat{c}_{1t}$ converges towards $\lim_{n \to 0} c_1^*(n) = R$. Contributing to the smooth process of convergence is the slow evolution in n_t, a product of a relatively small δ. Increasing δ results in n_t oscillating while maintaining an underlying process of convergence

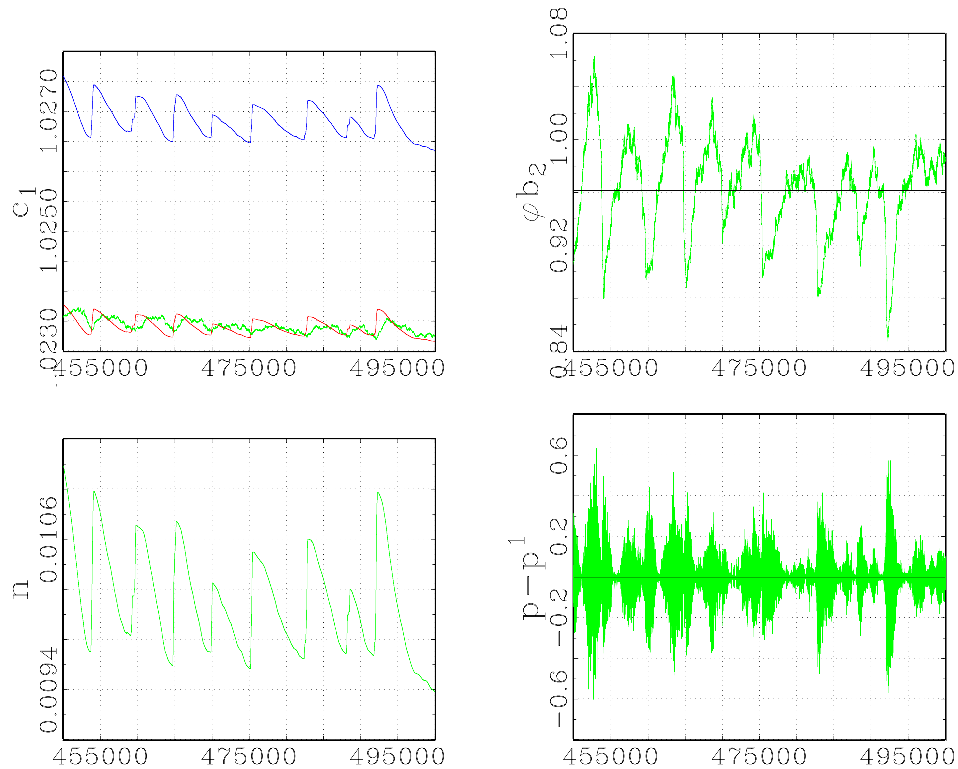


Figure 5: RD with high sensitivity to performance, with $\delta = 1$, produces oscillations in n_t overlaying its general decreasing trend.

towards zero, as captured in Figure 5.

The simulation presented in Figure 6 substitutes the long memory of $\mu_t = 1/t$ with $\mu_t = 1$. To better replicate the dynamic rules governing the phase space in Figure 2, the traders measure conditional variance using (22) and (23) rather than the experience-driven (31).[15] In this scenario, the convergence of n_t towards zero is halted with n_t hovering around 0.39. At time t, the d_{t+2} component of p_{t+1} remains unpredicted by the market. Payoffs are thus not perfectly forecastable and the realization produces realized profits that deviate from expectations. With a short memory, profit realizations generate movement in n_t from period to period that undermine the learning of

[15]The impact of how traders estimate conditional variance on the simulation is negligible.

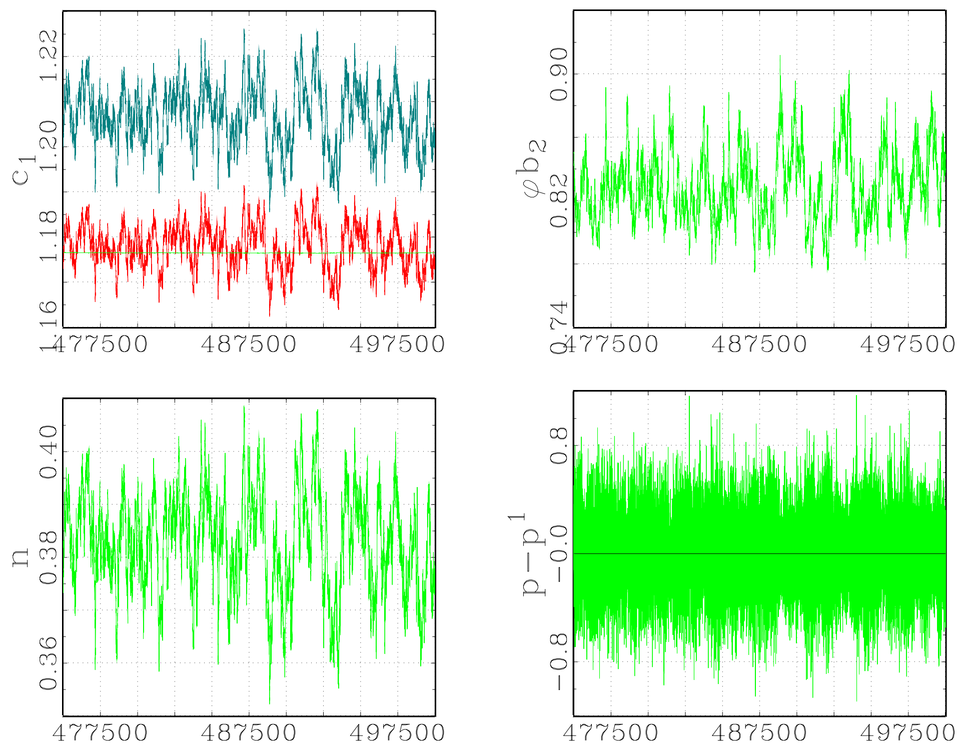


Figure 6: RD with $\mu_t = 1$ stabilizes n at a value above zero. The estimate $\hat{c}_{1t}$ is stable over time while $c_1^*(n_t)$ fluctuates rapidly with the fluctuations in n_t.

$\hat{c}_{1t}$ and ultimately the convergence of n_t.

Switching to constant gains in the updating of the market-based model parameters, with $\lambda_t = 0.01$, generates a different kind of non-convergence. As seen in Figure 7, after a period of learning $\hat{c}_{1t}$ settles into a stable distribution relative to $c_1^*(n_t)$, moving over time to track movement in $c_1^*(n_t)$. For sufficiently large n_t, the narrow distribution in $\hat{c}_{1t}$ favors the market-based model that allows quick adjustment in beliefs to the evolving market condition. The constant gain becomes a liability as n_t converges towards zero, the distribution in $\hat{c}_{1t}$ is at some point too wide to remain between c_1^- and c_1^+. The resulting mispricing substantially rewards the fundamental model, reversing the progress in n_t with $\hat{\pi}^F - \hat{\pi}^M$ remaining positive for some time while the small profits earned by the

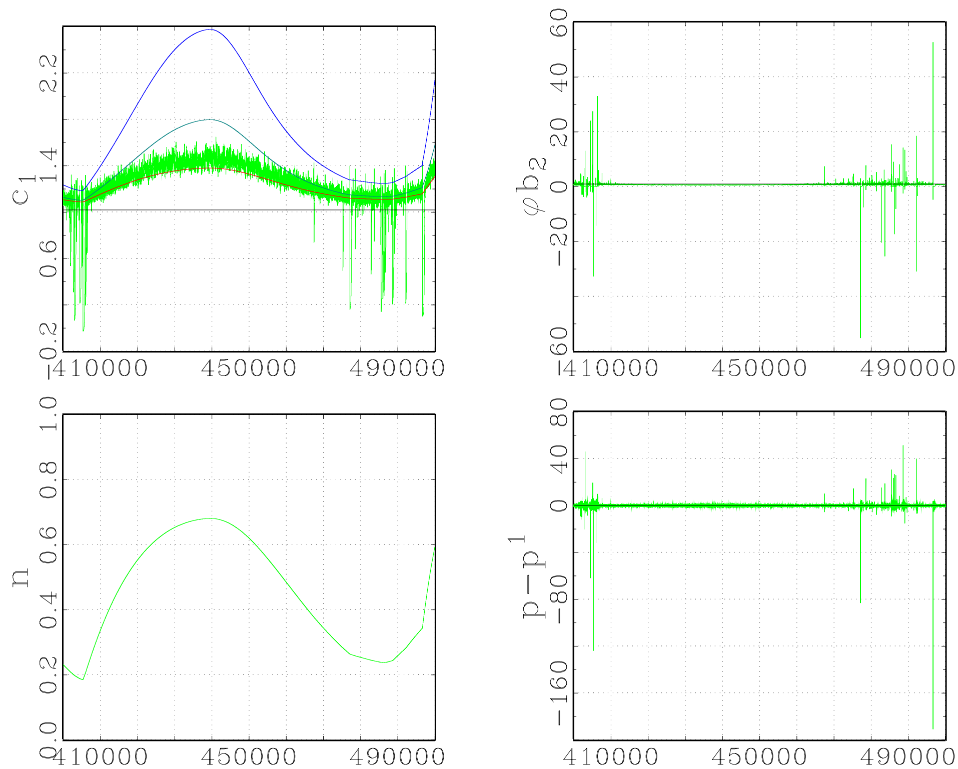


Figure 7: RD with $\lambda_t = 0.01$ generating small but consistent model error that produces long periods of near efficient pricing with inevitable bursts of mispricing.

market-based strategy take time to accumulate.

Relax the rationality of the trader by decoupling $\hat{c}_{2t}$ from $c_2^*(\hat{c}_{1t})$ so that the market-based strategy estimates both c_{1t} and c_{2t} through the learning process of (29) and (30). The consequence is two fold. First, the possible inconsistency between $\hat{c}_{1t}$ and $\hat{c}_{2t} \neq c_2^*(\hat{c}_{1t})$ introduces a new source of error in the market-based model. Second, there is additionally a price impact beyond simply adding to the magnitude of the error.

Figure 8 includes a middle row plotting $\hat{c}_{2t}$ on the left and $b_{1t} + \phi b_{2t}$ on the right. The latter captures that the first term of (37) is no longer constrained to zero. Under this environment, the market clearing pricing function is no longer consistent with the pricing function employed in

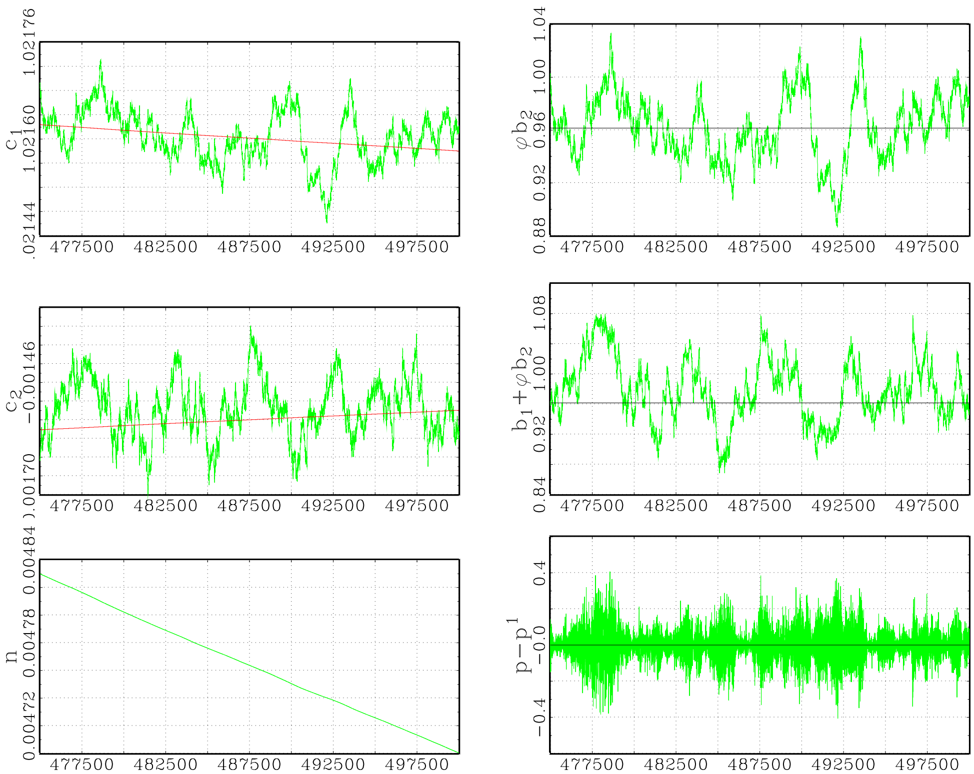
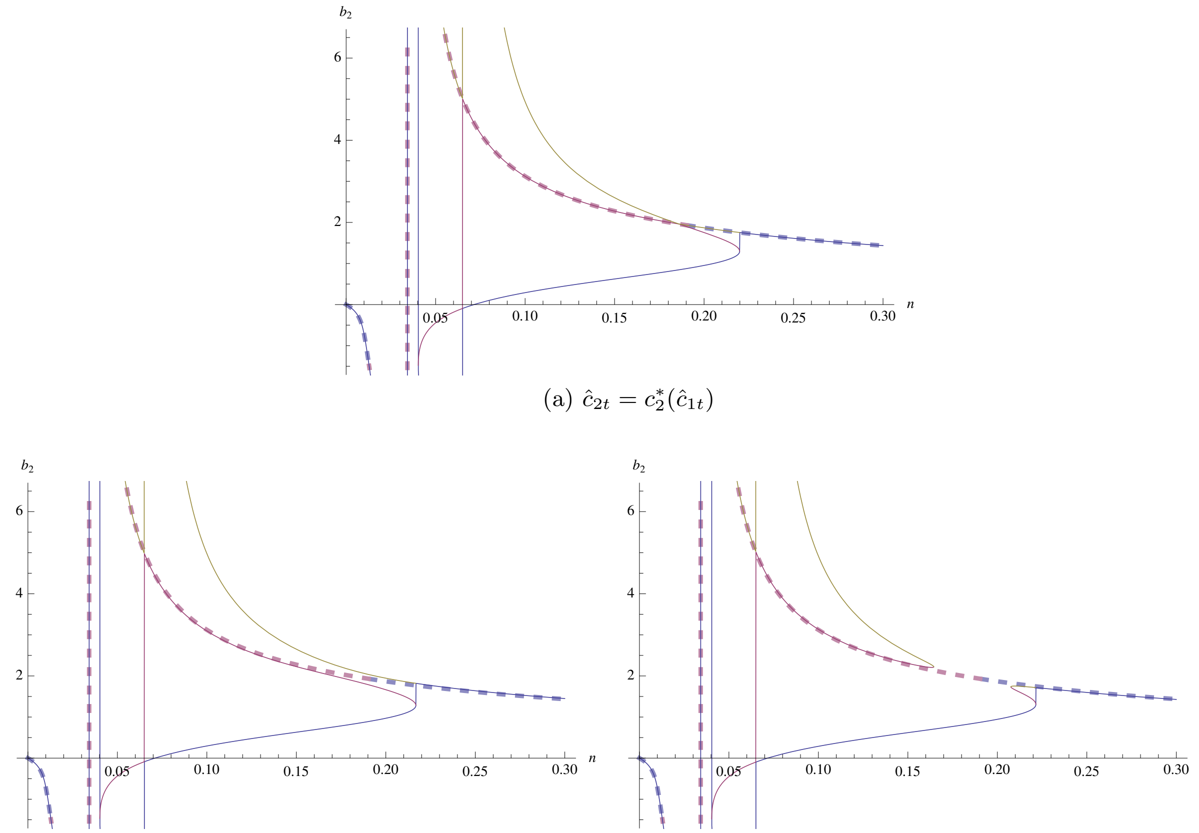


Figure 8: RD with $\hat{c}_{2t}$ allowed to differ from $c_2^*(\hat{c}_{1t})$ producing error in the pricing of observable and unobservable components of price.

the fundamental model, introducing a new source of error. The consequence of this is seen in the forecast errors in (15) and (16). For (15), the coefficient on $\phi d_t + \epsilon_{t+1}$ is no longer zero and the same is true for the coefficient on d_t in (16). The market does not properly price even the observable component of price. Relative to the base simulation, price deviations from the efficient price show considerably more volatility clustering with greater volatility that coincide with deviations from $b_{t1} + \phi b_{2t} = \phi/(R - \phi)$.

A more sophisticated fundamental trader accounts for the market-based population induced distortion to the price, as though deviations from $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$ are known and accounted for. This sophisticated fundamental trader behavior introduces multiple market clearing prices at



(a) $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$

(b) $\hat{c}_{2t} > c_2^*(\hat{c}_{1t})$

(c) $\hat{c}_{2t} < c_2^*(\hat{c}_{1t})$

Figure 9: Roots of the $b_2(n_t, \hat{c}_{1t}, \hat{c}_{2t})$ solution in the presence of rational fundamental traders who account for the deviation from $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$. $\hat{c}_{1t} = 1.1$. The root for the $b_2(n_t, \hat{c}_{1t})$ solution is included as a dashed line. For $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$, for all values of n_t, one of the roots of $b_2(n_t, \hat{c}_{1t}, \hat{c}_{2t})$ is equal to the root of $b_2(n_t, \hat{c}_{1t})$. For $\hat{c}_{2t} > c_2^*(\hat{c}_{1t})$, the point of tangency that ensures continuity in the roots of $b_2(n_t, \hat{c}_{1t}, \hat{c}_{2t})$ that match the root of $b_2(n_t, \hat{c}_{1t})$ is lost. For $\hat{c}_{2t} < c_2^*(\hat{c}_{1t})$, a gap opens between the two points of intersection in two of the roots tracing $b_2(n_t, \hat{c}_{1t})$.

low values of n_t, as depicted in Figure 9. Rather than imposing discipline in the pricing of the asset, the pricing errors are larger than with the model populated by less sophisticated fundamental traders.

The three frames of Figure 9 are generated by solving for the market clearing $b_2(n_t, \hat{c}_{1t}, \hat{c}_{2t})$ using the simulation default parameters. Each frame displays a different condition relating $\hat{c}_{2t}$ to $c_2^*(\hat{c}_{1t})$. When $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$ is imposed before solving for the price, only one non-imaginary root exists, the $b_2(n_t, \hat{c}_{1t})$ solution. This solution is included in each frame as a dashed line. As n decreases from above towards $\tilde{n} = 0.034$, the solution for $b_2 \to \infty$. The three real roots obtained when solving for $b_2(n_t, \hat{c}_{1t}, \hat{c}_{2t})$ are depicted under the three scenarios of $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$, $\hat{c}_{2t} > c_2^*(\hat{c}_{1t})$, and $\hat{c}_{2t} < c_2^*(\hat{c}_{1t})$. All three real roots are potential solutions. For $\hat{c}_{2t} = c_2^*(\hat{c}_{1t})$, segments of the three roots combine to produce a curve that overlaps with the $b_2(n_t, \hat{c}_{1t})$ solution. For $\hat{c}_{2t} > c_2^*(\hat{c}_{1t})$, a range of n open for which the two roots close to the $b_2(n_t, \hat{c}_{1t})$ solution become imaginary, leaving as the only real price coefficient a root that that does not produce a smooth market clearing price function in n. The multiple solutions means that the market cannot reasonably be simulated.

The traders estimate the conditional variance of the pricing error associated with their chosen model. The ratio of the estimated variance, $\hat{\sigma}^2_{Mt}/\hat{\sigma}^2_{Ft}$, affects the coefficients of the pricing equation, b_{1t} and b_{2t}. The baseline updating mechanism of (31) has the traders updating the estimated variance based on observation. In a well-behaved market such as that seen in Figure 4, substituting the REE(n_t) variance values, $\sigma^2_{Ft} = \sigma^*_F(n_t)^2$ and $\sigma^2_{Mt} = \sigma^*_M(n_t)^2$ of (22) and (23), has little impact since the estimates track closely to the REE(n_t) values. The ratio $\sigma^*_F(n)^2/\sigma^*_M(n)^2$ is monotonically increasing in n. At the default simulation parameter values, the ratio ranges from 1.52 to 3.08 for n_t near zero and $n_t = 1$, respectively.

Away from the REE(n_t), the impact on how traders estimating the conditional variance impacts market behavior and convergence can be seen in (33) and (34). The inverse ratio appears in the formulas for c_1^+ and $\tilde{c}_1$. Increasing relative uncertainty among the employers of the market-based strategy decreases their price impact (because they take a smaller position). The greater stability is reflected in the increased distances between $c_1^*(n_t)$ and $c_1^+(n_t)$ and between $c_1^+(n_t)$ and $\tilde{c}_1(n_t)$. Incorporating (35) and (36) into the model accomplishes this feat by introducing the error in $\hat{c}_{1t}$ into the the concurrent time t market-based trader uncertainty. Larger estimates of c_{1t} feed greater market-based trader uncertainty, attenuating market-based trader demand. The dashed lines designated $c_1^{+'}$ and $\tilde{c}_1^{'}$ in Figure 2 that are everywhere above the corresponding c_1^+ and $\tilde{c}_1$ capture the consequence. There is a value $\bar{n}^{'}$ such that for $n_t > \bar{n}^{'}$ the invalid price region does not exist. Thus, regardless of how large is $\hat{c}_{1t} > R$, there exists a positive market clearing price. Similarly, there is a value of $n^{+'}$ such that for $n_t > n^{+'}$ the market-based strategy is always profitable.

4 Conclusion

Data overcomes deficiencies in trader knowledge when traders rely on increasingly long histories to inform their decisions, producing a well-behaved, though not necessarily efficient, market. Regimes in which traders place greater emphasis on more recent outcomes undermine market efficiency and allow other deficiencies to affect the price.

If the trader strategy-adoption choice is best captured by the Discrete Choice Dynamic, the resulting fixed point is a rational expectations equilibrium at which the market-based trading strategy reliably outperforms the fundamental strategy. The pricing function that clears the market is a stable linear combination of public and private information. The fixed point can be achieved with long memory. It is characterized by price predictability that goes unexploited as the un-modeled component of individual trader strategy adoption has traders continue to use the fundamental strategy despite its inferior performance. The model fails to converge to the fixed point when traders employ finite memory. In this case, the market is prone to frequent sudden substantial pricing errors where even just random dividend realizations can lead to sudden jumps in the population between the two strategies.

If the evolution in strategy-adoption is better captured by the Replicator Dynamic process, then the market is without a fixed point. Instead, it offers an opportunity for the market to grow increasingly less reliant on fundamental traders. If this potential is realized, it is only because the traders employ increasingly long data histories to improve their ability to extract private information from market observables and because the population as a whole uses similarly long histories when evaluating performance. The resulting market is asymptotically well-behaved, generating only small price deviations from full efficiency, though these deviations are clustered in magnitude.

Decisions by traders to place greater emphasis on current data undermine convergence, thereby undermining market efficiency. The nature of the market deviation depends on the traders. For short memory in evaluating relative performance the market is characterized by a non-degenerative population of fundamental traders and a price function that generates prices close to the REE price appropriate for the population split. Short memory in estimating the parameters of the market-based model produces a market that is reasonably well-behaved for the majority of the time with sudden bursts of mispricing. The mispricing appears when the population, based on historical performance, crosses a threshold to become over-reliant on the market-based strategy relative to its accuracy. Correct pricing returns when the population reverses trend and increasingly adopts the fundamental strategy, driven by the performance during the period of mispricing.

Other sources of error remain present when traders rely on shorter histories. Inconsistency between the coefficients of the pricing model introduces error not only for the traders using the model but also undermines the price forecast of fundamental traders. All errors are made less extreme if the market-based traders employ some caution when their information calls for substantial positions. This calls for a more sophisticated strategy whereby the market-based traders exercise restraint when the market-derived information is suspect due to the size of the position.

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Part I

Proof of Proposition 3

Proof. Under the regularity conditions (see Marcet and Sargent (1989b), p342-343), the stability of the learning process can be established from the stability of $T(c) - c$ where $T(c)$ maps c into the projection coefficients. From (20) and (21),

$$c_1 = \frac{R}{R-\phi}\frac{1}{b_2}$$

and

$$c_2 = \frac{\phi}{R-\phi}(R-c_1)$$

so that, according to (11),

$$T(c_1) = \frac{nR + (1-n)(R-c_1)\frac{\sigma_F^2}{\sigma_M^2}}{n\beta}$$

and

$$T(c_2) = -\frac{\phi}{R-\phi}\left(\frac{nR(1-\beta) + (1-n)(R-c_1)\frac{\sigma_F^2}{\sigma_M^2}}{n\beta}\right)$$

The eigenvalues of the Jacobian, $\frac{\partial[T(\mathbf{c})-\mathbf{c}]}{\partial \mathbf{c}}$, are $\left\{-1,\ -1-\frac{1-n}{n}\frac{1}{\beta}\frac{\sigma_F^2}{\sigma_M^2}\right\}$, which are both less than zero. The learning process is thus locally stable so that $\Pr(|\mathbf{c}_t - \mathbf{c}^*| > \delta) \xrightarrow{a} 0$ for $\delta > 0$. $\square$