The Impact of Short-Selling Constraints on Financial Market Stability in a Heterogeneous Agents Model

Mikhail Anufriev
Jan Tuinstra

The impact of short-selling constraints on financial market stability in a heterogeneous agents model[*]

Mikhail Anufriev [a,b][†]   Jan Tuinstra [b]

Abstract

Recent turmoil on global financial markets has led to a discussion on which policy measures should or could be taken to stabilize financial markets. One such a measure that resurfaced is the imposition of short-selling constraints. It is conjectured that these short-selling constraints reduce speculative trading and thereby have the potential to stabilize volatile financial markets. The purpose of the current paper is to investigate this conjecture in a standard asset pricing model with heterogeneous beliefs. We model short-selling constraints by imposing trading costs for selling an asset short. We find that the local stability properties of the fundamental rational expectations equilibrium do not change when trading costs for short-selling are introduced. However, when the asset is overvalued, costs on short-selling increase mispricing and price volatility.

[a] *Economics Discipline Group, Business School, University of Technology, Sydney, 1-59 Quay Street, Haymarket, NSW2001, Australia*

[b] *CeNDEF, Amsterdam School of Economics, University of Amsterdam, Valckenierstraat 65–67, NL-1018 XE Amsterdam, Netherlands*

Keywords: Asset pricing model, Heterogeneous beliefs, Financial instability, Short-selling bans.

[†]Corresponding author: Mikhail.Anufriev@uts.edu.au.

1   Introduction

The practice of short-selling – borrowing a financial instrument from another investor to sell it immediately and close the position in the future by buying and returning the instrument – is widespread in financial markets. In fact, short-selling is the mirror image of a "long position", where an investor buys an asset which did not belong to him before. While a long position can be thought of as a bet on the increase of the assets' value (with dividend yield and opportunity costs taken into account), short-selling allows investors to bet on a fall in stock prices. Some people have argued that such betting may increase volatility of financial markets and even lead to the incidence of crashes. A proposed policy would then restrict short-selling. In this paper we investigate consequences of such a restriction in a heterogeneous agents model of a financial market and show that it may increase in mispricing as well as price volatility.

The historical account of Galbraith (1954) provides evidence that short sales were common during the market crash of 1929. As short-sellers were often blamed for the crash, the Securities and Exchange Commission (SEC) introduced the so-called "uptick rule" in 1938, which prohibited the selling short "on a downtick", i.e., at prices lower than the previous transaction price. Curiously enough, the uptick rule was removed on July 6, 2007, right before the market crash of $2008 - 2009$ began. Fig. 1 shows the evolution of the S&P500 index and indicates the end of the uptick rule period by the dotted vertical line in the left part of the figure. Since its removal, calls to restore the uptick rule have been recurrent. We show the dates of the statements by different practitioners, authority experts, congressmen and senators for restoring the uptick rule. The calls did not remain unanswered and in the fall of 2008 – at the peak of the credit crisis – the SEC temporarily prohibited short-selling in 799 different financial companies. The SEC's chairman, Christopher Cox, argued that: "The emergency order temporarily banning short selling of financial stocks will restore equilibrium to markets."[1] The period for which the short-selling ban was imposed is indicated by two vertical lines in the right part of the figure. Even more stringent policies have been adopted in other countries, see Beber and Pagano (2013) for an overview. It is not clear, however, whether such a ban on short-selling has actually been helpful in stabilizing financial markets. According to Boehmer, Jones, and Zhang (2009) the price for the banned stocks sharply increased when the ban was announced, but gradually decreased during the ban period. The whole S&P500 index continued to fall during the short-sell ban as well as afterwards, see Fig. 1.

The traditional academic view on constraints on short-selling is that they may lead to overpricing of the asset.[2] Miller (1977), for example, argues that the equi-

---

[1]See http://www.sec.gov/news/press/2008/2008-211.htm.

[2]Apart from the legal constraints, short-selling may also be constrained because it may be
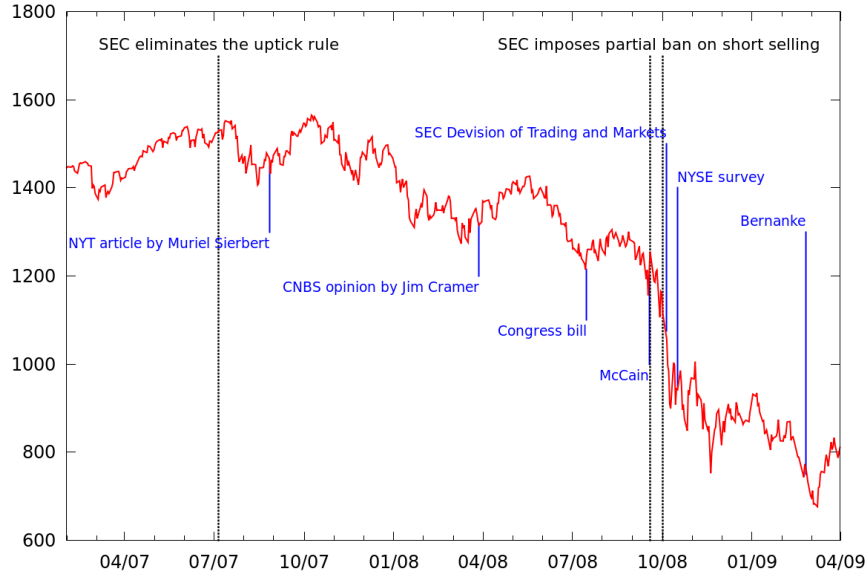
Figure 1: S&P 500 and the uptick rule. The end of the "uptick rule", the period between 19 September 2008 and 2 October 2008, when the short-sales ban for 799 financial stocks was in place and the major calls for reinstatement of the rule are shown.

librium price between demand and supply for a risky asset reflects an average view among heterogeneous investors about the asset's value. The investors with the most pessimistic view on the future price of the asset may sell the asset short at the equilibrium price. Therefore, the constraints on short selling effectively restrict the supply of shares, leading to a higher equilibrium price level than would emerge in the absence of constraints. In the more sophisticated, dynamic model of Harrison and Kreps (1978) risk-neutral investors have different expectations about the dividends of a certain asset and perfect foresight about beliefs of the other investors. In the absence of short-selling constraints, investors with different opinions take infinitely large, opposite positions. When the constraints are

---

costly, compared to taking a "long position". In particular, an investor willing to sell short should eventually deliver the shares to the buyer, and hence is required to "locate" the shares, i.e., to find another investor who is willing to lend these shares. (When shares have not been located, the operation is called "naked short-selling", which is subject to more strict regulations and is often banned, since it is believed to permit price manipulation.) In the absence of a centralized market for lending shares such an operation may be complicated. At least, it is costly, because short-selling requires not only paying a standard fee to the broker but also involves a commission (plus dividends) to the actual owner of the stock. Moreover, there is a recall risk of the lender wanting to recall a borrowed stock.

imposed, the price reflects the beliefs of the most optimistic investors, and due to speculative motives the actual price may even be higher.

However, since these first contributions other models have been developed that predict no mispricing or even underpricing as a consequence of short-selling constraints. Diamond and Verrecchia (1987) argue that since the constraints are common knowledge, financial market participants should take them into account both in their behavior as well as in their beliefs about the behavior of the other market participants. In their model of asymmetric information (based on Glosten and Milgrom (1985)) short-selling prevents some investors from desired trading. Even if not all private information is fully incorporated into the order flow, the fully rational and risk-neutral market-maker will take the existence of short-selling constraints into account, and will set bid and ask prices at the correct level. Bai, Chang, and Wang (2006) show that this result might change when rational traders are risk-averse. In this case uninformed traders will ask a premium for their higher perceived risk (because the short-selling constraints slow down price recovery), which leads to lower prices. But the model of Bai, Chang, and Wang (2006) may also result in the opposite prediction, as a consequence of smaller supply. Similarly, in a general equilibrium economy considered by Gallmeyer and Hollifield (2008) the short-sell constraints can lead either to overpricing or to underpricing depending on the intertemporal elasticity of substitution of the optimists. Notice that many of these results are obtained by assuming that investors are unboundedly rational. Laboratory experiments with paid human subjects show that short-selling constraints may lead to considerable mispricing, with the important reservation that relaxing the constraints reduces mispricing, but does not eliminate it completely, see, e.g., Haruvy and Noussair (2006).

Empirical research on short-sell bans tends to support the view that banned securities are overpriced. Jones and Lamont (2002) study data on the costs of short-selling between 1926 and 1933 and find that those assets which were expensive to sell short subsequently earned lower returns. Similarly, Chang, Cheng, and Yu (2007) examine the effect of revisions in the list of securities which cannot be sold short at the Hong Kong stock exchange. They find that inclusion of a stock to the list leads to an abnormal negative return, while exclusion from the list is associated with an abnormal positive return. These results imply that a short-selling ban leads to overpricing. Lamont and Thaler (2003) discuss 3Com/Palm and other examples of clear mispricing, where an arbitrage opportunity is obviously present. They attribute a failure to correct mispricing to the high cost of selling short. On the other hand, recent analysis of Beber and Pagano (2013) shows that, with the exception of the US, there is no evidence of overpricing of the banned stocks during the recent wave of short-selling constraints. They compute the cumulative abnormal return (with respect to the market) after the day the ban is introduced

for the stocks in the countries where the ban was imposed. Comparison of the cumulative returns of the stocks subject to a ban with the remaining stocks shows that the effect of ban on the stock price was positive during the first 30 days after it was introduced, but changed sign afterwards, even if the restrictions were not yet relaxed. These results indicate that the ban leads to a price increase for the banned stocks but only in the short run (which is also consistent with the US data).

Most of the models discussed above are static in nature and assume that investors are fully rational. This assumption of full rationality has been challenged on theoretical as well as empirical grounds. Theoretically, it can be argued that to actually compute rational beliefs, agents would need to know the precise structure and laws of motion for the economy, even though this structure depends on other agents' beliefs, see, e.g., Evans and Honkapohja (2001). Empirically, some important market regularities, such as recurrent periods of speculative bubbles and crashes, fat tails of the return distribution, excess volatility, long memory and volatility clustering are difficult to explain with models with fully rational investors. Moreover, there is an abundance of experimental evidence that suggests that theoretical models with fully rational agents do not even provide accurate descriptions of the behavior of relatively simple laboratory markets (see, e.g., Smith, Suchanek, and Williams, 1988, Lei, Noussair, and Plott, 2001, Hommes, Sonnemans, Tuinstra, and van de Velden, 2005 and Anufriev and Hommes, 2012).

An alternative approach is to consider models of behavioral finance (see, e.g., Shleifer, 2000 and Barberis and Thaler, 2003 for reviews) or, closely related, heterogeneous agents models (HAMs, see Hommes, 2006 and LeBaron, 2006 for reviews). In HAMs, for example, traders choose between different heuristics or rules of thumb when making an investment decision. Typically, heuristics that turned out to be more successful in the (recent) past will be used by more traders. Such models are also successful empirically (by reproducing many of the empirical regularities discussed above, see Lux, 2009), and therefore become an increasingly accepted alternative to the traditional models with a fully rational, representative agent. In this paper we investigate the impact of introducing short-selling constraints in such a heterogeneous agents model.

We take the well-known and widely used asset pricing model with heterogeneous beliefs from Brock and Hommes (1998) as our benchmark model. Traders in this model have to decide every period how much to buy or sell of an inelastically supplied risky asset, and they base their decision on one of a number of behavioral prediction strategies (e.g., a *fundamentalist* or a *trend following / chartist* prediction strategy). As new data become available agents not only update their forecasts but they also switch from one prediction strategy to another depending on past performance of those strategies. Such a low-dimensional heterogeneous

agents model is able to generate the type of dynamics typically observed in financial markets, in particular when traders are sensitive to differences in profitability between different prediction strategies.

We investigate the impact of imposing short-selling constraints in this framework, by analytical as well as numerical methods. Specifically we assume that traders need to pay additional 'trading costs' when they take a short position. We find that the imposition of these costs for short-selling does not affect the *local* stability properties of the fundamental steady state, that is, the financial market is neither stabilized nor destabilized due to these costs. However, if the price dynamics are volatile to begin with, the costs for short-selling affect the *global* dynamics and may lead to even more volatile price dynamics. The intuition for this result is that the implicit constraint on short-selling reduces the potential of the financial market to quickly correct mispricing. Limited liquidity in the market in each time period contributes to this effect. Furthermore, the temporary mispricing gets reinforced by the population dynamics.

A recent, independent study by Dercole and Radi (2012) gives results that are qualitatively similar to ours. They also study the effect of imposing short-selling constraints in the Brock and Hommes (1998) framework, but their model differs from ours in two respects. First, their analysis is restricted to the case of a full ban on short-selling, whereas our model allows for a wide range of intermediate settings, with the full ban as a limiting case. Second, the ban in their model is imposed only in periods in which the asset price decreases. They therefore focus on the effects the up-tick rule, mentioned above, has on price stability, whereas we are interested in the more general effects of trading costs of short-selling on market dynamics.

The rest of the paper is organized as follows. In the next section we extend the Brock-Hommes model to the case of positive outside supply and costly short selling. The dynamics of the model for the typical and familiar case of fundamentalists versus chartists are studied in Section 3. Section 4 concludes the paper.

2 An asset pricing model with heterogeneous beliefs and trading costs for short-selling

In this section we extend the well known heterogeneous beliefs asset pricing model of Brock and Hommes (1998) to allow for short selling constraints, which we model by introducing trading costs for selling short. In Section 2.1 we derive an individual trader's demand for the risky asset given his or her beliefs and the trading costs. In Section 2.2 we discuss aggregate demand for the risky asset and determine the market clearing price for the risky asset. Finally, Section 2.3 – following Brock

and Hommes (1998) – introduces evolutionary selection between the different belief types.

2.1 Individual asset demand

Consider a financial market where traders can invest their wealth in two assets, an inelastically supplied risky asset ("stock"), and a riskless asset ("bond"). The riskless asset is in perfect elastic supply and yields gross return $R = 1 + r_f$; its price is normalized to 1. The risky asset has ex-dividend price p_t and pays random dividend y_t in period t. The dividend is assumed to be identically and independently distributed with mean value $\overline{y}$. The supply of the risky asset is inelastic and equal to S units per trader, where S is a non-negative constant.[3]

All traders are myopic mean-variance maximizers, with the same risk aversion coefficient a. Moreover, traders have correct knowledge of the dividend process and homogeneous beliefs about the variance of asset prices, denoted σ^2. The only source of heterogeneity stems from the fact that traders may differ in their beliefs about the future price level. More specifically, there are H distinct belief types, or prediction strategies, with price expectations of type $h \in \mathcal{H} = \{1, \dots, H\}$ given by $\mathrm{E}_{h,t}[p_{t+1}]$.

Denote the number of shares of the risky asset purchased at time t by a trader of type h by $A_{h,t}$. In the case when $A_{h,t} < 0$, which may be optimal when the trader believes the price of the risky asset will go down, trader h *sells* the risky asset *short*. Trader h selects $A_{h,t}$ in order to solve

$$\max_{A_{h,t}} \left(\mathrm{E}_{h,t}[W_{h,t+1}] - \frac{a}{2} \mathrm{V}_{h,t}[W_{h,t+1}] \right), \tag{1}$$

where $W_{h,t+1}$ is wealth of a trader of type h in period $t+1$, which is subject to an individual intertemporal budget constraint. This budget constraint is given by

$$W_{h,t+1} = RW_{h,t} + (p_{t+1} + y_{t+1} - Rp_t) A_{h,t} - R\tau(A_{h,t}).$$

This intertemporal constraint deviates from the standard wealth constraint (see, e.g., Brock and Hommes (1998)) because it explicitly takes into account that market regulation or institutional arrangements may lead to additional *trading costs*, represented by $\tau(A_{h,t})$.[4] The money spend on trading costs cannot be invested in

---

[3]Note that, apart from introducing trading costs, we also extend Brock and Hommes (1998) by allowing for positive outside supply of the risky asset, $S > 0$. In Hommes, Huang, and Wang (2005) the case $S > 0$ is also studied but in a setting with an auctioneer, whereas we assume market clearing. We will get back to this issue in Section 3.1.

[4]We thank Gaetano Gaballo and an anonymous referee for suggesting to provide microfoundations for the short selling constraints, which we do by introducing these trading costs.

the riskless asset and therefore next period's wealth is decreased by $R\tau(A_{h,t})$. We specify these trading costs as

$$
\tau(A_{h,t}) =
\begin{cases}
0 & \text{if } A_{h,t} \geq 0 \\
T\,|A_{h,t}| & \text{if } A_{h,t} < 0
\end{cases},
$$

where T is a non-negative constant.[5] The case $T = 0$ corresponds to the original model of Brock and Hommes (1998). When $T > 0$, the trader needs to pay trading costs but only when he or she sells the risky asset short. Trading costs imply that the marginal benefits for selling one unit of the risky asset short drop from Rp_t to $R(p_t - T)$, whereas marginal (expected) costs are still $\mathrm{E}_{h,t}[p_{t+1}] + \overline{y}$. In the presence of these costs the trader therefore has to expect a higher capital gain in order for short-selling to be profitable. In particular, if T is sufficiently high, short-selling is effectively ruled out and in that case the trading costs are equivalent with a full ban on short-selling.

Given our assumptions, the objective function (1) for trader h, as a function of $A_{h,t}$, can be written as

$$
\Phi(A_{h,t}) =
\begin{cases}
RW_{h,t} + (\mathrm{E}_{h,t}[p_{t+1}] + \overline{y} - Rp_t)A_{h,t} - \frac{a}{2}\sigma^2 A_{h,t}^2 & \text{if } A_{h,t} \geq 0 \\
RW_{h,t} + (\mathrm{E}_{h,t}[p_{t+1}] + \overline{y} - Rp_t)A_{h,t} - RT\,|A_{h,t}| - \frac{a}{2}\sigma^2 A_{h,t}^2 & \text{if } A_{h,t} < 0
\end{cases}.
\tag{2}
$$

For $T = 0$ this objective function is maximized at

$$
A_{h,t}^* = \frac{\mathrm{E}_{h,t}[p_{t+1}] + \overline{y} - Rp_t}{a\,\sigma^2},
\tag{3}
$$

which is the standard mean-variance demand function.

If trading costs are positive, however, the solution is more complicated. First notice that, as long as $A_{h,t}^*$ from (3) is non-negative, it will still maximize (2). Non-negativity of $A_{h,t}^*$ is equivalent with

$$
p_t \leq p_t^h \equiv \frac{\mathrm{E}_{h,t}[p_{t+1}] + \overline{y}}{R}.
$$

The precise *cut-off value* p_t^h depends on the expectations of belief type h. Belief types that are more optimistic about the future asset price will have higher cut-off values and higher demand for the asset. They will therefore be more rarely

---

[5]Alternative specifications of trading costs, for example $\tau(A) = T_2 A^2$ for $A < 0$ and $\tau(A) = 0$ for $A \geq 0$, are also possible, see footnote 6.
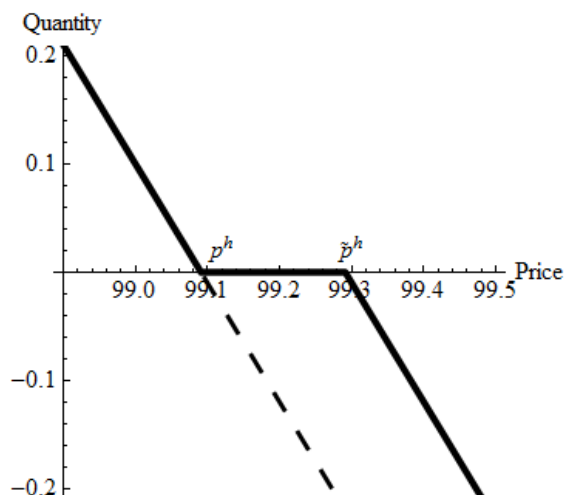
Figure 2: Individual demand as a function of the price of the risky asset. For $T = 0$, the demand function is linearly decreasing in the price p (the dashed line). For $T > 0$ the individual demand function is piecewise linear (the solid line) with zero demand for prices between p^h and $\tilde{p}^h = p^h + T$.

affected by the trading costs. If $p_t > p_t^h$ traders have the incentive to go short since they expect a decrease in the asset price. However, since they now have to pay an additional cost of T per unit sold short, the expected decrease in the asset price should be higher by exactly an additional T units for short-selling to become profitable. It follows that individual demand for trader h, as a function of the current market price p_t, becomes

$$
A_{h,t}(p_t) = \begin{cases} \frac{1}{a\sigma^2}\left(\mathrm{E}_{h,t}\left[p_{t+1}\right] + \overline{y} - Rp_t\right) & \text{if } p_t \leq p_t^h \\[2mm] 0 & \text{if } p_t^h < p_t \leq p_t^h + T \\[2mm] \frac{1}{a\sigma^2}\left(\mathrm{E}_{h,t}\left[p_{t+1}\right] + \overline{y} - R\left(p_t - T\right)\right) & \text{if } p_t > p_t^h + T \end{cases} \quad . \quad (4)
$$

Fig. 2 gives an example of the individual demand for trader h as a function of p_t (for given values of $\mathrm{E}_{h,t}\left[p_{t+1}\right]$ and the parameters $\overline{y}$, R, T and $a\sigma^2$). The figure shows that the individual demand curve consists of three linear pieces. For $p_t \leq p_t^h$ the individual demand coincides with the standard mean-variance demand function $A_{h,t}^*$. When $p_t > p_t^h$ the trading costs become effective and the trader will go short only when expected benefits outweigh costs. Thus, when the expected capital gain from short-selling, $p_t - p_t^h$, is smaller than the trading cost per unit, T, the trader will take a zero position, which corresponds to the flat section of the individual demand curve. When the expected capital gain is larger than the

9

trading cost, $p_t - p_t^h > T$, the trader will sell the asset short.[6] Note that for sufficiently high T there will never be short-selling of the risky asset.[7]

2.2 Asset market equilibrium

Let the fraction of traders of type h at time t be given by $n_{h,t} \geq 0$, with $\sum_{h=1}^{H} n_{h,t} = 1$. Aggregate demand (per trader) at time t for the risky asset then equals $A_t(p_t) = \sum_{h=1}^{H} n_{h,t} A_{h,t}(p_t)$, with $A_{h,t}(p_t)$ given by (4). In every period t the market equilibrium price p_t will be such that the market for the risky asset clears, that is, it solves

$$\sum_{h=1}^{H} n_{h,t} A_{h,t}(p_t) = S, \tag{5}$$

where $S \geq 0$ is the average outside supply.

First we consider the case where all traders are fully rational, and that there is common knowledge of rationality. Since traders then have identical price expectations individual asset demand will be the same for each trader. In equilibrium each trader will hold exactly S units of the risky asset and no trader sells the asset short. The only non-exploding equilibrium solution to (5) in this case is $p_t = p_f$, for all t, with p_f given by

$$p_f = \frac{\overline{y}}{r_f} - \frac{a\,\sigma^2}{r_f} S. \tag{6}$$

We refer to p_f as the *fundamental price*. When outside supply S is zero, this is simply the discounted value of the future stream of dividends. Under positive supply, risk-averse investors require a risk premium to hold the risky asset, which is reflected in the last term of (6).

It will be convenient to rewrite the individual demand function (4) in terms of

---

[6] Note that the slopes of the first and third pieces of the individual demand function are the same and equal to $-\frac{R}{a\sigma^2}$. It is easy to show that for the alternative specification of *quadratic* trading costs the resulting individual demand curve will have just one kink. Specifically, if $\tau(A) = T_2 A^2$ for $A < 0$ and $\tau(A) = 0$ for $A \geq 0$, then the demand curve coincides with $A_{h,t}^*$ for $p_t \leq p_t^h$ and is flatter with slope $-\frac{R}{a\sigma^2 + 2T_2}$ for $p_t > p_t^h$. We investigated this case in an earlier version of the current paper, and also studied the case when costs have both linear and quadratic components. The results are qualitatively similar to those reported in this paper.

[7]Therefore, a full ban on short-selling, as considered for example in Dercole and Radi (2012), is a special case of our model for $T \to \infty$.

the deviation of the price from the fundamental, $x_t = p_t - p^f$. This gives

$$
A_{h,t}(x_t) = \begin{cases} \frac{1}{a\sigma^2}\left(\mathrm{E}_{h,t}\left[x_{t+1}\right] - Rx_t\right) + S & \text{if } x_t \leq x_t^h \\ 0 & \text{if } x_t^h < x_t \leq x_t^h + T \quad . \quad (7) \\ \frac{1}{a\sigma^2}\left(\mathrm{E}_{h,t}\left[x_{t+1}\right] - R\left(x_t - T\right)\right) + S & \text{if } x_t > x_t^h + T \end{cases}
$$

where the cut-off value in deviations is

$$
x_t^h = \frac{\mathrm{E}_{h,t}\left[x_{t+1}\right] + a\,\sigma^2 S}{R}. \tag{8}
$$

If there is diversity of beliefs some traders may sell the asset short in equilibrium and it becomes more difficult to determine the market equilibrium price. First note that the aggregate demand function $A_t(x_t) = \sum_h n_{h,t} A_{h,t}(x_t)$ is the weighted sum of H piecewise-linear demand functions (for an illustration with $H = 2$, see Fig. 3). Each of these piece-wise linear demand functions is continuous and decreasing in x_t and has two kink points, one in x_t^h and one in $x_t^h + T$. Therefore the aggregate excess demand function, which consequently may have up to $2H$ different kink points, will also be continuous and decreasing in x_t. Moreover, it will be *strictly* decreasing as long as demand is strictly positive (the aggregate demand curve has a flat portion only if $x_t \in \left[x_t^h, x_t^h + T\right]$ for each belief type h, which would imply that aggregate demand is zero). The following result immediately follows from these considerations.

Proposition 2.1. *If $S > 0$ there exists a unique solution x_t to $A_t(x_t) = S$.*

To characterize this market equilibrium price we partition, for a given price deviation x, the set of belief types $\mathcal{H}$ as follows:

$$
P(x) = \left\{h \in \mathcal{H} \mid x \leq x_t^h\right\}, \quad Z(x) = \left\{h \in \mathcal{H} \mid x_t^h < x \leq x_t^h + T\right\}, \quad \text{and}
$$
$$
N(x) = \left\{h \in \mathcal{H} \mid x > x_t^h + T\right\}.
$$

That is, the set P (Z, N) consists of the trader types that have positive (zero, negative) demand for the risky asset. Aggregate demand at price x, $A_t(x)$, can now be written as

$$
A_t(x) = \sum_{h \in P(x)} n_{h,t}\left(\frac{\mathrm{E}_{h,t}\left[x_{t+1}\right] - Rx}{a\sigma^2} + S\right) + \sum_{h \in N(x)} n_{h,t}\left(\frac{\mathrm{E}_{h,t}\left[x_{t+1}\right] - R\left(x - T\right)}{a\sigma^2} + S\right).
$$
$$
\tag{9}
$$

Solving the market clearing equation $A_t(x_t) = S$ we find that the market price (in deviations) is characterized by

$$x_t = \frac{1}{R \sum_{h \notin Z(x_t)} n_{h,t}} \tag{10}$$

$$\times \left[\sum_{h \in P(x_t)} n_{h,t} \left(E_{h,t}[x_{t+1}] \right) + \sum_{h \in N(x_t)} n_{h,t} \left(E_{h,t}[x_{t+1}] + RT \right) - a\sigma^2 S \sum_{h \in Z(x)} n_{h,t} \right].$$

Note however, that x_t is still implicitly defined by (10) since the right-hand side also depends upon x_t through the definition of the sets $P(x_t)$, $Z(x_t)$ and $N(x_t)$. Below we will derive the market equilibrium price x_t explicitly for some special cases. Appendix A contains the description of an algorithm for finding the market equilibrium price for an arbitrary number of belief types.

2.2.1 Market equilibrium in absence of trading costs

If $T = 0$ we have $P(x_t) = \mathcal{H}$ and equation (10) reduces to

$$x_t = \frac{1}{R} \sum_{h=1}^{H} n_{h,t} \, E_{h,t}[x_{t+1}], \tag{11}$$

that is, the realized price deviation is equal to the (discounted) weighted average of price expectations for the next period. Note that this holds for any number of belief types H.

2.2.2 Market equilibrium when there are two belief types

For the case of two belief types, $H = 2$, it is also possible to derive the price explicitly. Let us assume that at time t belief type 1 is more optimistic about next period's asset price than belief type 2, i.e., $E_{1,t}[x_{t+1}] > E_{2,t}[x_{t+1}]$. In a market with two belief types, and given that $S > 0$, the traders of the optimistic type must have positive demand for the risky asset at the equilibrium price. Therefore only three cases are possible: either the pessimistic type (belief type 2) has positive demand at the equilibrium price, or it has zero demand at the equilibrium price, or it has negative demand at the equilibrium price. These three cases are shown in the three panels of Fig. 3, which graphically illustrates the market equilibrium in the model with heterogeneous expectations (see the caption for an explanation).

In the first case (left panel of Fig. 3) both belief types have a positive position in the equilibrium and the relevant market equilibrium price is given by (11) for $H = 2$, that is

$$x_t^{NC} = \frac{1}{R} \left(n_{1,t} \, E_{1,t}[x_{t+1}] + n_{2,t} \, E_{2,t}[x_{t+1}] \right). \tag{12}$$
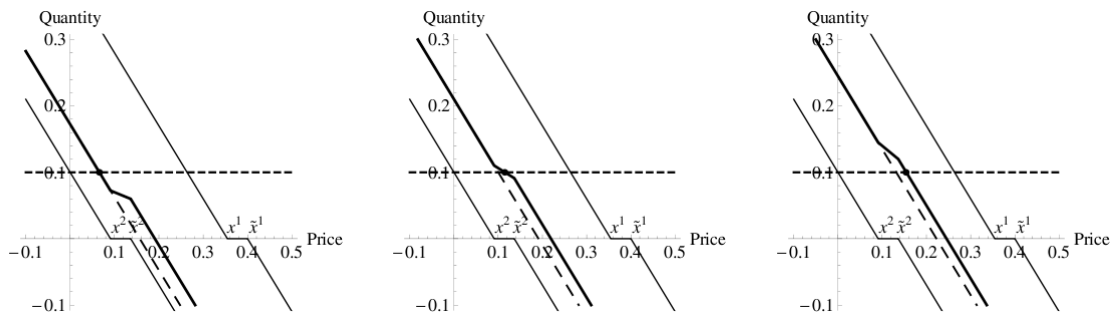
Figure 3: Temporary market equilibrium. Demand functions of the two different types are shown as thin solid lines. Aggregate demand (thick dashed or thick solid line, when short-selling has no or positive costs, respectively) is the vertical average of these two demand functions weighted by the fractions of the two types. Equilibrium price (in deviations) is obtained as the x-coordinate of the intersection of the aggregate demand with supply (horizontal dashed line). Equilibrium positions of the two types are obtained as the y-coordinates of the corresponding demand functions evaluated in the equilibrium price. Three cases discussed in the text are illustrated. *Left panel:* Both types have positive demand at the equilibrium price. *Middle panel:* The pessimistic type has zero demand at the equilibrium. *Right panel:* The pessimistic type has negative demand at the equilibrium.

This is only relevant when the second type's equilibrium position is positive, that is, when $x_t^{NC} < x_t^2$, or equivalently, when $\mathrm{E}_{1,t}[x_{t+1}] - \mathrm{E}_{2,t}[x_{t+1}] < a\sigma^2 S/n_{1,t}$.

In the second case (middle panel of Fig. 3) the pessimistic trader type 2 has a zero position in the equilibrium. The equilibrium price can then be found from (10) as

$$x_t^F = \frac{1}{R}\left(\mathrm{E}_{1,t}[x_{t+1}] - \frac{n_{2,t}}{n_{1,t}}a\sigma^2 S\right). \tag{13}$$

This expression for the equilibrium price is relevant when $x_t^F \in [x_t^2, x_t^2 + T)$. The condition $x_t^F < x_t^2 + T$ is equivalent with $\mathrm{E}_{1,t}[x_{t+1}] - \mathrm{E}_{2,t}[x_{t+1}] < a\sigma^2 S/n_{1,t} + RT$.

Finally, the right panel of Fig. 3 illustrates the case where the equilibrium allocation is such that pessimistic belief type sells the risky asset short. From (10) the price then follows as

$$x_t^C = \frac{1}{R}\left(n_{1,t}\,\mathrm{E}_{1,t}[x_{t+1}] + n_{2,t}\left(\mathrm{E}_{2,t}[x_{t+1}] + RT\right)\right).$$

The three cases discussed above are the ones when type 1 traders are more optimistic than type 2 traders. The two remaining cases, which can be studied

13

in a similar way, are those where type 2 traders are pessimistic and have zero or negative equilibrium position in the risky asset, respectively.

The next proposition summarizes the market equilibrium price if there are two belief types.

Proposition 2.2. *Consider the model with short-selling constraints and two belief types. The equilibrium price for period t, x_t, is given as follows*

$$
x_t = \begin{cases}
\frac{1}{R}\left(n_{1,t}\,\mathrm{E}_{1,t}\left[x_{t+1}\right] + n_{2,t}\left(\mathrm{E}_{2,t}\left[x_{t+1}\right] + RT\right)\right) & \text{if } \triangle\mathrm{E}_t \geq \frac{a\sigma^2 S}{n_{1,t}} + RT \\[2mm]
\frac{1}{R}\left(\mathrm{E}_{1,t}\left[x_{t+1}\right] - \frac{n_{2,t}}{n_{1,t}}a\sigma^2 S\right) & \text{if } \frac{a\sigma^2 S}{n_{1,t}} \leq \triangle\mathrm{E}_t < \frac{a\sigma^2 S}{n_{1,t}} + RT \\[2mm]
\frac{1}{R}\left(n_{1,t}\,\mathrm{E}_{1,t}\left[x_{t+1}\right] + n_{2,t}\,\mathrm{E}_{2,t}\left[x_{t+1}\right]\right) & \text{if } -\frac{a\sigma^2 S}{n_{2,t}} \leq \triangle\mathrm{E}_t < \frac{a\sigma^2 S}{n_{1,t}} \\[2mm]
\frac{1}{R}\left(\mathrm{E}_{2,t}\left[x_{t+1}\right] - \frac{n_{1,t}}{n_{2,t}}a\sigma^2 S\right) & \text{if } -\frac{a\sigma^2 S}{n_{2,t}} - RT \leq \triangle\mathrm{E}_t < -\frac{a\sigma^2 S}{n_{2,t}} \\[2mm]
\frac{1}{R}\left(n_{1,t}\left(\mathrm{E}_{1,t}\left[x_{t+1}\right] + RT\right) + n_{2,t}\,\mathrm{E}_{2,t}\left[x_{t+1}\right]\right) & \text{if } \triangle\mathrm{E}_t < -\frac{a\sigma^2 S}{n_{2,t}} - RT
\end{cases} \tag{14}
$$

where $\triangle\mathrm{E}_t = \mathrm{E}_{1,t}\left[x_{t+1}\right] - \mathrm{E}_{2,t}\left[x_{t+1}\right]$.

Fig. 4 gives a schematic representation of the five different regions, corresponding to the different expressions for the equilibrium price x_t in (14). For the full ban on short-selling, i.e., when trading costs $T \to \infty$, the most left and the most right regions disappear and only the three central regions need to be considered.

2.2.3 The instantaneous price increase from introducing trading costs

Before introducing the last element of our model (the evolution of fractions of belief types), it is useful to address a comparative statics question. Consider a financial authority that aims to discourage short-selling through an unanticipated increase of trading costs in period t, after expectations and the distribution of traders over belief types have already been determined. *What short-run impact does this policy have on the market price, and how strong will this impact be?*

From Proposition 2.2 it follows that the current distribution of the traders over the belief types is pivotal for this instantaneous effect of the increase in trading costs. Consider the stylized scenario where at time t trading costs are increased from $T = 0$ to a value that implies a full ban on short-selling.[8] Moreover, let type 1 traders be more optimistic than type 2 traders, that is $\triangle\mathrm{E}_t > 0$.

---

[8]To fully understand the quantitative effects for the intermediate situation – where trading

1st type is constr. | None is constrained | 2nd type is constr. $\triangle E_t$

neg. pos. | zero pos. | | zero pos. | neg. pos.

$-\frac{a\sigma^2 S}{n_{2,t}} - RT$ $\quad -\frac{a\sigma^2 S}{n_{2,t}} \quad 0$ $\qquad \frac{a\sigma^2 S}{n_{1,t}} \quad \frac{a\sigma^2 S}{n_{1,t}} + RT$
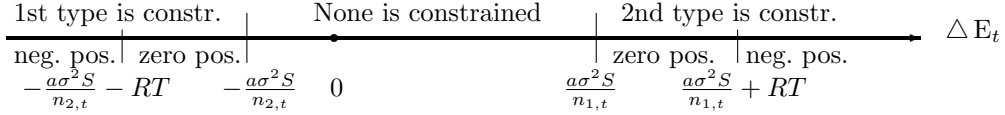
Figure 4: Illustration of the five different regions for equilibrium market price presented in Proposition 2.2.

In the absence of trading costs the equilibrium price, x_t^{NC}, will be given by (12). For high trading costs the price equals either x_t^{NC} or x_t^F (as in (13)). In particular (see Proposition 2.2 or Fig. 4), increased trading costs will affect the equilibrium price if and only if $\triangle E_t > a\sigma^2 / (1 - n_{2,t})$, that is, when the pessimistic traders are sufficiently pessimistic (that is, $\triangle E_t$ is high), and/or when the fraction of pessimistic traders $n_{2,t}$ is *small*. This latter effect can be explained by considering the market equilibrium equation (5). Intuitively, the higher the fraction of a certain type is, the more homogeneous the population of traders is, and the closer the individual equilibrium holdings of this type of traders is to the average available supply $S > 0$. Hence, when the fraction of pessimistic traders is high enough they may still hold a positive number of shares (although less than S) at the equilibrium and an increase in trading costs for short-selling will not have an effect. (Compare the left and right panels of Fig. 3.)

Provided that $\triangle E_t > a\sigma^2 / (1 - n_{2,t})$ the instantaneous price increase due to the introduction of trading costs for short-selling is equal to

$$x_t^F - x_t^{NC} = \frac{n_{2,t}}{R} \left(\triangle E_t - \frac{a\sigma^2 S}{1 - n_{2,t}} \right). \tag{15}$$

The impact of the introduction of trading costs gets stronger with $\triangle E_t$, that is, when the constrained traders become more pessimistic. The effect of the relative number of pessimistic traders, $n_{2,t}$, is ambiguous however. When the fraction $n_{2,t}$ is just low enough to guarantee that trading costs will have an effect, an even smaller fraction will make the increase in the price larger (see the term in the brackets which increases as $n_{2,t}$ gets smaller). However, the first term of (15) decreases with $n_{2,t}$, implying that when $n_{2,t}$ is very small, so will be the price increase.

Miller (1977) argued that short-selling constraints eliminate the pessimistic investors' opinion from the pricing equation, increasing market-clearing prices. The discussion above not only quantifies the instantaneous effect of introducing short-sell constraints, but also illustrates that the effect on the price exists only under certain conditions (in particular, when the fraction of pessimistic traders is

---

costs marginally increase in period t – would require a tedious study of several cases, corresponding to the different regions in which the equilibrium price lies before and after the change in trading costs. This analysis leads to a similar conclusion about the role of the fraction of pessimistic traders.

low enough). Note that in the longer run the distribution of types will change after the introduction of trading costs, making it more difficult to evaluate the impact of these costs. The remainder of the paper is devoted to understanding these long run effects.

2.3 Updating of belief types

Following Brock and Hommes (1998) we assume that, at the end of every trading round traders may update their belief type or prediction strategy on the basis of the performance of these different types. Performance of the different types is measured by the net return generated by those belief types. Excess return of holding the risky asset at time t is the difference between the return on the risky asset, $p_t + y_t$, and the return on the risk free asset, which is $R(p_{t-1} - T)$ or Rp_{t-1}, depending upon whether the risky asset was sold short or not. Written in terms of price deviations excess return therefore becomes

$$
r_t = \begin{cases}
\tilde{r}_t = x_t - Rx_{t-1} + \delta_t + a\sigma^2 S & \text{if } A_{h,t-1} > 0 \\[2mm]
\tilde{r}_t + RT = x_t - R(x_{t-1} - T) + \delta_t + a\sigma^2 S & \text{if } A_{h,t-1} < 0
\end{cases}
\tag{16}
$$

where $\delta_t = y_t - \overline{y}$ is a shock due to the dividend realization. Individual holdings $A_{h,t-1}$ are given by (7). We then specify performance of belief type h, in period t, as

$$
U_{h,t} = r_t A_{h,t-1} - C_h,
$$

where C_h is the information cost associated with strategy h, which is assumed to be constant over time.

Similar to the position of the trader, the performance measure can therefore fall in three regimes. In particular, we have

$$
U_{h,t} = \begin{cases}
\dfrac{1}{a\sigma^2} \left(\mathrm{E}_{h,t-1}\left[x_t\right] - Rx_{t-1} + a\sigma^2 S\right) & \\
\quad \times \left(x_t - Rx_{t-1} + \delta_t + a\sigma^2 S\right) - C_h, & \text{if } x_{t-1} \leq x^h_{t-1} \\[3mm]
-C_h & \text{if } x^h_{t-1} < x_{t-1} \leq x^h_{t-1} + T, \\[3mm]
\dfrac{1}{a\sigma^2} \left(\mathrm{E}_{h,t-1}[x_t] - R(x_{t-1} - T) + a\sigma^2 S\right) & \\
\quad \times \left(x_t - R(x_{t-1} - T) + \delta_t + a\sigma^2 S\right) - C_h & \text{if } x_{t-1} > x^h_{t-1} + T
\end{cases}
$$

In the remainder we will focus on the deterministic skeleton of the model, that is, we take $\delta_t \equiv 0$ for all t. The fraction of traders choosing belief type h in period $t+1$ depends upon the relative performance of this type, as measured

by $U_{h,t}$. This *evolutionary competition* between belief types can be modeled in different ways. Following Brock and Hommes (1998) we use the so-called *logit dynamics*, which has become one of the standards ways to model switching between types in heterogeneous agent models of financial markets. Other models, such as *reinforcement learning* or *replicator dynamics* typically lead to similar qualitative results. The logit dynamics are specified as

$$n_{h,t+1} = \frac{\exp\left[\beta U_{h,t}\right]}{\sum_{h'=1}^{H} \exp\left[\beta U_{h',t}\right]}. \tag{17}$$

Note that, for period $t + 1$, the new fractions of traders, $n_{h,t+1}$ are determined on the basis of past return, r_t and their positions *two periods* ago, $A_{h,t-1}$. The parameter $\beta \geq 0$ is the *intensity of choice* measuring the sensitivity of agents with respect to the difference in past performances of the different types. If the intensity of choice goes to infinity, all traders always switch to the most successful type of the previous period. At the opposite extreme, with $\beta = 0$, agents are equally distributed over the different belief types, independent of the past performance of these types. For the special case of zero outside supply and no trading costs Brock and Hommes (1998) find that for many specifications of the belief types the dynamics of the model depends on the value of β. We extend these results to the positive supply case below.

The full heterogeneous beliefs model can now be described by the implicit pricing equation (10) and the evolution of fraction (17). If there are only two belief types (as in the application from the next section) the price dynamics are given by (14), whereas the switching dynamics (17) becomes

$$n_{2,t+1} = \frac{1}{1 + \exp\left[\beta \triangle U_t\right]}, \qquad n_{1,t+1} = 1 - n_{2,t+1}. \tag{18}$$

The performance differential $\triangle U_t = U_{1,t} - U_{2,t}$ can be written as

$$\triangle U_t = \begin{cases} \tilde{r}_t \triangle A_{t-1} - RTA_{2,t-1} - \triangle C & \text{if } \triangle \mathrm{E}_{t-1} \geq \frac{a\sigma^2 S}{n_{1,t-1}} + RT \\ \tilde{r}_t \triangle A_{t-1} - \triangle C & \text{if } -\frac{a\sigma^2 S}{n_{2,t-1}} - RT \leq \triangle \mathrm{E}_{t-1} < \frac{a\sigma^2 S}{n_{1,t-1}} + RT \\ \tilde{r}_t \triangle A_{t-1} + RTA_{1,t-1} - \triangle C & \text{if } \triangle \mathrm{E}_{t-1} < -\frac{a\sigma^2 S}{n_{2,t-1}} - RT \end{cases}$$

$$\tag{19}$$

where $\tilde{r}_t$ is excess return for the risky asset when $A_{h,t} \geq 0$ and $\triangle A_{t-1} = A_{1,t-1} - A_{2,t-1}$, with $A_{h,t-1}$ given by (7) and $\triangle C = C_1 - C_2$. Recall that, if there are two trader types, determination of the equilibrium price and the equilibrium demands of the two types requires considering five different regions, see Fig. 4 and

Proposition 2.2. It is then straightforward to check that we have

$$
\triangle A_{t-1} = \begin{cases}
\frac{1}{a\sigma^2}\left(\mathrm{E}_{1,t-1}\left[x_t\right] - \mathrm{E}_{2,t-1}\left[x_t\right] - RT\right) & \text{if } \triangle\mathrm{E}_{t-1} \geq \frac{a\sigma^2 S}{n_{1,t-1}} + RT \\[2mm]
\frac{1}{a\sigma^2}\left(\mathrm{E}_{1,t-1}\left[x_t\right] - Rx_{t-1}\right) + S & \text{if } \frac{a\sigma^2 S}{n_{1,t-1}} \leq \triangle\mathrm{E}_{t-1} < \frac{a\sigma^2 S}{n_{1,t-1}} + RT \\[2mm]
\frac{1}{a\sigma^2}\left(\mathrm{E}_{1,t-1}\left[x_t\right] - \mathrm{E}_{2,t-1}\left[x_t\right]\right) & \text{if } -\frac{a\sigma^2 S}{n_{2,t-1}} \leq \triangle\mathrm{E}_{t-1} < \frac{a\sigma^2 S}{n_{1,t-1}} \\[2mm]
-\frac{1}{a\sigma^2}\left(\mathrm{E}_{2,t-1}\left[x_t\right] - Rx_{t-1}\right) - S & \text{if } -\frac{a\sigma^2 S}{n_{2,t-1}} - RT \leq \triangle\mathrm{E}_{t-1} < -\frac{a\sigma^2 S}{n_{2,t-1}} \\[2mm]
\frac{1}{a\sigma^2}\left(\mathrm{E}_{1,t-1}\left[x_t\right] - \mathrm{E}_{2,t-1}\left[x_t\right] + RT\right) & \text{if } \triangle\mathrm{E}_{t-1} < -\frac{a\sigma^2 S}{n_{2,t-1}} - RT
\end{cases}
\tag{20}
$$

Note that only in two of the five cases (the first and last one) trading costs are being paid by one of the types. The model with two types is now described by equations (14), (18), (19) and (20).

3   A stylized asset market model with fundamentalists and chartists

In this section we analyze the model from the previous section for a stylized but typical and often studied application of the heterogeneous beliefs model put forward in Brock and Hommes (1998). This application involves two belief types, each representative of an important class of belief types that can be encountered on actual financial markets.[9] First, the *fundamentalists* believe that the price will return to its fundamental value in the next trading period, that is, they predict $\mathrm{E}_{1,t}[p_{t+1}] = p_f$. In terms of deviations this can be written as

$$
\mathrm{E}_{1,t}\left[x_{t+1}\right] = 0 \,.
\tag{21}
$$

In addition, fundamentalists have to pay 'information' costs $C > 0$ in order to carry out the fundamental analysis necessary to obtain the fundamental forecast. Secondly, *chartists* or *trend extrapolators* use past data to forecast the future price and believe that any mispricing will continue. That is, they predict $\mathrm{E}_{2,t}[p_{t+1}] = p_f + g\,(p_{t-1} - p_f)$, which in terms of deviations can be written as

$$
\mathrm{E}_{2,t}\left[x_{t+1}\right] = gx_{t-1},
\tag{22}
$$

where $g > 0$ is the extrapolation coefficient. The forecast of chartists is assumed to be available for free.

---

[9]More general formulations of these two belief types can be found in Gaunersdorfer, Hommes, and Wagener (2008) and Anufriev and Panchenko (2009).

In Section 3.1 we investigate the model without trading costs. We will use this model (which is equivalent to the model studied in Brock and Hommes (1998), except that we allow for positive exogenous supply of shares, $S > 0$) as a benchmark against which the model with trading costs for short-selling can be compared. In Section 3.2 we study the effect of trading costs in this asset market model on *(i)* the existence, location and local stability of the steady states of the model; and *(ii)* global dynamics of asset prices.

3.1 Asset price dynamics in the benchmark model

Given (21) and (22) and letting $n_t = n_{2,t}$ be the number of chartists in period t, the price dynamics in deviations (11) reduces to

$$x_t = n_t \frac{g}{R} x_{t-1} \,. \tag{23}$$

This price deviation may increase over time when the chartists are strongly extrapolating the price, $g > R$, and when their fraction is large enough. From (18), (19) and (20) we obtain

$$n_t = \left[1 + \exp\left(-\beta \left[\frac{g\, x_{t-3}}{a\sigma^2} \left(x_{t-1} - Rx_{t-2} + \delta_{t-1} + a\sigma^2 S \right) + C \right] \right) \right]^{-1} . \tag{24}$$

If we consider the so-called "deterministic skeleton", that is, take $\delta_t \equiv 0$, the system consisting of (23) and (24) reduces to a 3-dimensional deterministic dynamical system. The next result characterizes the steady states of the deterministic skeleton and its local stability properties. It generalizes Lemma 2 from Brock and Hommes (1998) to the case of positive supply, $S > 0$.[10]

Proposition 3.1. *Consider the system* (23)–(24), *with* $\delta_t \equiv 0$. *Let* $n^{eq} = 1/(1 + e^{-\beta C})$ *and* $n^* = R/g$. *Let* x_+ *and* x_- *denote the solutions (when they exist) to*

$$\frac{1}{n^*} = 1 + \exp\left[-\beta \left(\frac{gx}{a\sigma^2} \left(-r_f x + a\sigma^2 S \right) + C \right) \right], \tag{25}$$

with $x_+ \geq x_-$. *Then:*

1. *for* $0 < g < R$, *the fundamental steady state* $E_1 = (0, n^{eq})$ *is the unique, globally stable steady state.*

---

[10]Hommes, Huang, and Wang (2005) also consider positive supply in an asset pricing model. In their case, however, the asset price is not determined by equilibrium between supply and demand, but by a market maker.

2. for $R < g < 2R$, we introduce two constants $0 < \beta^{SN} < \beta^*$ as follows

$$\beta^{SN} = -\frac{4r_f \ln\frac{g-R}{R}}{4r_f C + a\sigma^2 g S^2} \qquad and \qquad \beta^* = -\frac{1}{C}\ln\frac{g-R}{R}. \qquad (26)$$

We have:

(a) $0 \leq \beta < \beta^{SN}$: the fundamental steady state $E_1 = (0, n^{eq})$ is globally stable;

(b) $\beta^{SN} < \beta < \beta^*$: two non-fundamental steady states $E_2 = (x_+, n^*)$ and $E_3 = (x_-, n^*)$ exist. The steady states E_1 and E_2 are locally stable, the steady state E_3 is unstable. For every β it holds that $x_+ > x_- > 0$.

(c) $\beta > \beta^*$: the fundamental steady state E_1 is unstable, and for every β it holds that $x_+ > 0 > x_-$. The two non-fundamental steady states E_2 and E_3 are locally stable for β small enough; they lose their stability through a Neimark-Sacker bifurcation when β increases.

3. for $g > 2R$, there exist three steady states $E_1 = (0, n^{eq})$, $E_2 = (x_+, n^*)$ and $E_3 = (x_-, n^*)$; the fundamental steady state E_1 is unstable.

Proof. See Appendix B. □

An important finding is that the heterogeneous beliefs asset pricing model can have multiple steady states. The second case of this proposition, dealing with intermediate values of the extrapolation coefficient, $g \in (R, 2R)$, allows for both stability and instability of the fundamental steady state, depending on the intensity of choice. This is illustrated in the bifurcation diagrams in Fig. 5, where we choose parameter values $S = 0.1$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, $a\sigma^2 = 1$ and $C = 1$.

The left panel shows the theoretical result of the second case from Proposition 3.1. The dynamical system exhibits a saddle-node bifurcation at $\beta = \beta^{SN}$ in which two non-fundamental steady states are created and a transcritical bifurcation at $\beta = \beta^*$ where the fundamental steady state loses its stability. Furthermore, when β becomes larger, both non-fundamental steady states lose stability through a Neimark-Sacker bifurcation leading to an invariant closed curve and (quasi-)periodic dynamics. For large values of β, the dynamics of the system are illustrated by the numerical bifurcation diagram, displayed in the right panel of Fig. 5. The two different colors in the bifurcation diagrams correspond to different attractors for the same intensity of choice, but with different initial conditions (starting above and below the fundamental price, respectively). For values of β for which the non-fundamental steady states are unstable the asset price exhibits endogenously generated bubbles and crashes (recall that these are simulations for the model with $\delta_t = 0$ for all t).
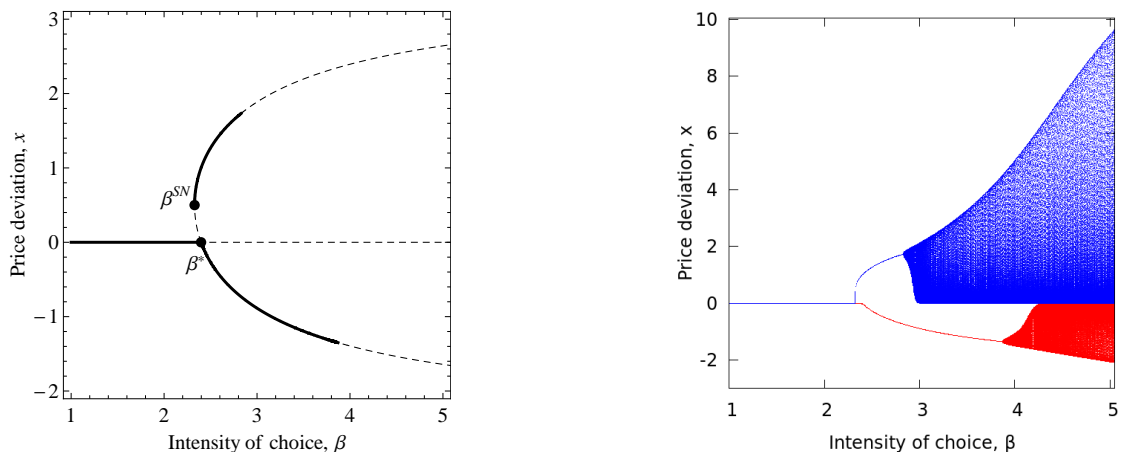
20

Figure 5: Bifurcation diagram for the model with fundamentalists and chartists in absence of trading costs. *Left Panel:* Theoretical diagram illustrating Prop. 3.1, case *2..* The thick curves show the locally stable steady states, the dotted curves show the unstable steady states. *Right panel:* The numerical version of the bifurcation diagram. For each $\beta \in [1, 5]$, 2000 points after 1000 transitory periods are shown. Two attractors are shown by the points of different colour. The parameters are: $S = 0.1$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$.

Comparing these results with the case of zero supply, $S = 0$, studied in Brock and Hommes (1998), one observes that positive supply breaks the symmetry between the upper and lower attractors.[11] When supply is positive agents have a positive position in the risky asset and require a positive return at the fundamental steady state. As a result, the asset price may respond much more strongly to a positive deviation from the fundamental price than to a negative deviation of equal size.

The dynamics in the model is illustrated in Fig. 6 for $\beta = 4$. The left (right) panels show the dynamics on the upper (lower) attractor. From top to bottom the panels show the evolution of prices, fractions of fundamentalists and the positions of both fundamentalists and chartists, respectively. On the upper attractor the price is initially growing, generating positive return. The chartists expect a further price rise and hold the shares of the risky asset, the fundamentalists expect a devaluation and their positions are negative. The relative fraction of chartists is high, not only because fundamentalists pay information costs, but also because chartists' expectations of positive returns are confirmed. The mechanism leading to a crash is endogenous to the model and works as follows. As prices become larger,

---

[11]When $S = 0$ the dynamics exhibits a pitchfork bifurcation at $\beta^* = \beta^{SN}$. Moreover, the bifurcation diagram is symmetric with respect to the line $x = 0$.
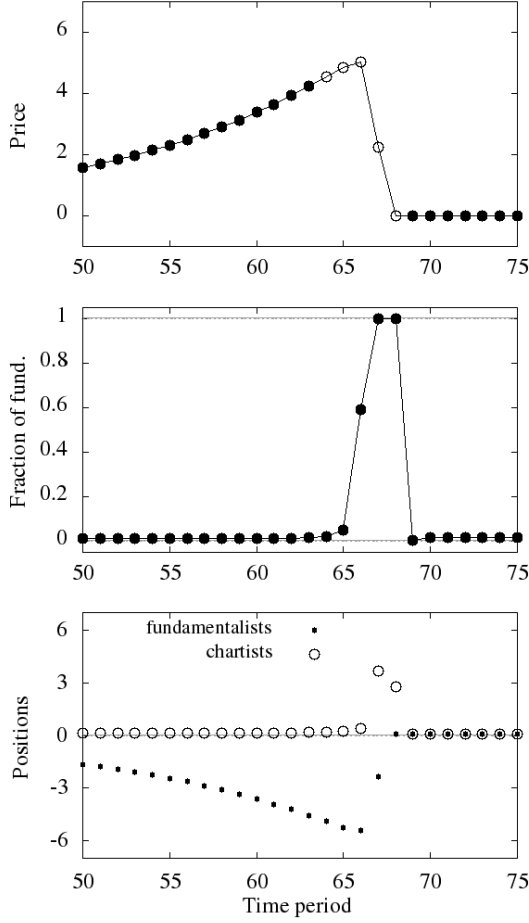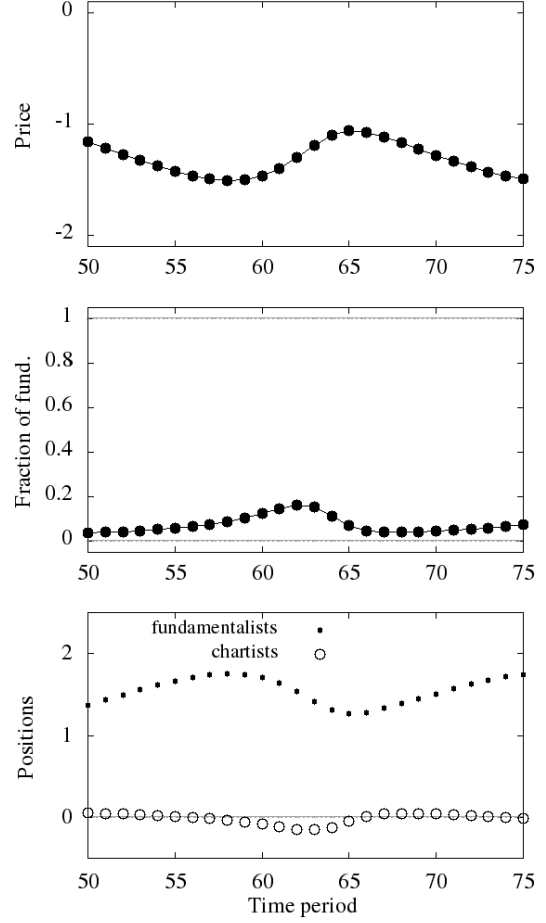
Dynamics on the upper attractor.   Dynamics on the lower attractor.

Figure 6: Dynamics in the benchmark model with fundamentalists and chartists without trading costs. *Upper panels:* prices. Empty disks correspond to the period with negative excess return. *Middle panels:* fraction of fundamentalists. *Lower panels:* positions of fundamentalists (points) and chartists (empty disks). Parameters are: $S = 1$, $\beta = 4$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$.

the capital gain cannot compensate for the lower dividend yield. Return decreases and eventually becomes negative (the periods of negative return are indicated by the empty disks for the price in the top panel). The performance of fundamentalists improves, their fraction increases, and returns continue to decrease. Eventually the increase in the number of fundamentalists is so substantial that the market crashes and prices return to their fundamental value. Here the performance of fundamentalists, relative to that of chartists, is lower due to the information costs and the story repeats.

On the lower attractor we observe cyclical dynamics for the same value of β. On this trajectory the asset is undervalued and the return is always positive: dividend yield outweighs the capital gain effect. Chartists have smaller positions than fundamentalists, and may even go short during some periods, but dominate the market due to costs for the fundamental strategy. An initial fall in prices is reinforced by chartists but is reversed due to their bad relative performance.

3.2 Asset price dynamics with trading costs

We now introduce trading costs in the setting of Section 3.1. Recall that we have fundamentalists (with $E_{1,t}[x_{t+1}] = 0$) and chartists (with $E_{2,t}[x_{t+1}] = gx_{t-1}$), so that the expectation differential is given by $\triangle E_t[x_{t+1}] = -gx_{t-1}$.[12] Substituting these values for $E_{1,t}[x_{t+1}]$, $E_{2,t}[x_{t+1}]$ and $\triangle E_t[x_{t+1}]$ into equations (14), (18), (19) and (20) one obtains the relevant dynamical system (see Appendix C for the resulting equations). In what follows we will investigate the existence of steady states and their local stability properties, as well as the global dynamics of asset prices and how these dynamics depend upon the trading costs T.

3.2.1 Impact of trading costs on steady states

Recall that, for strictly positive but finite values of T and two belief types, there are five different regions, depending upon which belief type, if any, is selling short (see Fig. 4). Steady states may exist in each of these five regions and, as a further complication, explicit solutions to the conditions that implicitly define the steady states in these regions typically do not exist. Therefore we need to resort to numerical methods.

Our findings concerning steady states are summarized in Fig. 7. The left panels show bifurcation diagrams, with the intensity of choice β as the bifurcation parameter, for the case with trading costs (red curves, for $T = 0.1$, $T = 0.2$ and $T = 0.3$, respectively) and for the case without trading costs (black curves, which are identical to the curves in the left panel of Fig. 5). Dashed (black and red) curves correspond to unstable steady states, whereas the solid curves represent steady states that are locally stable. The right panels of Fig. 7 present the same bifurcation curves, but in (n, x)-space. These panels demarcate the five different regions in which the steady state can lie, and depict the position of the steady states in these regions (for different values of β). For a given value of n (not

---

[12]We abstract from the fact that traders in the market may start to use different prediction strategies due to the introduction of trading costs. For example, fundamentalists may believe that prices will take longer to return to the fundamental value under the new market conditions. For the moment we will only consider how the trading costs have an impact on the distribution of traders over the existing belief types. In Section 4 we discuss how this can be extended.
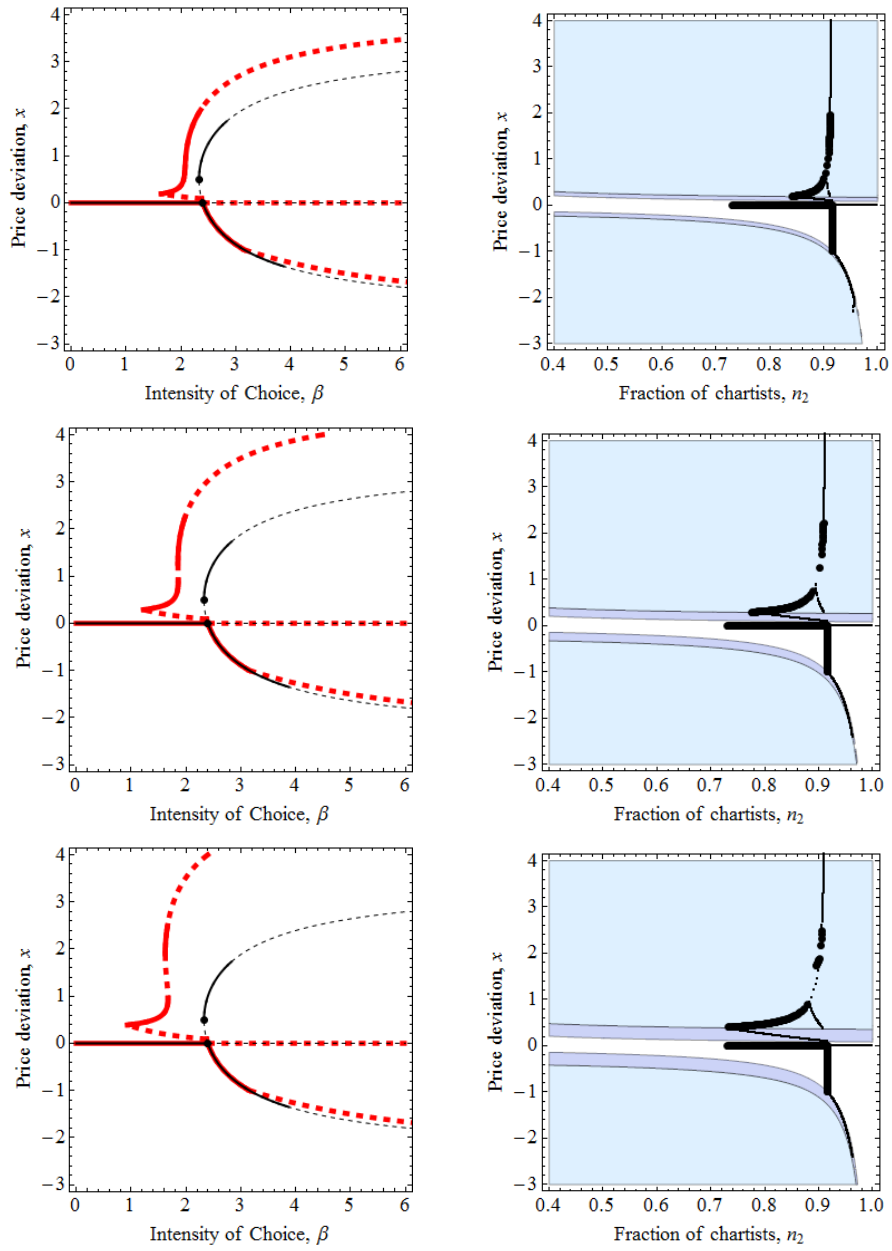
Figure 7: Bifurcation diagrams (left panels) and phase spaces of the steady states corresponding to different $\beta \in [0, 50]$ (right panels) of the asset pricing model with fundamentalists and chartists for $S = 0.1$. *Upper panels: $T = 0.1$; Middle panels: $T = 0.2$; Lower panels: $T = 0.3$.*

too close to 1), moving from low (negative) to high (positive) values of x one

passes the regions where chartists have negative individual demand (lower blue area), chartists have zero demand (lower dark blue area), all traders have positive individual demand (white area), fundamentalists have zero demand (upper dark blue area) and fundamentalists have negative individual demand (upper light blue area). Unstable steady states correspond to the thin black curves and stable steady states are represented by the thick black curves.

Several important observations can be made from these bifurcation diagrams. First, the fundamental steady state (with $x^* = 0$) exists in both the benchmark model and the model with trading costs. Moreover, the critical value β^* for which this fundamental steady state loses stability through a transcritical bifurcation is not affected by the trading costs. This can be easily understood from the fact that at the fundamental steady state all traders have a positive position ($A_{h,t} = S > 0$ for all h). Trading costs therefore do not have any impact on the existence and local stability of the fundamental steady state. This is consistent with the panels on the right of Fig. 7. The fundamental steady state, corresponding to the horizontal black line at $x = 0$, becomes unstable for one particular value of the fraction of chartists, n^{eq} (this critical value of n^{eq} is characterized by $\beta = \beta^*$ from Proposition 3.1), which does not depend on the level of the trading costs.

However, the existence of the non-fundamental steady states does depend, to a substantial extent, on the trading costs. In particular, the non-fundamental steady states emerge for a substantially lower value of the intensity of choice β when $T > 0$. Higher trading costs therefore increase the space of parameters for which mispricing may occur. The right hand panels of Fig. 7 show that these non-fundamental steady states emerge in the area where trading costs are effective and the fundamentalist traders have a zero or negative position in equilibrium.

Related to this, an increase in trading costs, for a given value of β, increases the distance between the upper non-fundamental steady state and the fundamental steady state and therefore increases mispricing in this sense as well. The underlying mechanism is the following. If the price deviation is sufficiently high chartists will buy the risky asset and fundamentalists will sell the asset short. The introduction or increase of trading costs decreases the return on selling the asset short and the fundamentalists therefore require a higher equilibrium price to still be willing to take a negative position in the risky asset.

Note that the location of the lower non-fundamental steady state, when it lies below the fundamental steady state, is only marginally affected by trading costs. In fact, from Fig. 7 it follows that the lower non-fundamental steady state is identical to the one in the benchmark model for a considerable range of values of β. The reason for this is the following. In the lower non-fundamental steady state fundamentalists believe that the asset is undervalued with respect to its fundamental price and will buy the asset. However, since the fraction of chartists

is much higher than the fraction of the fundamentalists, due to the information costs the latter have to pay, chartists may still take a positive position in the risky asset without violating market clearing condition (5). Only for higher values of β the trading costs play a role and chartists holdings of the risky asset in the lower non-fundamental steady state become zero. This leads to a difference between the lower non-fundamental steady state in the model with trading costs and the one in the benchmark model, as the left panels of Fig. 7 illustrate. The price will be higher with trading costs (that is, the absolute price deviation will be smaller) since the chartists now take a zero, instead of a negative, position. Consequently fundamentalists hold fewer shares, which they are willing to do for a smaller (negative) price deviation.

Finally, the critical values of the intensity of choice β related to the secondary bifurcations of the non-fundamental steady-states, that is, the values of β for which these steady states lose stability and endogenous fluctuations emerge, are smaller for higher T. This effect is present both in the upper and the lower non-fundamental steady state. In fact, the right panels of Fig. 7 show that the lower fundamental steady state becomes unstable as soon as the chartists holdings of the risky asset become zero or negative. This implies that introducing trading costs may decrease mispricing, provided the market is at the lower non-fundamental steady state, but only at the expense of destabilizing the market. Note that the upper non-fundamental steady state, where mispricing is increased by the introduction of trading costs, is still locally stable in part of the region where the fundamentalists behavior is influenced by the trading costs.

Summarizing these findings we find that, although existence and local stability of the fundamental steady state are not affected by trading costs, existence, location and local stability of the non-fundamental steady states are. This suggests that the introduction of trading costs may destabilize the financial market and increase volatility, or at least lead to a higher degree of mispricing of the risky asset.

3.2.2 Global dynamics in the presence of trading costs

Fig. 8 shows two bifurcation diagrams obtained by numerical simulations. The left panel presents the bifurcation diagram with respect to β for $T = 0.1$. This diagram is generated with the same parameter values as the right panel of Fig. 5, except for the trading costs. For comparison, the maximal price deviation from Fig. 5 has been superimposed upon the left bifurcation diagram in Fig. 8. The bifurcation diagram is consistent with the steady state analysis from above: for the upper attractor the introduction of trading costs leads to a substantial increase in mispricing and volatility for a wide range of values of the intensity of choice β. The quantitative effect of trading costs on the lower attractor is clearly smaller
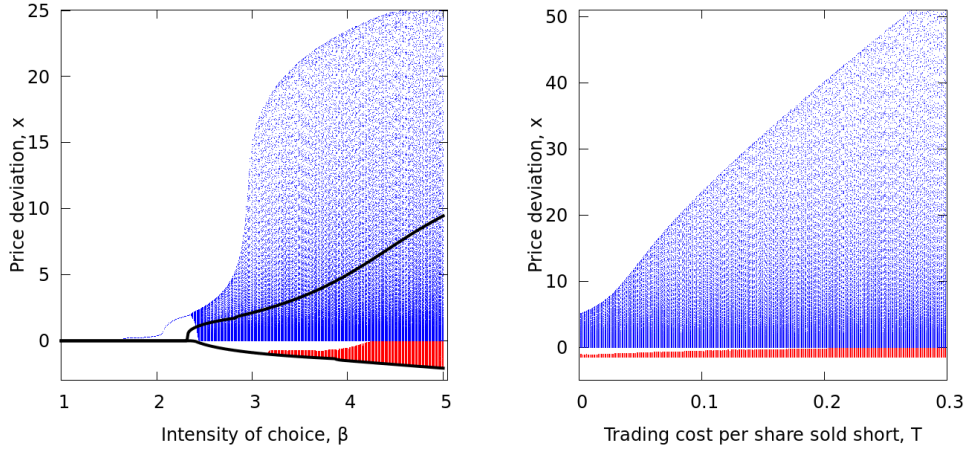
Figure 8: Numerically computed bifurcation diagrams for the model with fundamentalists and chartists with trading costs. For each simulation, 2000 points after 1000 transitory periods are shown. Two attractors are shown by the points of different colour. *Left panel:* β is changing between 1 and 5 and $T = 0.1$. The black curves corresponding to the maximum price deviation for the case without trading costs are superimposed for comparison. *Right panel:* T is changing between 0 and 0.3 and $\beta = 4$. Other parameters are: $S = 0.1$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$.

(although the lower non-fundamental steady state becomes unstable for a much smaller value of β in the presence of trading costs, as can be seen from comparing the left panel of Fig. 8 with the right panel of Fig. 5).

The right panel of Fig. 8 shows a bifurcation diagram with respect to trading costs T, for a fixed value of $\beta = 4$. Again, an increase in trading costs T leads to a substantial increase in volatility, in particular along the upper attractor.

Fig. 9 shows bifurcation diagrams based upon numerical simulations for the same market with a random disturbance, $\varepsilon_t \sim N\left(0, (0.01)^2\right)$, added to the realized price in every period. The two panels of Fig. 9 show the average price of the risky asset (left panel) and the standard deviation of that price (right panel) after a transitory period of 1000 periods for simulations of this stochastic model with different values of β. The red points show the results for the model with trading costs ($T = 0.1$) and the blue points the results for the model without these costs. Fig. 9 is consistent with our earlier findings that trading costs typically increase mispricing and volatility substantially. It also shows that the dynamics of the deterministic version of the model give a good representation of the dynamics in the stochastic model. Finally, Fig. 9 helps in understanding which attractor of the deterministic model is more relevant in case several of these attractors coexist.
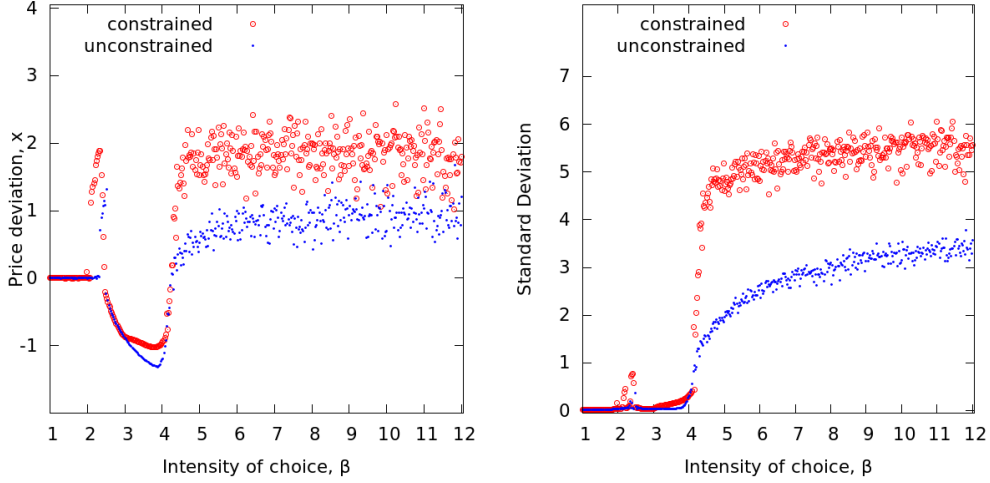
27

Figure 9: Effect of trading costs on the average (left panel) and standard deviation (right panel) of the price for the risky asset for the benchmark model (blue points) and ($T = 0.1$, red points) in the presence of noise. Other parameters are: $S = 0.1$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$. For each $\beta \in [1, 12]$, the statistics are computed for 2000 points after 1000 transitory periods. At each time step of the simulation a random disturbance $\varepsilon_t \sim N\left(0, (0.01)^2\right)$ is added to the realized price.

When only the lower non-fundamental steady state is locally stable, which happens for intermediate values of β, the dynamics of the stochastic model are attracted to that steady state. However, for values of the intensity of choice β such that all steady states of the (deterministic) dynamical system are unstable, and two coexisting attractors exist, dynamics in the stochastic model typically settle on the upper, more volatile, attractor.

How do the time series of prices, fractions and positions of the different traders in the presence of trading costs compare with that of the benchmark model? Fig. 10 shows the trajectories for the same value of $\beta = 4$ and for the same initial conditions as in Fig. 6 but now for the model with trading costs. In particular for the upper attractor this leads to a more extreme bubble and crash pattern, with the price deviation increasing up to around 20 (instead of around 5 in the case without trading costs). The introduction of trading costs for short-selling therefore inhibits the market's potential to correct mispricing.

28
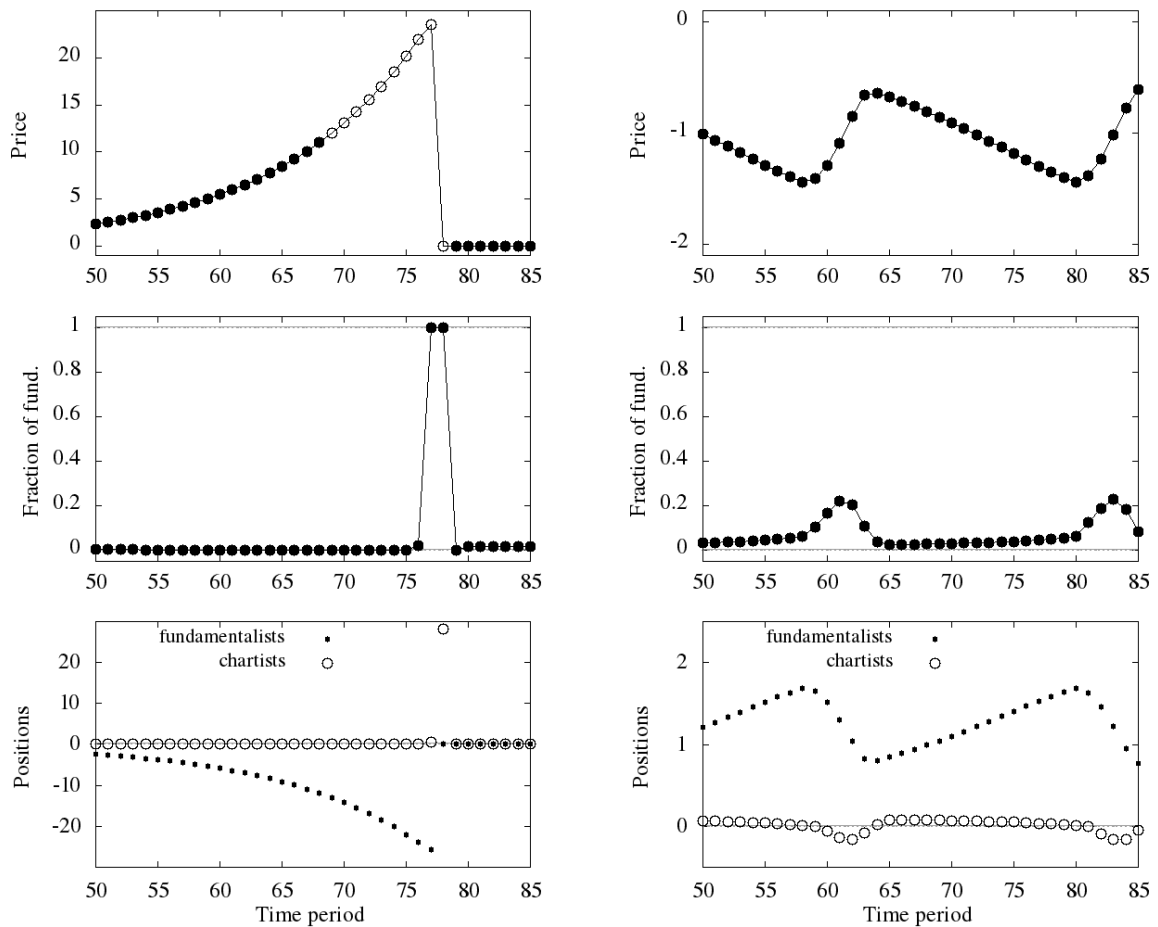
Dynamics on the upper attractor.   Dynamics on the lower attractor.

Figure 10: Dynamics in the model with fundamentalists and chartists and trading costs, $T = 0.1$. *Upper panels:* prices. An empty disks correspond to the period with negative excess return. *Middle panels:* fraction of fundamentalists. *Lower panels:* positions of fundamentalists (points) and chartists (empty disks). Parameters are: $S = 1$, $\beta = 4$, $r_f = 0.1$, $\bar{y} = 10$, $g = 1.2$, and $C = 1$.

4   Conclusion

In this paper we have analyzed the quantitative consequences of imposing short-selling constraints for asset-pricing dynamics in a model with heterogeneous beliefs. Most of the existing literature points out that short-selling restrictions may lead to systematic overvaluation of the security. The intuition for that was provided by Miller (1977) who shows, in a two-period setting, that a diversity of expectations among investors leads to overpricing. Our model formalizes this intuition and

extends it to a dynamic setting.

In our model the demand of myopic investors depends on their expectations of the future price. Expectations are heterogeneous and agents are allowed to switch between different forecasting rules over time. As is well known, the dynamics of such a model depends on the intensity of choice. For low values of this parameter the dynamics converge to the fundamental steady state. For high values of the intensity of choice the model may exhibit price oscillations with excess volatility.

We introduce short-selling constraints in this environment by imposing trading costs for selling the risky asset short. Since trading costs do not have to be paid at the fundamental steady state, where all traders hold a positive amount of the risky asset, existence and local stability of that steady state are unaffected by these costs. However, typically non-fundamental steady states also exist in this environment, and their existence and local stability depends crucially on trading costs. In particular, when there are trading costs for short-selling, these non-fundamental steady states emerge for a wider range of parameters of the underlying model, they may correspond to a much larger degree of mispricing – in particular when the asset is overvalued – and they lose stability for lower values of the intensity of choice parameter. Introducing trading costs for short-selling may therefore very well increase mispricing and price volatility, a feature which is quite robust and confirmed by our numerical simulations.

To study the effect of short-selling constraints, we have deliberately chosen a model with heterogeneous expectations, capable of generating the patterns of bubbles and crashes that financial authorities aim to prevent or mitigate by restricting short-selling.[13] Some features of this model may influence our findings. Consequently, in future research we would like to extend the model in several directions. First, the myopic agents of the model do not take the short-sell constraints into account while forming their expectations. While we believe that such an assumption is quite reasonable in the framework of boundedly rational agents, it would be also interesting to analyze the model with some fraction of rational agents, who take the short-sell constraints into account. Alternatively, one might look at a larger set of belief types and investigate whether the introduction of trading costs changes the (steady state) distribution of the population of traders over these belief types. Second, the constraints which we analyzed are individual, while on the real markets there are many aggregate constraints. For example, the total amount of shares available for short sales is, in reality, limited. The effect of that type of constraints can be analyzed in a large scale agent-based version of the current model.

Finally, the effect of short-selling constraints is closely related to the role of

---

[13]Our paper therefore also contributes to the small but growing literature on using heterogeneous agents models to evaluate regulatory policies, see, e.g., Westerhoff (2008).

margin requirements. Indeed, in a real market selling a share short requires providing some collateral to the broker. If the price of an asset rises, the investor who is short should cover his nominal losses to an extent which depends on the margin requirement. It is not surprising that among the most important questions discussed in the literature on margin requirements is their role in market volatility and the prevention of bubbles. Two opposing points of views can be found in the literature. On the one hand, Seguin and Jarrell (1993) and Hsieh and Miller (1990) argue that margin requirements are empirically irrelevant for price behavior, whereas, e.g., Garbade (1982) and Hardouvelis and Theodossiou (2002) provide theoretical arguments why an increase in margin requirements is beneficial for market stability. Again, with an agent-based extension of the model presented here we plan to analyze the joint effect of short-selling constraints and margin requirements on financial market dynamics and price volatility.

References

ANUFRIEV, M., AND C. HOMMES (2012): "Evolutionary Selection of Individual Expectations and Aggregate Outcomes in Asset Pricing Experiments," *American Economic Journal: Microeconomics*, 4(4), 35 – 64.

ANUFRIEV, M., AND V. PANCHENKO (2009): "Asset prices, traders' behavior and market design," *Journal of Economic Dynamics & Control*, 33, 1073 – 1090.

BAI, Y., E. CHANG, AND J. WANG (2006): "Asset prices under short-sale constraints," Discussion paper, MIT Sloan School of Management, http://web.mit.edu/wangj/www/pap/BCW_061112.pdf.

BARBERIS, N., AND R. THALER (2003): "A survey of behavioral finance," in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. Stultz. North-Holland (Handbooks in Economics Series), Amsterdam.

BEBER, A., AND M. PAGANO (2013): "Short-selling bans around the world: evidence from the $2007 - 09$ crisis," *Journal of Finance*, 68(1), 343–381.

BOEHMER, E., C. JONES, AND X. ZHANG (2009): "Shackling short sellers: the 2008 shorting ban," Discussion paper, Columbia Business School, http://www2.gsb.columbia.edu/faculty/cjones/ShortingBan.pdf.

BROCK, W., AND C. HOMMES (1998): "Heterogeneous beliefs and routes to chaos in a simple asset pricing model," *Journal of Economic Dynamics & Control*, 22(8-9), 1235–1274.

CHANG, E., J. CHENG, AND Y. YU (2007): "Short-sales constraints and price discovery: evidence from the Hong Kong market," *Journal of Finance*, 62(5), 2097–2121.

DERCOLE, F., AND D. RADI (2012): "Constraints on short-selling and asset price adjustment to private information," Discussion paper, mimeo.

DIAMOND, D., AND R. VERRECCHIA (1987): "Constraints on short-selling and asset price adjustment to private information," *Journal of Financial Economics*, 18(2), 277–311.

EVANS, G., AND S. HONKAPOHJA (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press.

GALBRAITH, J. (1954): *The Great Crash* 1929. Boston: Houghton Mifflin.

GALLMEYER, M., AND B. HOLLIFIELD (2008): "An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy," *Review of Finance*, 12(2), 323–364.

GARBADE, K. (1982): "Federal Reserve margin requirements: A regulatory initiative to inhibit speculative bubbles," in *Crises in the Economic and Financial Structure*, ed. by P. Wachtel, pp. 317–36. Free Press.

GAUNERSDORFER, A., C. HOMMES, AND F. WAGENER (2008): "Bifurcation routes to volatility clustering under evolutionary learning," *Journal of Economic Behavior & Organization*, 67(1), 27 – 47.

GLOSTEN, L., AND P. MILGROM (1985): "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 14(1), 71–100.

HARDOUVELIS, G., AND P. THEODOSSIOU (2002): "The asymmetric relation between initial margin requirements and stock market volatility across bull and bear markets," *Review of Financial Studies*, 15(5), 1525–1559.

HARRISON, J., AND D. KREPS (1978): "Speculative investor behavior in a stock market with heterogeneous expectations," *The Quarterly Journal of Economics*, 92(2), 323–336.

HARUVY, E., AND C. NOUSSAIR (2006): "The effect of short selling on bubbles and crashes in experimental spot asset markets," *Journal of Finance*, 61(3), 1119–1157.

HOMMES, C. (2006): "Heterogeneous agent models in economics and finance," in *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics*, ed. by K. Judd, and L. Tesfatsion. Elsevier/North-Holland (Handbooks in Economics Series).

HOMMES, C., H. HUANG, AND D. WANG (2005): "A robust rational route to randomness in a simple asset pricing model," *Journal of Economic Dynamics & Control*, 29(6), 1043–1072.

HOMMES, C., J. SONNEMANS, J. TUINSTRA, AND H. VAN DE VELDEN (2005): "Coordination of Expectations in Asset Pricing Experiments," *Review of Financial Studies*, 18(3), 955–980.

HSIEH, D., AND M. MILLER (1990): "Margin regulation and stock market volatility," *Journal of Finance*, 45(1), 3–29.

JONES, C., AND O. LAMONT (2002): "Short-sale constraints and stock returns," *Journal of Financial Economics*, 66(2-3), 207–239.

LAMONT, O., AND R. THALER (2003): "Can the market add and subtract? Mispricing in tech stock carve-outs," *Journal of Political Economy*, 111(2), 227–268.

LEBARON, B. (2006): "Agent-based computational finance," in *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics*, ed. by K. Judd, and L. Tesfatsion. Elsevier/North-Holland (Handbooks in Economics Series).

LEI, V., C. NOUSSAIR, AND C. PLOTT (2001): "Nonspeculative bubbles in experimental asset markets: lack of common knowledge of rationality vs. actual irrationality," *Econometrica*, 69(4), 831–859.

LUX, T. (2009): "Stochastic behavioral asset pricing models and the stylized facts," in *Handbook of Financial Markets: Dynamics and Evolution*, ed. by T. Hens, and K. Schenk-Hoppé, p. 161. North-Holland.

MILLER, E. M. (1977): "Risk, uncertainty, and divergence of opinion," *Journal of Finance*, 32(4), 1151–1168.

SEGUIN, P., AND G. JARRELL (1993): "The irrelevance of margin: Evidence from the crash of '87," *Journal of Finance*, 48(4), 1457–1473.

SHLEIFER, A. (2000): *Inefficient Markets: An introduction to Behavioral Finance*. Oxford University Press, USA.

SMITH, V., G. SUCHANEK, AND A. WILLIAMS (1988): "Bubbles, crashes, and endogenous expectations in experimental spot asset markets," *Econometrica*, 56(5), 1119–1151.

WESTERHOFF, F. (2008): "The Use of Agent-Based Financial Market Models to Test the Effectiveness of Regulatory Policies," *Jahrbücher für Nationalökonomie und Statistik*, 228, 195–227.

APPENDIX

A Market equilibrium when there are more than two belief types

The procedure outlined in Section 2.2 to determine the market equilibrium price can be straightforwardly extended to more than two belief types. The problem, however, that the number of different regions that are needed to describe the market equilibrium price becomes very high for more than two belief types. For example, for $H = 3$ we already have 19 instead of 5 different regions for which we have to determine the market equilibrium price (given that in equilibrium always at least one belief type has to have strictly positive demand – and therefore the set $P(x_t)$ can never be empty – in general there will be $3^H - 2^H$ different regions when there are H different belief types). Since this is too complicated to work with we outline an algorithm below to determine the market equilibrium price for more than two belief types that can be used in numerical simulations.

The algorithm to compute the market equilibrium price x_t consists of the following steps.

1. For any period t we can, without loss of generality, order the types on the basis of their expectations, $\mathrm{E}_{h,t}[x_{t+1}]$, from the most optimistic type ($h = 1$) to the least optimistic type ($h = H$). This implies that the cut-off values (8) satisfy

$$x_t^H \leq x_t^{H-1} \leq \cdots \leq x_t^2 \leq x_t^1. \tag{27}$$

Note that the ordering of the set of second kink points, which are given by $x_t^h + T$ for $h \in H$, is the same as in (27). The aggregate demand function will have (up to) $2H$ kink points, which we denote z_t^1, ..., z_t^{2H} and rank from high to low as follows:

$$x_t^H = z_t^{2H} \leq z_t^{2H-1} \leq \ldots \leq z_t^2 \leq z_t^1 = x_t^1 + T \tag{28}$$

2. For an arbitrary point x, we determine the sets $P(x)$, $N(x)$ and $Z(x)$ as follows. If we find $h', h'' \in \mathcal{H}$ such that $x_t^{h'+1} < x \leq x_t^{h'}$ and $x_t^{h''+1} + T < x \leq x_t^{h''} + T$, we have $P(x) = \{1, 2, \ldots, h'\}$ and $N(x) = \{h''+1, \ldots, H\}$. Moreover, if $x \leq x_t^H$ (that is, the price is so low that even the most pessimistic trader type wants to buy the asset) we have $P(x) = \mathcal{H}$ and if $x > x_t^1$ (the price is so high that even the most optimistic belief type wants to sell it) we have $P(x) = \emptyset$. Similarly, we have $N(x) = \mathcal{H}$ if $x > x_t^1 + T$ and $N(x) = \emptyset$ if $x_t \leq x_t^H + T$. Finally, we can find $Z(x)$ as the complement of $P(x) \cup N(x)$.

3. We find the equilibrium by evaluating aggregate demand (9) in the kink points given in (28). First compute $A_t\left(z_t^{2H}\right)$. If $A_t\left(z_t^{2H}\right) \leq S$ we know that the market equilibrium satisfies $x_t \leq z_t^{2H}$ and we apply (10) for $P(x) = \mathcal{H}$ to find the market equilibrium price x (which, in this case, will be given by (11)). If $A_t\left(z_t^{2H}\right) > S$, we proceed to the next kink point z_t^{2H-1}. If $A_t\left(z_t^{2H-1}\right) \leq S$ the market equilibrium price satisfies $x \in \left(z_t^{2H}, z_t^{2H-1}\right]$. Using Step 2. we determine the relevant sets $P(x)$, $Z(x)$ and $N(x)$ and find the equilibrium by applying (10) for these given sets and the algorithm stops. If $A_t\left(z_t^{2H-1}\right) > S$ we proceed to kink point z_t^{2H-2}, and so on.

This algorithm will eventually stop and find an equilibrium because aggregate demand will never increase when we move from kink point z_t^k to kink point z_t^{k-1} and in the largest kink point $z_t^1 = x_t^1 + T$ all types have non-positive demands and therefore $A_t\left(z_t^1\right) < S$. Notice that, depending on the expectations of the different belief types, there might be a different ordering of kink points, (27). This prevents us from writing one explicit formula for the equilibrium price.

B   Proof of Proposition 3.1

System (23)-(24) can be written as 3D system in terms of variables x_t and n_t, which are price deviation and the fraction of chartists after the trading[14] in time t, respectively, and a new variable $y_t = x_{t-1}$.

$$
\begin{cases}
x_{t+1} = g n_t x_t / R \\
n_{t+1} = \left(1 + \exp\left\{-\beta\left[\dfrac{g\, y_t}{a\sigma^2}\left(\dfrac{g\, n_t x_t}{R} - R x_t + a\sigma^2 S\right) + C\right]\right\}\right)^{-1} \\
y_{t+1} = x_t.
\end{cases}
$$

Let (x^*, n^*, y^*) denote an arbitrary steady-state. The first equation implies that either $x^* = 0$ or $n^* = R/g$. The first case corresponds to the fundamental steady state, whereas in the second case the non-fundamental steady states may arise. Defining $F(z) := (1 + \exp\{\beta[z - C]\})^{-1}$ and

$$
z\left(x_t, n_t, y_t\right) = -\frac{g\, y_t}{a\sigma^2}\left(\frac{g n_t x_t}{R} - R x_t + a\sigma^2 S\right),
$$

the Jacobian matrix at an arbitrary steady state (x^*, n^*, y^*) is given by

$$
J(x^*, n^*, y^*) = \begin{pmatrix} g n^*/R & g x^*/R & 0 \\ F'(z^*)\frac{\partial z}{\partial x_t} & F'(z^*)\frac{\partial z}{\partial n_t} & F'(z^*)\frac{\partial z}{\partial y_t} \\ 1 & 0 & 0 \end{pmatrix}, \tag{29}
$$

---

[14]Note that to satisfy a convention of writing the dynamical system as variables at time $t+1$ which are functions of variables at time t, we have changed the timing, so that $n_t = n_{2,t+1}$.

where

$$z^* = -\frac{g\,y^*}{a\sigma^2}\left(\frac{gn^*x^*}{R} - Rx^* + a\sigma^2 S\right) = -\frac{g\,(x^*)^2}{a\sigma^2}\left(-r_f + a\sigma^2 S\right).$$

The derivatives are

$$F'(z) = -\frac{\beta\exp(\beta(z-C))}{(1+\exp(\beta(z-C)))^2} = -\beta\left(F(z)^{-1} - 1\right)F(z)^2,$$

$$\frac{\partial z}{\partial x_t} = -\frac{gy_t}{a\sigma^2}\left(\frac{gn_t}{R} - R\right), \quad \frac{\partial z}{\partial n_t} = -\frac{gy_t}{a\sigma^2}\frac{gx_t}{R}, \quad \frac{\partial z}{\partial y_t} = -\frac{g}{a\sigma^2}\left(\frac{gn_tx_t}{R} - Rx_t + a\sigma^2 S\right).$$

Fundamental steady state. As we found above the system has a fundamental steady state. Using the second equation of the system we find that the fraction in this steady state is equal to n_2^{eq}. Since $n_2^{eq} \in [1/2, 1)$, the fundamental steady state always exists. When $g < R$ we obtain from (23) that $|x_{t+1}| < \kappa|x_t|$ for any t with $\kappa = g/R < 1$. This proves the global stability result of the first case. The Jacobian at the fundamental steady state is equal to

$$J(0, n_2^{eq}, 0) = \begin{pmatrix} gn_2^{eq}/R & 0 & 0 \\ 0 & 0 & -F'(0)\,gS \\ 1 & 0 & 0 \end{pmatrix}$$

and the eigenvalues are 0, 0 and gn_2^{eq}/R. When $g > 2R$ we obtain that the last eigenvalue is greater than 1 and the fundamental steady state is unstable. When $R < g < 2R$, we find that the fundamental steady state is locally stable for $\beta < \beta^*$, with β^* defined as in (26).

Non fundamental steady states. When $n^* = R/g$ the non-fundamental steady state may exist, but only if $n^* \in [0, 1]$. Furthermore, equation (25) has to have real solutions. The function on the right-hand side of this equation is bell-shaped and is shown for two values of β in Fig. 11. It is easy to check that the function attains its maximum at the point $x^m = a\sigma^2 S/(2r_f)$ where its value is $(1 + \exp[-\beta(C + a\sigma^2 gS^2/(4r_f))])^{-1}$. Equating the maximum with R/g we find that the two non-fundamental steady states emerge for $\beta = \beta^{SN}$ defined in (26). For $\beta > \beta^{SN}$ the two solutions are given by

$$x_\pm^* = \frac{a\sigma^2 S}{2r_f} \pm \frac{\sqrt{a\sigma^2}\sqrt{a\sigma^2 gS^2\beta + 4r_f\left(\beta C + \ln\left(\frac{g}{R} - 1\right)\right)}}{2r_f\sqrt{g\beta}}.$$

When β increases the value at x^m increases, so that for $\beta > \beta^{SN}$ the two non-fundamental steady states always exist. Notice that since the point of tangency,
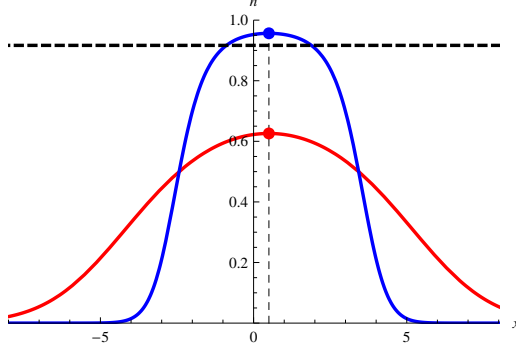
Figure 11: Emergence of the non-fundamental steady-states in $R < g < 2R$ case. The dashed line shows the ratio $R/g \in (1/2, 1)$ from the LHS of (25). Two bell-shape curves illustrate the function from the RHS of (25) for two values of β. When $\beta = 0.5$ (red) there are no steady-states, while for $\beta = 3$ (blue) two steady-states have been created.

$x^m > 0$, initially (i.e., for small β), $x_+^* > x_-^* > 0$. At $\beta = \beta^*$ the second term under the square root in the numerator becomes zero and, obviously, $x_- = 0$. At this moment the steady state $E_3 = (x_-, n_2^*)$ coincides with the fundamental steady state. The Jacobian at the non-fundamental steady state is given by

$$J(x, n^*, x) = \begin{pmatrix} 1 & gx/R & 0 \\ kxr_f & -kgx^2/R & -k(a\sigma^2 S - r_f x) \\ 1 & 0 & 0 \end{pmatrix}, \tag{30}$$

where

$$k = \frac{g}{a\sigma^2} F'\left(-\frac{g\,x}{a\sigma^2}\left(a\sigma^2 S - r_f x\right)\right) = -\beta \frac{g - R}{a\sigma^2} \frac{R}{g},$$

and in the last equality we used that $n_2^* = F(z)$. The characteristic equation is

$$\lambda^3 - \lambda^2\left(1 - \frac{gkx^2}{R}\right) - gkx^2\lambda + \frac{gkx}{R}\left(a\sigma^2 S - r_f x\right) = 0.$$

Take $\beta = \beta^{sn}$, when $x_+ = x_- = a\sigma^2 S/(2r_f)$. Calculations show that $\lambda = 1$ is one of the eigenvalues, while the two others are given by

$$\lambda_\pm = \frac{-(a\sigma^2)^2 gkS^2 \pm \sqrt{(a\sigma^2)^2 gkS^2\left(16r_f^3 R + (a\sigma^2)^2 gkS^2\right)}}{8r_f^2 R}.$$

When $S = 0$ both eigenvalues $\lambda_\pm$ are zeros. Hence, when S is positive but small enough, they are within the unit circle. Therefore, when β gets larger than β^{sn},

38

one of the non-fundamental steady states should be unstable while the other one is stable. At $\beta = \beta^* > \beta^{sn}$ the lower non-fundamental steady state, E_3, undergoes a bifurcation by coinciding with the fundamental steady state, which is stable for $\beta < \beta^*$. We then conclude that it is E_3 that is unstable and E_2 that is stable for β slightly higher than the bifurcation value at β^{sn}. When $\beta \to \infty$ the steady states approach the values

$$\frac{a\sigma^2 S}{2r_f} \pm \frac{\sqrt{a\sigma^2}\sqrt{a\sigma^2 g S^2 + 4r_f C}}{2r_f\sqrt{g}},$$

whereas $k \to -\infty$. It implies that the sum of eigenvalues diverges (it is equal to $gkx^2/R - 1$ from the characteristic equation), implying that they lose stability.

C   Market dynamics with trading costs and fundamentalists versus chartists

Recall that we denote $n_t = n_{2,t}$, the fraction of chartists. The price dynamics in deviations, given in general by (14), when substituting $\mathrm{E}_{1,t}[x_{t+1}] = 0$, $\mathrm{E}_{2,t}[x_{t+1}] = gx_{t-1}$ and $\triangle \mathrm{E}_t[x_{t+1}] = -gx_{t-1}$ becomes

$$x_t = \begin{cases} \frac{n_t}{R}\left(gx_{t-1} + RT\right) & \text{if } x_{t-1} \leq -\frac{a\sigma^2 S}{g(1-n_t)} - \frac{RT}{g} \\[2mm] -\frac{1}{R}\frac{n_t}{1-n_t}a\sigma^2 S & \text{if } -\frac{a\sigma^2 S}{g(1-n_t)} - \frac{RT}{g} < x_{t-1} \leq -\frac{a\sigma^2 S}{g(1-n_t)} \\[2mm] \frac{1}{R}n_t gx_{t-1} & \text{if } -\frac{a\sigma^2 S}{g(1-n_t)} \leq x_{t-1} \leq \frac{a\sigma^2 S}{gn_t} \\[2mm] \frac{1}{R}\left(gx_{t-1} - \frac{1-n_t}{n_t}a\sigma^2 S\right) & \text{if } \frac{a\sigma^2 S}{gn_t} \leq x_{t-1} \leq \frac{a\sigma^2 S}{gn_t} + \frac{RT}{g} \\[2mm] \frac{1}{R}\left((1-n_t)RT + n_t gx_{t-1}\right) & \text{if } x_{t-1} > \frac{a\sigma^2 S}{gn_t} + \frac{RT}{g} \end{cases} \tag{31}$$

The fraction of chartists evolves according to

$$n_{t+1} = \left(1 + \exp\left\{\beta \triangle U_t\right\}\right)^{-1}, \tag{32}$$

where $\triangle U_t = U_{1,t} - U_{2,t}$ is given by (19), which in this case becomes (note that the exogenously given cost differential is $\triangle C = C_1 - C_2 = C$)

$$\triangle U_t = \begin{cases} \tilde{r}_t \triangle A_{t-1} - RTA_{2,t-1} - C & \text{if } x_{t-2} \leq -\frac{a\sigma^2 S}{(1-n_{t-1})g} - \frac{RT}{g} \\[2mm] \tilde{r}_t \triangle A_{t-1} - C & \text{if } -\frac{a\sigma^2 S}{g(1-n_{t-1})} - \frac{RT}{g} < x_{t-2} \leq \frac{a\sigma^2 S}{gn_{t-1}} + \frac{RT}{g} \\[2mm] \tilde{r}_t \triangle A_{t-1} + RTA_{1,t-1} - C & \text{if } x_{t-2} > \frac{a\sigma^2 S}{gn_{t-1}} + \frac{RT}{g} \end{cases}$$

$$(33)$$

with $A_{1,t} = -\frac{1}{a\sigma^2} R\left(x_{t-1} - T\right) + S$, $A_{2,t} = \frac{1}{a\sigma^2}\left(gx_{t-2} - R\left(x_{t-1} - T\right)\right) + S$ and

$$
\triangle A_t = \begin{cases}
-\frac{1}{a\sigma^2}\left(gx_{t-2} + RT\right) & \text{if } x_{t-2} \leq -\frac{a\sigma^2 S}{g(1-n_{t-1})} - \frac{RT}{g} \\[2mm]
-\frac{1}{a\sigma^2} Rx_{t-1} + S & \text{if } -\frac{a\sigma^2 S}{g(1-n_{t-1})} - \frac{RT}{g} < x_{t-2} \leq -\frac{a\sigma^2 S}{g(1-n_{t-1})} \\[2mm]
-\frac{1}{a\sigma^2} gx_{t-2} & \text{if } -\frac{a\sigma^2 S}{g(1-n_{t-1})} < x_{t-2} \leq \frac{a\sigma^2 S}{gn_{t-1}} \\[2mm]
-\left(\frac{1}{a\sigma^2}\left(gx_{t-2} - Rx_{t-1}\right) + S\right) & \text{if } \frac{a\sigma^2 S}{gn_{t-1}} < x_{t-2} \leq \frac{a\sigma^2 S}{gn_{t-1}} + \frac{RT}{g} \\[2mm]
-\frac{1}{a\sigma^2}\left(gx_{t-2} - RT\right) & \text{if } x_{t-2} > \frac{a\sigma^2 S}{gn_{t-1}} + \frac{RT}{g}
\end{cases}
$$

$$(34)$$

The full dynamical system is now given by equations (31), (32), (33) and (34). Note that for $T = 0$ this system reduces to (23)-(24) studied in Subsection 3.1.

As another special case, consider $T \to \infty$. This is equivalent with a full ban on short selling and it implies that only three of the five regions remain. We then obtain equilibrium price

$$
x_t = \begin{cases}
-\frac{1}{R}\frac{n_t}{1-n_t}a\sigma^2 S & \text{if } x_{t-1} \leq -\frac{a\sigma^2 S}{g(1-n_t)} \\[2mm]
\frac{1}{R}n_t gx_{t-1} & \text{if } -\frac{a\sigma^2 S}{g(1-n_t)} < x_{t-1} \leq \frac{a\sigma^2 S}{gn_t} \\[2mm]
\frac{1}{R}\left(gx_{t-1} - \frac{1-n_t}{n_t}a\sigma^2 S\right) & \text{if } x_{t-1} > \frac{a\sigma^2 S}{gn_t}
\end{cases}
$$

As above, let us denote by n_t the fraction of chartists (i.e., $n_t = n_{2,t}$) in period t. This fraction evolves according to

$$n_{t+1} = \left(1 + \exp\{\beta \triangle U_t\}\right)^{-1},$$

where $\triangle U_t = \tilde{r}_t \triangle A_{t-1} - C$ with

$$
\triangle A_t = \begin{cases}
-\frac{1}{a\sigma^2} Rx_{t-1} + S & \text{if } x_{t-2} \leq -\frac{a\sigma^2 S}{g(1-n_{t-1})} \\[2mm]
-\frac{1}{a\sigma^2} gx_{t-2} & \text{if } -\frac{a\sigma^2 S}{g(1-n_{t-1})} < x_{t-2} \leq \frac{a\sigma^2 S}{gn_{t-1}} \\[2mm]
-\left(\frac{1}{a\sigma^2}\left(gx_{t-2} - Rx_{t-1}\right) + S\right) & \text{if } x_{t-2} > \frac{a\sigma^2 S}{gn_{t-1}}
\end{cases}
$$